Analytical Solution Of The Characteristic Equation Of A Three-Dimensional Dynamic Chain

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***Abstract*— This paper examines the dynamic diagram of a mechanical system consisting of a sequence of three mass points in one-dimensional, rectilinear, free motion, interacting with each other via elastic and damping elements (a three-dimensional, homogeneous dynamic chain). The mathematical model of the classical system's dynamics is based on the Lagrange method of the second kind, where [units] are the generalized values of the deviation of a point mass from the structure of the initial coordinates. The equations of motion of the three-dimensional dynamic chain are transformed into a sixth-order canonical matrix form in phase coordinates. A sixth-degree characteristic equation is constructed, and its solution is analytically determined based on the parameters of the dynamic chain. The solution is represented in the complex plane, where the distribution of roots ensures the chain's properties, and the subsequent analytical design procedure selects the variable parameters.**

Keywords*— *dynamic chain; mathematical model; characteristic equation; root distribution; dynamic design

I. INTRODUCTION

This work is a continuation of the article [1], where a two-dimensional dynamic chain was considered, and is devoted to the search for analytical solutions for a three-dimensional dynamic chain. Exact, analytical methods for solving dynamic problems, having a number of advantages [2, 3], are developed along with universal, approximate, numerical methods, complementing each other [4, 5]. Analytical methods are constructed heuristically on the basis of symmetry, asymmetry, similarity, analogy [6, 7, 8] and are presented in an ordered mathematical form in the form of special, singular matrices and corresponding determinants [9, 10, 11]. An example of the effective application of this approach is the scientific direction in the analytical study of the dynamics of random processes in complex systems, the calculation scheme of which is an asymmetric Markov chain and the corresponding Kolmogorov differential equations [12, 13, 14, 15, 16]. The objective of this work is to apply a similar heuristic approach to the analytical modeling and design of technical systems whose computational model is a three-dimensional dynamic chain. This objective is achieved by using the following algorithm:

- constructing a computational model of the technical system as a three-dimensional dynamic chain;
- creating a mathematical model of the three-dimensional dynamic chain in the form of Lagrange equations;
- reducing the mathematical model to a sixth-order canonical matrix form in phase coordinates;
- generating a characteristic determinant based on the canonical matrix form;
- transforming the characteristic equation of the sixth-order canonical matrix model to an algebraic equation of the sixth degree;
- determining the coefficients of the algebraic equation in the parameters of the three-dimensional dynamic chain;
- analytical solution of the characteristic equation in the parameters of the three-dimensional dynamic chain;
- verification of the roots of the characteristic equation of the three-dimensional dynamic target;
- analysis of the distribution of the roots of the characteristic equation in the complex plane.

II. Statement of the Problem

A three-dimensional dynamic chain is considered, the computational model for which is shown in Fig. 1.

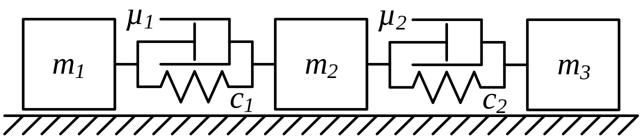

Fig. 1. *Computational model of a three-dimensional dynamic chain (m_1, m_2, m_3 are the masses of the chain links; c_1, c_2 are the elasticity coefficients; μ_1, μ_2 are the damping coefficients)*

The deviations of the chain links from their natural configuration, $q_1(t)$, $q_2(t)$, $q_3(t)$ are taken as generalized coordinates.

A mathematical model of the free motion of a three-dimensional dynamic chain in canonical matrix form and the corresponding characteristic equation are required.

Find an analytical partial solution to the characteristic equation and establish the distribution of roots in the complex plane, technically feasible by selecting the design parameters of the three-dimensional dynamic chain.

