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Anisotropic Plates: New Analysis For Onset Of Buckling

Girish Kumar Ramaiah[1] and KazaVijayakumar[2*]
[1]Department of Mechanical Engineering, GITAM University, Visakhapatnam, 560045, India
Email: gkramaiah@gmail.com
[2]Department of Aerospace Engineering, Indian Institute of Science Bangalore, 560012, India
*Email kazavijayakumar@gmail.com

***Abstract*—Analysis of buckling of anisotropic plates within small deformation theory of elasticity is considered. Kirchhoff's vertical displacement as a face variable introduced in our recently published monograph, 'Poisson Theory of Elastic Plates' is used here for the analysis of onset of buckling of anisotropic plates. Due to coupling of extension problems subjected to in-plane compressive loads, the analysis is presented here in terms of polynomial functions as used in the above cited monograph. Due to difficulty in generating the software for the higher order polynomial functions, Fourier Sinusoidal series for higher order corrections of six stress components is suggested for proper estimation of buckling loads.**

Keywords—Elasticity; Plates; Anisotropy; Buckling; Poisson Theory

I. Introduction

In extension problems, in-plane stresses in terms of Airy's stress function are independent of material constants. The displacements are, however, dependent on material constants and solutions for displacements are through satisfying the compatibility conditions (three in the interior and three on the boundary). Analysis is confined, here, to displacement-based theories. Initial displacement variables $[u, v]_0$ in extension problems and transverse shear stresses in the auxiliary bending problems are independent of thickness-wise coordinate (z). Kirchhoff displacement w_0 is generally a domain variable in the reported two-dimensional theories. In-plane displacements $[u, v]_0$ are domain variables and solutions for them remain same for both static and z-integrated equilibrium equations. Kirchhoff's w_0 is treated as a face variable in our recently published monograph [1] in resolving seventeen-decade old Poisson-Kirchhoff boundary conditions paradox. Its utility as a face variable is initiated here in the analysis of buckling problems.

II PRELIMINARIES

For simplicity in presentation, a square anisotropic plate bounded within $0 \le X, Y \le a$, $Z= \pm h$ planes with reference to Cartesian co-ordinate system (X, Y, Z) is considered. Coordinates x= X/a, y=Y/a, z=Z/h, displacements (u, v, w) = (U, V, W)/h, and half-thickness ratio α = (h/a) in non-dimensional form are used. Equilibrium equations in stress components are (with 3-D stress components as functions of coordinates x, y, and z)

$$\alpha\,(\sigma_{x,x} + \tau_{xy,y}) + \tau_{xz,z} = 0 \qquad (1a)$$
$$\alpha\,(\sigma_{y,y} + \tau_{xy,x}) + \tau_{yz,z} = 0 \qquad (1b)$$
$$\alpha\,(\tau_{xz,x} + \tau_{yz,y}) + \sigma_{z,z} = 0 \qquad (2)$$

in which suffix after ',' denotes partial derivative operator.

Stress-Strain and Strain-Displacement Relations

In displacement-based models, stress components are expressed in terms of displacements, via, six stress-strain constitutive relations, and six strain-displacement relations.

In the present study, these relations are confined to the classical small deformation theory of elasticity.

In a recent monograph [1], preliminary solutions of transverse stresses in bending are governed by Poisson equation. They become dependent on material constants through the solution of in-plane equilibrium equations in terms of displacements. Here, it is convenient to denote displacements [u, v, w] as [u_i], (i = 1, 2, 3), in-plane stresses [σ_x, σ_y, τ_{xy}] and transverse stresses [τ_{xz}, τ_{yz}, σ_z] as [σ_i], [σ_{3+i}], (i = 1, 2, 3), respectively. With the corresponding notation for strains, strain-displacement relations are

$$[\varepsilon_1, \varepsilon_2, \varepsilon_3] = \alpha\, [u_{,x}, v_{,y}, u_{,y} + v_{,x}] \qquad (3)$$

$$[\varepsilon_4, \varepsilon_5, \varepsilon_6] = [u_{,z} + \alpha\, w_{,x}, v_{,z} + \alpha\, w_{,y}\, , w_{,z}] \qquad (4)$$

The material of the plate is homogeneous and anisotropic with monoclinic symmetry. Strain-stress relations are in terms of compliances [S_{ij}] with the usual summation convention of repeated suffix denoting summation over specified integer values:

$$\varepsilon_i = S_{ij}\, \sigma_j \;(i, j = 1, 2, 3, 6) \qquad (5)$$

$$\varepsilon_r = S_{rs}\, \sigma_s \;(r, s = 4, 5) \qquad (6)$$

From semi-inverted above relations, Stress-strain relations with [Q_{ij}] are

$$\sigma_i = Q_{ij}\, [\varepsilon_j - S_{j6}\, \sigma_z] \;(i, j = 1, 2. 3) \qquad (7)$$

$$\sigma_r = Q_{rs}\, \varepsilon_s \qquad (r, s = 4, 5) \qquad (8)$$

With σ_i in equations (7), equations (1) in terms of strains become

$$\alpha\, [Q_{1j}(\varepsilon_j - S_{j6}\, \sigma_z)_{,x} + Q_{3j}(\varepsilon_j - S_{j6}\, \sigma_z)_{,y}] + \tau_{xz,z} = 0 \qquad (9a)$$

$$\alpha\, [Q_{2j}\, (\varepsilon_j - S_{j6}\, \sigma_z)_{,y} + Q_{3j}\, (\varepsilon_j - S_{j6}\, \sigma_z)_{,x}] + \tau_{yz,z} = 0 \qquad (9b)$$

Note that σ_{60} (i.e., σ_{z0}) does not participate in the equilibrium equations but ε_{60} (i.e, ε_z) is required to nullify errors later in the transverse shear strain-displacement relations due to $w_1 = z\, \varepsilon_{z0}$.

Thickness-wise (z-) polynomial distribution functions $f_n(z)$ are generated through recurrence relations with $f_0 = 1$, $f_{2n+!,z} = f_{2n}$, $f_{2n+2,z} = - f_{2n+1}$ such that $f_{2n+2}(\pm1) = 0$. They are up to n = 3

$$[f_n] = [z, \tfrac{1}{2}(1 - z^2), \tfrac{1}{2}z(1 - z^2/3)] \qquad (10)$$

Displacements [u, v, w] are expressed as

$$[u, v, w] = f_n\, (z)\, [u_n, v_n, w_n],\; n = 0, 1.,.(11)$$

To keep associated 2-D variable as a free variable, it is necessary to replace f_{2i+1} by f^*_{2i+1}, with $\beta_{2i+1} = [f_{2i+1}(1) / f_{2i-1}(1)]$ so that $f^*_{2i+1}(\pm1) = 0$. given by

$$f^*_{2i+1} = f_{2i+1} - \beta_{2i+1}\, f_{2i-1},\; i = 1, 2, ... \quad (12)$$

At the onset of buckling, in-plane displacements [u, v] are even functions of z and Kirchhoff displacement $w_0(x, y)$ in bending is kept as a face variable.

III Analysis of Buckling problems

In-plane displacements [u, v] = z [u_1, v_1] in bending deformation are considered in the on-set of buckling of a plate with corresponding in-plane distribution of strains in face parallel planes are $(\varepsilon_x, \varepsilon_y)_1 = \alpha\, (u_{1,x}\, , v_{1,y})$. Additional term due to large deflection of a plate in bending, from von Karman's theory [2], is

$$(u_1 + v_1)^2 / 2 = [(u_1^{\,2} + v_1^{\,2}) / 2 + u_1\, v_1] \;(13)$$

so that additional terms in strain-displacement relations are $\varepsilon_x = u_1^{\,2} / 2$, $\epsilon_y = v_1^{\,2} / 2$, $\gamma_{xy} = u_1\, v_1$. Above mentioned additional term is from z-integration of (- z) = c - z^2 / 2 such that it is zero at z = ±1 so that c = 1/2. Hence, second degree terms from von Karman theory are zero along the faces of the plate. It implies that the term containing z^2 corresponds to $f_2(z)w_2(x, y)$ in normal shear deformation theory [4, 5].