III. Mathematical Model

Free motion of a three-dimensional dynamic chain defined by a system of linear Lagrange differential equations:

$$m_1\ddot{q}_1 = -\dot{q}_1\mu_1 - q_1c_1 + \dot{q}_2\mu_1 + q_2c_1;$$
$$m_2\ddot{q}_2 = \dot{q}_1\mu_1 + q_1c_1 - \dot{q}_2(\mu_1+\mu_2) - q_2(c_1+c_2) + \dot{q}_3\mu_2 + q_3c_2;$$
$$m_3\ddot{q}_3 = \dot{q}_2\mu_2 + q_2c_2 - \dot{q}_3\mu_2 + q_3c_2;$$

New variables (phase coordinates) are introduced:

$$\dot{q}_1 = x_1,\ q_1 = x_2,\ \dot{x}_1 = \ddot{q}_1,\ \dot{x}_2 = x_1;$$
$$\dot{q}_2 = x_3,\ q_2 = x_4,\ \dot{x}_3 = \ddot{q}_2,\ \dot{x}_4 = x_3;$$
$$\dot{q}_3 = x_5,\ q_3 = x_6,\ \dot{x}_5 = \ddot{q}_3,\ \dot{x}_6 = x_5.$$

The mathematical model of a three-dimensional dynamic chain takes the following standard matrix form:

$$\frac{d}{dt}[X_i] = D\cdot\ [X_i] \qquad (i = 1,2,3,4,5,6).$$

where $[X_i]$ – is the column matrix of phase coordinates;

D – is a dynamic square matrix of sixth order:

$$[X_i] = \begin{Vmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{Vmatrix};\ D = \begin{Vmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{Vmatrix},$$

Here

$$d_{11} = \begin{Vmatrix} -\dfrac{\mu_1}{m_1} & -\dfrac{c_1}{m_1} \\ 1 & 0 \end{Vmatrix},\ d_{22} = \begin{Vmatrix} -\dfrac{\mu_1+\mu_2}{m_2} & -\dfrac{c_1+c_2}{m_2} \\ 1 & 0 \end{Vmatrix},$$

$$d_{33} = \begin{Vmatrix} -\dfrac{\mu_2}{m_3} & -\dfrac{c_2}{m_3} \\ 1 & 0 \end{Vmatrix},$$

$$d_{12} = \begin{Vmatrix} \dfrac{\mu_1}{m_1} & \dfrac{c_1}{m_1} \\ 0 & 0 \end{Vmatrix},\ d_{21} = \begin{Vmatrix} \dfrac{\mu_1}{m_2} & \dfrac{c_1}{m_2} \\ 0 & 0 \end{Vmatrix},\ d_{13} = d_{31} = \begin{Vmatrix} 0 & 0 \\ 0 & 0 \end{Vmatrix},$$

$$d_{23} = \begin{Vmatrix} \dfrac{\mu_2}{m_2} & \dfrac{c_2}{m_2} \\ 0 & 0 \end{Vmatrix},\ d_{32} = \begin{Vmatrix} \dfrac{\mu_2}{m_3} & \dfrac{c_2}{m_3} \\ 0 & 0 \end{Vmatrix}.$$

The corresponding dynamic matrix of the sixth-order characteristic determinant is formed:

$$\Delta(\lambda) = \det(D - \lambda E),$$

where E – is the unique sixth-order matrix.

The characteristic equation

$$\Delta(\lambda) = 0$$

is transformed into a sixth-order algebraic multiplication:

$$a_6\lambda^6 + a_5\lambda^5 + a_4\lambda^4 + a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0$$

Here, the coefficients of the algebraic equation are pairwise ordered sixth-order asymmetric determinants, composed by the columns of matrix D and the corresponding columns of matrix – E as follows:

$$a_6 = |0\;0\;0\;0\;0\;0|;$$

$$a_0 = |1\;2\;3\;4\;5\;6|;$$

$$\begin{aligned} a_5 = &|1\;0\;0\;0\;0\;0| + \\ +&|0\;2\;0\;0\;0\;0| + \\ +&|0\;0\;3\;0\;0\;0| + \\ +&|0\;0\;0\;4\;0\;0| + \\ +&|0\;0\;0\;0\;5\;0| + \\ +&|0\;0\;0\;0\;0\;6|; \end{aligned}$$

$$\begin{aligned} a_1 = &|0\;2\;3\;4\;5\;6|; \\ +&|1\;0\;3\;4\;5\;6| + \\ +&|1\;2\;0\;4\;5\;6| + \\ +&|1\;2\;3\;0\;5\;6| + \\ +&|1\;2\;3\;4\;0\;6| + \\ +&|1\;2\;3\;4\;5\;0|; \end{aligned}$$

$$\begin{aligned} a_4 = &|1\;2\;0\;0\;0\;0| + \\ +&|1\;0\;3\;0\;0\;0| + \\ +&|1\;0\;0\;4\;0\;0| + \\ +&|1\;0\;0\;0\;5\;0| + \\ +&|1\;0\;0\;0\;0\;6| + \\ +&|0\;2\;3\;0\;0\;0| + \\ +&|0\;2\;0\;4\;0\;0| + \\ +&|0\;2\;0\;0\;5\;0| + \\ +&|0\;2\;0\;0\;0\;6| + \\ +&|0\;0\;3\;4\;0\;0| + \\ +&|0\;0\;3\;0\;5\;0| + \\ +&|0\;0\;3\;0\;0\;6| + \\ +&|0\;0\;0\;4\;5\;0| + \\ +&|0\;0\;0\;4\;0\;6| + \\ +&|0\;0\;0\;0\;5\;6|; \end{aligned}$$

$$\begin{aligned} a_2 = &|0\;0\;3\;4\;5\;6| + \\ +&|0\;2\;0\;4\;5\;6| + \\ +&|0\;2\;3\;0\;5\;6| + \\ +&|0\;2\;3\;4\;0\;6| + \\ +&|0\;2\;3\;4\;5\;0| + \\ +&|1\;0\;0\;4\;5\;6| + \\ +&|1\;0\;3\;0\;5\;6| + \\ +&|1\;0\;3\;4\;0\;6| + \\ +&|1\;0\;3\;4\;5\;0| + \\ +&|1\;2\;0\;0\;5\;6| + \\ +&|1\;2\;0\;4\;0\;6| + \\ +&|1\;2\;0\;4\;5\;0| + \\ +&|1\;2\;3\;0\;0\;6| + \\ +&|1\;2\;3\;0\;5\;0| + \\ +&|1\;2\;3\;4\;0\;0|; \end{aligned}$$

$$\begin{aligned} a_3 = &|1\;2\;3\;0\;0\;0| + \\ +&|1\;2\;0\;4\;0\;0| + \\ +&|1\;2\;0\;0\;5\;0| + \\ +&|1\;2\;0\;0\;0\;6| + \\ +&|1\;0\;3\;4\;0\;0| + \\ +&|1\;0\;3\;0\;5\;0| + \\ +&|1\;0\;3\;0\;0\;6| + \\ +&|1\;0\;0\;4\;5\;0| + \\ +&|1\;0\;0\;4\;0\;6| + \\ +&|1\;0\;0\;0\;5\;6| + \\ +&|0\;2\;3\;4\;0\;0| + \\ +&|0\;2\;3\;0\;5\;0| + \\ +&|0\;2\;3\;0\;0\;6| + \\ +&|0\;2\;0\;4\;5\;0| + \\ +&|0\;2\;0\;4\;0\;6| + \\ +&|0\;2\;0\;0\;5\;6| + \\ +&|0\;0\;3\;4\;5\;0| + \\ +&|0\;0\;3\;4\;0\;6| + \\ +&|0\;0\;3\;0\;5\;6| + \\ +&|0\;0\;0\;4\;5\;6|; \end{aligned}$$