Onset of Buckling (New Analysis)

In the classical theory based on Kirchhoff's assumptions, the lateral deflection w_0 (x, y) is a domain variable. In a recent article [3] on fundamental Theories of Aeronautics/ Mechanical structures, past and present Reddy's work is extensively referred in the analysis of beams, plates, and shells. Reddy's third order shear deformation theory of plate in buckling is based on the following bending displacements along with $[u, v]_0$ of stretching problem

$$u = u_0(x, y) + z\,(1 - z^2/3)w_{2,x} - (z^3/3)\,\alpha w_{0,x} \quad (14)$$

$$v = v_0(x, y) + z\,(1 - z^2/3)w_{2,y} - (z^3/3)\alpha w_{0,y} \quad (15)$$

Above bending displacements differ from Reissner [4] and Ambartsumian [5] theories in which coefficient of each gradient of w_0 corresponds to Kirchhoff's theory (note that z can be replaced by any asymmetric function f(z) with $[f_{,z}]_{z=1} = 1$, but $(z^3/3)$ is not a good replacement of z. Each gradient of w_2 is from thickness-wise integration $\int(1 - z^2)dz$ of normal shear deformation theory. It may be useful in the analysis of post buckling behavior of the plate but not necessary for the onset of buckling.

Here, a theory based on our monograph [1] is proposed with Kirchhoff w_0 (x, y) as a face variable. In-plane displacements and strains of stretching and flexure problems are assumed at the onset of buckling in the form along with nonlinear term due to von Karman's theory

$$\varepsilon_1 = \alpha\, u_{0,x} + \alpha^2\ \{(z\psi_1 + f_3(z)\zeta_1)_{,xx} + \ z\,\zeta_{1,x}{}^2/2\} \quad (16a)$$

$$\varepsilon_2 = \alpha\, v_{0,y} + \alpha^2\ \{(z\psi_1 + f_3(z)\,\zeta_1)_{,yy} + \ z\,\zeta_{1,y}{}^2/2\} \quad (16b)$$

$$\varepsilon_3 = \alpha\,(u_{0,y} + v_{0,x}) + \alpha^2\ \{[(z\psi_1 + f_3(z)\zeta_1)_{,xy} + z\,\zeta_{1,x}\,\zeta_{1,y}] \quad (17)$$

$$\varepsilon_{3+i} = [0, 0, S_{66}\,\sigma_{60}], \quad i = 1,2,3 \quad (18)$$

With in-plane stresses of extension problem, onset of buckling is due to critical in-plane stress resultants $\lambda\,[N_x\,,\ N_y\,,\ N_{xy}] = \lambda\int[\sigma_x\,,\sigma_y\,,\tau_{xy}]\,dz$ through thickness of the plate (In fact, it is possible to consider, in general, scale factors $\lambda[\alpha_1\,,\alpha_2\,,\alpha_3]$ with each of α_i varying from 0 to 1 with at least one of them is 1). In the classical theory, in-plane stresses are independent of z. They normally over estimate critical buckling load if one considers exact solutions of these stresses. In fact, one can use the solutions of linear problems of extension and bending of plates from the monograph [1] with initial strains

$$\varepsilon_1 = \alpha\, u_{0,x} + \alpha^2\, z\,\Psi_{1,xx} \quad (19a)$$

$$\varepsilon_2 = \alpha\, v_{0,y} + \alpha^2\, z\,\Psi_{1,yy} \quad (19b)$$

$$\varepsilon_3 = \alpha\,(u_{0,y} + v_{0,x}) + 2\,\alpha^2\, z\,\Psi_{1,xy} \quad (20)$$

Note that governing equations of primary bending problem consist of a fourth order equation in Ψ_1 and a second order equation in $\varphi_1(x, y)$.

With known Ψ_1, onset of buckling is from solution of gradients of ζ_1 from nonlinear equations

$$(\zeta_{1,x})^2/2 = \Psi_{1,x} \quad (21a)$$

$$(\zeta_{1,y})^2/2] = \Psi_{1,y} \quad (21b)$$

In the increased nonlinear term due to $[\zeta_1 + \delta_1]$ is $[\zeta_{1,x}{}^2 + \delta_{1,x}{}^2]/2 + [\zeta_{1,x}\,\delta\zeta_{1,x}]$ in Eq.(21a) and similar expression in Eq. (21b). Neglecting squared terms, we get $\zeta_{1,} = \Psi_1$ in the in the interior of the plate.

Polynomial $f_k(z)$ Series solutions for stresses and strains [6]

In a primary extension problem, the plate is subjected to symmetric normal stress $\sigma_{z0} = q_0(x, y)/2$, asymmetric shear stresses $[\tau_{xz},\ \tau_{yz}] = \pm [T_{xz}(x, y), T_{yz}(x, y)]$ at the faces of the plate. Due to the out-off plane equilibrium equation, prescribed face shears $[T_{xz}, T_{yz}]$ have to be gradients of a harmonic function ψ_1 so that $[T_{xz}, T_{yz}] = -\alpha [\psi_{1,x}, \psi_{1,y}]$. Transverse shear stresses and normal stress satisfying face conditions are

$$[\tau_{xz}, \tau_{yz}] = -\alpha z [\psi_{1,x}, \psi_{1,y}], \sigma_{z0} = q_0(x, y)/2 \quad (22)$$

Above transverse stresses are independent of material constants and remain the same within the plate. One should note here that σ_{z0} does not participate in the equilibrium equations but contributes to the in-plane constitutive relations. $[\tau_{xz}, \tau_{yz}]$ in the above equation are related to in-plane displacements $[u_0, v_0]$ through equilibrium equations (1). From constitutive relation,

$$\varepsilon_{z0} = S_{6j} \sigma_{j0} + S_{66} q_0/2 \ (j = 1, 2, 3) \quad (23)$$

Correspondingly, vertical deflection w is linear in z and cannot be prescribed to be zero along the edge of the plate due to $S_{6j} \sigma_{j0}$ even if the faces are free of transverse stresses.

Preliminary analysis

In-plane equilibrium equations in the preliminary analysis with $[u, v] = [u_0(x, y), v_0(x, y)]$ are

$$\alpha [Q_{1j} (\varepsilon_{j0} - S_{j6} \sigma_{z0})_{,x} + Q_{3j} (\varepsilon_{j0} - S_{j6} \sigma_{z0})_{,y}] = \alpha\psi_{1,x} \quad (24a)$$
$$\alpha [Q_{2j} (\varepsilon_{j0} - S_{j6} \sigma_{z0})_{,y} + Q_{3j} (\varepsilon_{j0} - S_{j6} \sigma_{z0})_{,x}] = \alpha\psi_{1,y} \quad (24b)$$

subjected to suitable edge conditions along x (and y) constant edges.

Effect of $w = z\, \varepsilon_{z0}$

We consider higher-order in-plane displacement terms $f_2(z) [u_2, v_2]$ which induce transverse shear stresses $z [\tau_{xz1}, \tau_{yz1}]$, $f_2(z) \sigma_{z2}$ from constitutive relations, and $z w_1(x, y)$ (other than the known $z \varepsilon_{z0}$) due to strain-displacement relations in the domain of the plate.

In-plane displacements $[u_2, v_2]$ are related from transverse shear-strain relations and constitutive relations to $[\tau_{xz1}, \tau_{yz1}]$, even in the absence of induced w_1, in the form

$$\tau_{xz1} = -[Q_{44} (u_2 - \alpha \varepsilon_{z0,x}) + Q_{45} (v_2 - \alpha \varepsilon_{z0,y})] \quad (25a)$$
$$\tau_{yz1} = -[Q_{55} (v_2 - \alpha \varepsilon_{z0,y}) + Q_{45} (u_2 - \alpha \varepsilon_{z0,x})] \quad (25b)$$

Displacements consistent with shear stresses $[\tau_{xz1}, \tau_{yz1}]$ are

$$w = z (\varepsilon_{z0} + w_1), u = (u_0 + f_2 u_2), v = (v_0 + f_2 v_2) \quad (26)$$

and vertical stress $\sigma_z = \sigma_{z0} + f_2 \sigma_{z2}$.