$$\begin{aligned} a_3 = &|0\;0\;0\;4\;5\;6| + \\ +&|0\;0\;3\;0\;5\;6| + \\ +&|0\;0\;3\;4\;0\;6| + \\ +&|0\;0\;3\;4\;5\;0| + \\ +&|0\;2\;0\;0\;5\;6| + \\ +&|0\;2\;0\;4\;0\;6| + \\ +&|0\;2\;0\;4\;5\;0| + \\ +&|0\;2\;3\;0\;0\;6| + \\ +&|0\;2\;3\;0\;5\;0| + \\ +&|0\;2\;3\;4\;0\;0| + \\ +&|1\;0\;0\;0\;5\;6| + \\ +&|1\;0\;0\;4\;0\;6| + \\ +&|1\;0\;0\;4\;5\;0| + \\ +&|1\;0\;3\;0\;0\;6| + \\ +&|1\;0\;3\;0\;5\;0| + \\ +&|1\;0\;3\;4\;0\;0| + \\ +&|1\;2\;0\;0\;0\;6| + \\ +&|1\;2\;0\;0\;5\;0| + \\ +&|1\;2\;0\;4\;0\;0| + \\ +&|1\;2\;3\;0\;0\;0|. \end{aligned}$$

It seems appropriate to consider a homogeneous, three-dimensional dynamic chain, i.e.,

$$m_1 = m_2 = m_3 = m,\; c_1 = c_2 = c,\; \mu_1 = \mu_2 = \mu$$

and introduce mass-reduced elasticity and damping coefficients:

$$\bar{c} = \frac{c}{m},\; \bar{\mu} = \frac{\mu}{m}.$$

Then, due to the structural features of the dynamic chain, the coefficients of the characteristic equation take the form:

$$\begin{aligned} &a_6 = 1, && a_0 = 0, \\ &a_5 = 4\bar{\mu}, && a_1 = 0, \\ &a_4 = 4\bar{c} + 3\bar{\mu}^2, && a_2 = 3\bar{c}^2, \\ &a_3 = 6\bar{\mu}\cdot\,\bar{c}, && \end{aligned}$$

and the sixth-degree characteristic equation reduces to a fourth-degree equation:

$$\lambda^2\cdot\,(a_6\lambda^4 + a_5\lambda^3 + a_4\lambda^2 + a_3\lambda + a_2) = 0.$$

An algebraic fourth-degree equation admits a particular analytical solution [10, 11]:

$$\lambda_{3,4,5,6} = -\frac{a_5}{4} \pm$$
$$\pm \frac{1}{\sqrt{2}} \sqrt{\left[\frac{3}{2} \left(\frac{a_5}{2} \right)^2 - a_4 \right] \pm \sqrt{\left[\left(\frac{a_5}{2} \right)^2 - a_4 \right]^2 - 4a_2}},$$

subject to:

$$a_3 + \frac{a_5}{2} \left[\left(\frac{a_5}{2} \right)^2 - a_4 \right] = 0.$$

From which follows the particular analytical solution of the original characteristic equation of a homogeneous, three-dimensional dynamic chain:

$$\lambda_{1,2} = 0,\ \lambda_{3,4} = -\frac{1}{2}\bar{\mu} \pm i\bar{\mu}\frac{\sqrt{3}}{2},\ \lambda_{5,6} = -\frac{3}{2}\bar{\mu} \pm i\bar{\mu}\frac{\sqrt{3}}{2}$$

subject to: $\bar{c} = \bar{\mu}^2$, and also:

$$\lambda_{1,2} = 0,\ \lambda_{3,4} = \pm i\sqrt{\bar{c}},\ \lambda_{5,6} = \pm i\sqrt{3}\sqrt{\bar{c}}$$

subject to:

$$\bar{\mu} = 0.$$

Verification of the resulting analytical solutions is carried out by applying Vieta's theorem [17, 18]. Given the following conditions:

$$\bar{c} = \bar{\mu}^2$$

the coefficients of the characteristic equation take the form:

$$a_5 = 4\bar{\mu},\ a_4 = 7\bar{\mu}^2,\ a_3 = 6\bar{\mu}^3,\ a_2 = 3\bar{\mu}^4$$

and using Vieta's formulas, we obtain:

$$-a_5 = Tr[D] = \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 = -4\bar{\mu},$$
$$a_4 = \lambda_3\lambda_4 + \lambda_3\lambda_5 + \lambda_3\lambda_6 + \lambda_4\lambda_5 + \lambda_4\lambda_6 + \lambda_5\lambda_6 = 7\bar{\mu}^2,$$
$$-a_3 = \lambda_3\lambda_4\lambda_5 + \lambda_3\lambda_4\lambda_6 + \lambda_3\lambda_5\lambda_6 + \lambda_4\lambda_5\lambda_6 = -6\bar{\mu}^3,$$
$$a_2 = \lambda_3\lambda_4\lambda_5\lambda_6 = 3\bar{\mu}^4.$$