In the vertical deflection w, $w_1(x, y)$ is added to facilitate determination of $[u_2, v_2]$ from satisfying both static and z-integrated equilibrium equations.

In extending Poisson theory to extension problems, transverse stresses have to be independent of vertical displacement. Hence, $[u_2, v_2]$ are modified as

$$[u_2, v_2]^* = \{[u_2 - \alpha (\varepsilon_{z0} + w_1)_{,x},\ v_2 - \alpha (\varepsilon_{z0} + w_1)_{,y}]\} \quad (27)$$

so that transverse shear stresses from strain-displacement relations and constitutive relations are

$$[\tau_{xz1}, \tau_{yz1}^* = -[(Q_{44} u_2 + Q_{45} v_2), (Q_{55} v_2 + +Q_{45} u_2)] \quad (28)$$

Normal stress σ_{z2} from static equilibrium equation is

$$\sigma_{z2}^* = -\alpha [(Q_{44} u_2 + Q_{45} v_2)_{,x} + (Q_{55} v_2 + +Q_{45} u_2)_{,y}] \quad (29)$$

To keep $[\tau_{xz3}, \tau_{yz3}]$ as free variables in the integrated equilibrium equations, $f_3(z)$ is modified with $\beta_1 = 1/3$ as $f^*_3(z) = f_3(z) - \beta_1 z$ so that

$$[\tau_{xz}, \tau_{yz}]^{**} = z\,[\tau_{xz1}^*, \tau_{yz1}^*] + f_3\,[\tau_{xz3}, \tau_{yz3}] \quad (30)$$

with $\tau_{xz1}^* = (\tau_{xz1} - \beta_1\tau_{xz3})$ and $\tau_{yz1}^* = (\tau_{yz1} - \beta_1\tau_{yz3})]$.

From static equilibrium equation of transverse stresses, $\alpha\,[\tau_{xz1,x} + \tau_{yz1,y}]^* = \sigma_{z2}^*$ and $\alpha\,[\tau_{xz3,x} + \tau_{yz3,y}] = \sigma_{z4}$ so that $\sigma_{z2}^{**} = \sigma_{z2}^* - \beta_1\,\sigma_{z4}$ from which one gets (from coefficient of z)

$$\alpha\,[(Q_{44}\,u_2 + Q_{45}\,v_2)_{,x} + (Q_{55}\,v_2 + Q_{45}\,u_2)_{,y}] + \beta_1\sigma_{z4} = 0 \quad (31)$$

Strain-displacement relations from equations (27) give

$$\varepsilon_{x2}^* = \varepsilon_{x2} - \alpha^2\,(\varepsilon_{z0} + w_1)_{,xx} \quad (32a)$$
$$\varepsilon_{y2}^* = \varepsilon_{y2} - \alpha^2\,(\varepsilon_{z0} + w_1)_{,yy} \quad (32b)$$
$$\gamma_{xy2}^* = \gamma_{xy2} - 2\alpha^2\,(\varepsilon_{z0} + w_1)_{,xy} \quad (33)$$

Here also, $[u_2, v_2]$ are expressed in terms of gradients of two functions $[\psi_2, \varphi_2]$, like in bending problems, in the form

$$[u_2, v_2] = -\alpha\,[(\psi_{2,x} + \varphi_{2,y}), (\psi_{2,y} - \varphi_{2,x})]$$

Note that contribution of w_1 is the same as ψ_2 in $[u_2, v_2]^*$ in the integration of equilibrium equations since contributions of f_1 and $f_{2,z}$ are of opposite sign in strain-displacement relations whereas the corresponding contribution of f_1 and z-integrated $f_{2,z}$ are of the same sign. In-plane strains become, with ε_{i2}^* (i= 1, 2, 3) denoted by $\varepsilon_{x2}^*, \varepsilon_{y2}^*, \gamma_{xy2}^*$, respectively,

$$\varepsilon_{x2}^* = -\alpha^2\,(2\psi_{2,xx} + \varphi_{2,yx} + \alpha^2\varepsilon_{z0,xx}) \quad (34a)$$
$$\varepsilon_{y2}^* = -\alpha^2\,(2\psi_{2,yy} + \varphi_{2,yx} + \alpha^2\varepsilon_{z0,yy}) \quad (34b)$$
$$\gamma_{xy2}^* = -\alpha^2\,(4\psi_{2,xy} + \varphi_{2,xx} - \varphi_{2,yy} + 2\varepsilon_{z0,xy} \quad (34c)$$

Corresponding in-plane stresses are

$$\sigma_{i2}^* = Q_{ij}\,(\varepsilon_{j2}^* - S_{6j}\,\sigma_{z0})\ (i, j = 1, 2, 3) \quad (35)$$

From the integration of equilibrium equations, reactive transverse stresses are

$$\tau_{xz3}^* = \alpha\,[\sigma_{1,x} + \sigma_{3,y}]_2^* \quad (36a)$$
$$\tau_{yz3}^* = \alpha\,[\sigma_{2,y} + \sigma_{3,x}]_2^* \quad (36b)$$
$$\sigma_{z4} = \alpha\,(\tau_{xz3,x} + \tau_{yz3,y})^* \ \textit{(coefficient of } f_3\textit{)} \quad (37)$$

Noting that σ_{z4} from equation (31) is negative of the one from equation (37) due to $(f_3 + f_1) = 0$ at the faces of the plate, the equation governing in-plane displacements (u_2, v_2) is

$$\alpha\beta_1(\tau_{xz3,x} + \tau_{yz3,y})^* = \alpha[(Q_{44}\,u_2 + Q_{45}\,v_2)_{,x} + (Q_{55}v_2 + Q_{45}\,u_2)_{,y}] \quad (38)$$

The above equation is a fourth-order equation in ψ_2 to be solved along with harmonic function φ_2 with three conditions along x = constant edges (with analogue conditions along y = constant edges).

(i) $u_2^* = 0$ or $\sigma_{x2}^* = 0$ (39a)
(ii) $v_2^* = 0$ or $\tau_{xy2}^* = 0$ (39b)
(iii) $\varphi_2 = 0$ or $\tau_{xz3}^* = 0$ (39c)

Concerning solution of a 3-D problem, above analysis in the determination of $[u_2, v_2, \varepsilon_{z2}]$ is in error in the transverse strain-displacement relations due to $[\tau_{xz}, \tau_{yz}] = f_3(z)\,[\tau_{xz3}, \tau_{yz3}]$, and in the constitutive relations due to $f_4(z)\,\sigma_{z4}$.

With prescribed $[w, \tau_{xz}, \tau_{yz}] = \pm\,[w, \tau_{xz}, \tau_{yz}]_1$ along $z = \pm 1$ faces, induced or reactive σ_{z2} is parabolic from equilibrium equation of transverse stresses whereas in-plane displacements (u, v) or corresponding stresses are induced or prescribed parabolic distributions to be determined from z-integrated equilibrium equations.

Iterative Method: Higher-order corrections

$$\tau^*_{xz2n+1} = (\tau_{xz2n+1} - \beta_{2n-1}\tau_{xz2n-1}) \quad (40a)$$
$$\tau^*_{yz2n+1} = (\tau_{yz2n+1} - \beta_{2n-1}\tau_{yz2n-1}) \quad (40b)$$

$\sigma^*_{z2n+2} = \sigma_{z2n+2} - \beta_{2n-1}\, \sigma_{z2n}$ (41)

At the nth stage of iteration ($n \geq 1$), transverse stresses $[\tau_{xz}, \tau_{yz}]_{2n-1}$, and w_{2n-1} are known in the preceding stage. Concerning in-plane displacements, one should include additional terms such that they are consistent with known stresses $[\tau_{xz}, \tau_{yz}]_{2n-1}$ and are free to obtain stresses $[\tau_{xz}, \tau_{yz}]_{2n+1}$, σ_{z2n+2} and w_{2n+1}. We have from constitutive relations,

$\gamma_{xz2n-1} = S_{44}\, \tau_{xz2n-1} + S_{45}\, \tau_{yz2n-1}$ (42a)
$\gamma_{yz2n-1} = S_{55}\, \tau_{yz2n-1} + S_{45}\, \tau_{xz2n-1}$ (42b)