Given the following conditions:

$$\bar{\mu} = 0$$

the coefficients of the characteristic equation take the form:

$$a_5 = 0,\ a_4 = 4\bar{c},\ a_3 = 0,\ a_2 = 3\bar{c}^2$$

and using Vieta's formulas, we obtain:

$$-a_5 = \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 = 0,$$
$$a_4 = \lambda_3\lambda_4 + \lambda_3\lambda_5 + \lambda_3\lambda_6 + \lambda_4\lambda_5 + \lambda_4\lambda_6 + \lambda_5\lambda_6 = 4\bar{c},$$
$$-a_3 = \lambda_3\lambda_4\lambda_5 + \lambda_3\lambda_4\lambda_6 + \lambda_3\lambda_5\lambda_6 + \lambda_4\lambda_5\lambda_6 = 0,$$
$$a_2 = \lambda_3\lambda_4\lambda_5\lambda_6 = 3\bar{c}^2.$$

Analytical design of a dynamic chain, using modal control theory [19], is based on analyzing and ensuring the required distribution of the roots of the characteristic equation in the complex plane by selecting variable parameters of the technical system.

In the case where the parameters of a homogeneous, three-dimensional dynamic chain are related by the condition:

$$\bar{c} = \bar{\mu}^2,$$

the distribution of roots λ_i (i=1,2,3,4,5,6) in the complex plane is shown in Figure 2:

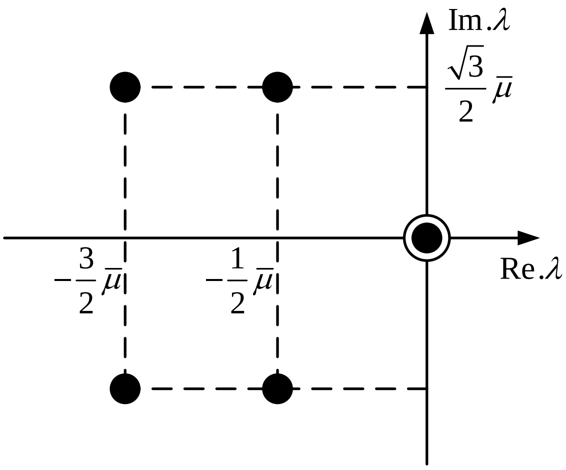

Fig. 2. *Double zero and two complex conjugate roots with a negative real part and an equal imaginary part.*

Here, the dynamic chain is unstable overall, with transient processes damped by the superposition of oscillations with equal frequencies (resonance).

In the case where:

$$\bar{\mu} = 0,$$

the corresponding distribution of roots is shown in Figure 3:

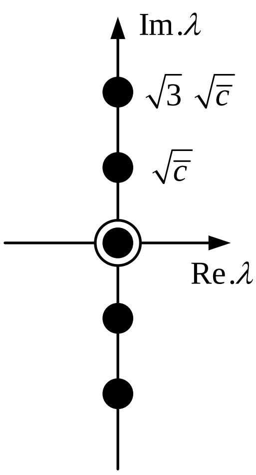

Fig. 3. *Double zero and two different imaginary roots.*

Here, the dynamic chain as a whole is unstable, and transient processes are undamped during superpositions of oscillations with different frequencies (beating).

Thus, the obtained particular analytical solution of the characteristic equation of the sixth degree allows for the implementation of analytical design and modeling of transient processes in a three-dimensional, defining dynamic target, using modal control.

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