Modified displacements and the corresponding derived quantities denoted with * are with w_{2n-1} as correction to ε_{z2n-2} due to $[u, v]_{2n}$

$u^*_{2n} = u_{2n} - \alpha\, (\varepsilon_{z2n-2} + w_{2n-1})_{,x} + \gamma_{xz2n-1}$(43a)
$v^*_{2n} = v_{2n} - \alpha\, (\varepsilon_{z2n-2} + w_{2n-1})_{,y} + \gamma_{yz2n1}$(43b)

Strain-displacement relations give

$\varepsilon^*_{x2n} = \varepsilon_{x2n} - \alpha^2(\varepsilon_{z2n-2} + w_{2n-1})_{,xx} + \alpha\, \gamma_{xz2n-1,x}$ (44a)
$\varepsilon^*_{y2n} = \varepsilon_{y2n} - \alpha^2(\varepsilon_{z2n-2} + w_{2n-1})_{,yy} + \alpha\, \gamma_{yz2n-1,y}$ (44b)
$\gamma^*_{xy2n} = \gamma_{xy2n} - 2\alpha^2(\varepsilon_{z2n-2} + w_{2n-1})_{,xy} + \alpha(\gamma_{xz,y} + \gamma_{yz,x})_{2n-1}$ (44c)
$\gamma^*_{xz2n-1} = \gamma_{xz2n-1} - (u_{2n-2} + u_{2n})$ (45a)
$\gamma^*_{yz2n-1} = \gamma_{yz2n-1} - (v_{2n-2} + v_{2n})$ (45b)

In-plane stresses and transverse shear stresses from constitutive relations are

$[\sigma^*_i]_{2n} = [Q_{ij}\, \varepsilon^*_j]_{2n}$ ($i, j = 1, 2, 3$) (46)
$\tau^*_{xz2n-1} = \tau_{xz2n-1} - (Q_{44}\, u + Q_{45}\, v)_{2n}$ (47a)
$\tau^*_{yz2n-1} = \tau_{yz2n-1} - (Q_{55}\, v + Q_{45}\, u)_{2n}$ (47b)

One gets from equations (1, 2, 40, 41) noting that $\sigma_{z2n}^* = (\sigma_{z2n} - \beta_{2n-1}\, \sigma_{z2n+2})$

$\alpha[(Q_{44}u_{2n}+Q_{45}\, v_{2n})_{,x}+(Q_{54}u_{2n}+Q_{55}v_{2n})_{,y}] + \beta_{2n-1}\sigma_{z2n+2} = 0$ (48)

(Note that w_{2n-1} is not present in the above equation)

For the use of $[u^*, v^*]_{2n}$ in the integration of equilibrium equations, displacements $[u, v]_{2n}$ are expressed in the form

$[u, v]_{2n} = -\alpha\, [\psi_{2n,x}, \psi_{2n,y}]$ (49)

Contributions of ψ_{2n} and w_{2n-1} in $[u^*, v^*]_{2n}$ are the same in giving corrections to w (x, y, z) and transverse stresses (in fact, the contribution of w_{2n-1} is through strain-displacement relations in static equilibrium equations, and through constitutive relations in through-thickness integration of equilibrium equations). Hence, w_{2n-1} in $[u^*, v^*]_{2n}$ is replaced by ψ_{2n} (to be independent of w_{2n-1} used in strain-displacement relations) so that $[u^*, v^*, \varepsilon^*_x, \varepsilon^*_y, \gamma^*_{xy}]_{2n}$ are

$u^*_{2n} = (2\, u_{2n} + \gamma_{xz2n-1} - \alpha\, \varepsilon_{z2n-2,x})$ (50a)
$v^*_{2n} = (2\, v_{2n} + \gamma_{yz2n-1} - \alpha\, \varepsilon_{z2n-2,y})$ (50b)
$\varepsilon^*_{x2n} = (2\, \varepsilon_{x2n} + \alpha\, \gamma_{xz2n-1,x} - \alpha^2\varepsilon_{z2n-2,xx})$ (51a)
$\varepsilon^*_{y2n} = (2\, \varepsilon_{y2n} + \alpha\, \gamma_{yz2n-1,y} - \alpha^2\varepsilon_{z2n-2,yy})$ (51b)
$\gamma^*_{xy2n}=[2\gamma_{xy2n}+\alpha(\gamma_{xz2n-1,y}+\gamma_{yz2n-1,x}) - 2\alpha^2\varepsilon_{z2n-2,xy}]$ (51c)

(Note that the role of w_{2n-1} is in its contribution to the integrated equilibrium equations.)

From the integration of equilibrium equations using the strains in equations (50), reactive transverse stresses are

$\tau^*_{xz2n+1}=\alpha[Q_{1j}(\varepsilon^*_j - S_{j6}\sigma_z)_{,x} + Q_{3j}(\varepsilon^*_j - S_{j6}\sigma_z)_{,y}]_{2n}$ ($j=1, 2, 3$) (52a)
$\tau^*_{yz2n+1}=\alpha[Q_{2j}(\varepsilon^*_j - S_{j6}\sigma_z)_{,y} + Q_{3j}(\varepsilon^*_j - S_{j6}\sigma_z)_{,x}]_{2n}$ ($j=1, 2, 3$) (52b)
$\sigma_{z2n+2} = -\alpha\, (\tau^*_{xz,x} + \tau^*_{yz,y})_{2n+1}$ (53)

One equation governing in-plane displacements $(u, v)_{2n}$, noting that σ_{z2n+2} from equation (48) is negative of the one from equation (53) due to $(f_{2n+1,zz} + f_{2n-1}) = 0$, is given by

$$\alpha\, \beta_{2n-1} (\tau^*_{xz,x} + \tau^*_{yz,y})_{2n+1} = \alpha\, [(Q_{44}\, u + +Q_{45}\, v)_{,x} + (Q_{54} u + Q_{55} v)_{,y}]_{2n} \qquad (54)$$

With the second equation $v_{2n,x} = u_{2n,y}$, the above equation becomes a fourth-order equation in ψ_{2n} to be solved along with harmonic function φ_{2n} with three conditions along constant x = constant edges (with analogous conditions along y = constant edge)

(i)	$(u_{2n}$ or $\sigma_{2n})^* = 0$,	(55a)
(ii)	$(v_{2n}$ or $\tau_{xy2n})^* = 0$,	(55b)
(iii)	$\tau_{xz2n+1}{}^* = 0$	(55c)

In principle, one may continue the iterative procedure until specified accuracy is achieved. However, it is not easy to develop software for the generation of polynomial f(z) functions involved in the evaluation of necessary β_{2n-1} to keep face shears as free variables.

Use of Sinusoidal series

It is convenient for generation of higher order polynomial z-distribution terms to express the basic function z in Fourier sine series in the form with $\lambda_{2n-1} = 2/[(2n\text{-}1)\,\pi]$,

$$z = \sum A_{2n-1} \sin (z/\lambda_{2n-1}) \text{ (sum on n)} \qquad (56)$$

in which $A_{2n-1} = \int \sin (z/\lambda_{2n-1})\, z\, dz = \lambda_{2n-1}{}^2$.

One gets each polynomial $f_k(z)$ function (k = 1, 2, 3…..) from successive integrations

$$f_{2k-1}(z) = \sum \lambda_{2n-1}{}^{2k} \sin (z/\lambda_{2n-1}) \text{ (sum on n)} \qquad (57a)$$

$$f_{2k}(z) = \sum \lambda_{2n-1}{}^{2k+1} \cos (z/\lambda_{2n-1}) \text{ (sum on n)} \qquad (57b)$$

One should note that replacing z with one term approximation $[\lambda_1{}^2 \sin (z/\lambda_1)]$ is an approximation but better than replacing with $z^3/3$ in Reddy's third order shear deformation theory.

Above each polynomial $f_k(z)$ function in infinite series of sinusoidal functions can be used to overcome difficulty of generating software with polynomial functions in the above presented iterative method. (However, two-term polynomial solutions may be adequate for design purposes.) The required analysis was presented earlier for isotropic plate [7, 8] and can be extended for anisotropic plates.

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