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An Improved Archimedes' Approximation to pi

A Linear Combination

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***Abstract*— Archimedes used the perimeter of inscribed and circumscribed regular polygons to obtain lower and upper bounds for the number pi. He started with two regular hexagons and he doubled their sides from 6 to 12, 24, 48, until 96. Using the perimeter of a 96-sided regular polygon, Archimedes obtained bounds for the number pi: 3+10/71<pi<3+1/7. His algorithm can be implemented as a recurrence formula called the Borchardt-Pfaff-Schwab method. Dörrie proposed an improvement on this algorithm that produces a narrower interval which encapsulates pi. Here a linear combination of the bounds is realized to obtain an improved accuracy approximation.**

Keywords—area; regular polygons; hexagon; the number pi; approximation; algorithm; fraction

I. INTRODUCTION

Archimedes developed the first algorithm for the approximation of the number pi (π). In his work entitled "*On the Measurements of a Circle*" Archimedes demonstrated that the perimeter of a convex polygon inscribed in a circle is less than the circumference of the circle. In a similar way, the circumference of the circle is less than the perimeter of any circumscribed convex polygon. Using this property, he started with inscribed and circumscribed regular hexagons as an initial value of the estimation process. Their perimeters are readily calculated. He then demonstrated how to calculate the perimeters of regular polygons of twice as many sides. Consequently, he used a series of inscribed and circumscribed regular polygons applied to the same circle. He used a circle with diameter d=1 (r=1/2) [1,2]. The length of its circumference is |C|=π.

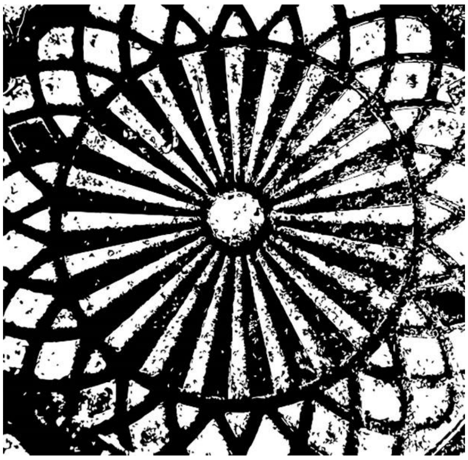

Fig. 1. An (art) illustration of the Archimedes method with an inscribed regular 48-gon.

His method is a recursive process. Let p_k and P_k denote the length of the perimeters of regular polygons of k sides that are inscribed and circumscribed, respectively. Consequently, the number pi is bounded by the generated values p and P: p<π<P, where

$$P_{2k} = \frac{2p_kP_k}{p_k+P_k},\ p_{2k} = \sqrt{p_kP_{2k}}.$$

Archimedes' recurrence formula realizes the Archimedes algorithm. This technique is applied to produce successive approximations to the number pi. The algorithm is also called the Borchardt-Pfaff-Schwab (BPS) algorithm. Archimedes showed that using the perimeters of 96-sided regular polygons, the following estimation for the number pi is obtained: S<π<T, where S=3+10/71, and T=3+1/7 [1].

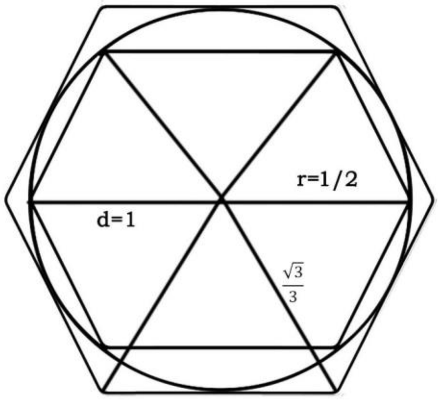

Fig. 2. Archimedes' initial hexagons with sides 1/2 and √3/3. The circle with diameter d= 1 (r=1/2) and |C|=π.

Thus, we have bounds for the number pi: $S < \pi < T$; the question is how to combine S and T to obtain a better approximation using these bounds. Very often, it is suggested to take the arithmetic average of these values, i.e., T+(S-T)/2 (=3.14185…). Here we propose to determine the parameter x in such way that it satisfies the following relation:

$$\pi = x * S + (1 - x) * T.$$

The solution of this equation is $x = \frac{\pi - T}{S - T} \approx 0.6284511 \cdots$ [3]. The number x is irrational and non-algebraic as its formula is expressed using the number pi. We can apply a continued fraction to find the best rational approximations for this number [4]. The following continued fraction has been generated (9 terms) u=[0;1,1,1,2,4,6,1,1,3] to approximate x. Table 1 presents the corresponding numerators and denominators taken for the continued fraction u. The table shows the value for the assumed representation: x=numerator/denominator (a truncated fraction). For example, for 0/1, we have T, for 1/2 we have the halved interval, i.e. T+(S-T)/2. The reasonable approach is to use x=22/35, as the next term is 137/218 and is more complicated. With this value for x (x=22/35) we have

$$\pi \approx \frac{22 * S + 13 * T}{35} = 3.141592411 \ldots.$$

Thus, using a simple linear combination we improved the accuracy of the Archimedes' estimation for the number pi. Of course, in such an approach we have to know the value of the number pi [3].

TABLE 1. The approximations from the continued fraction of the value x (Numerator/Denominator) & π.

Numerator	Denominator	Approximation of pi
0	1	3.1428571430
1	1	3.1408450700
1	2	3.1418511070
2	3	3.1415157612
5	8	3.1415995980
22	35	3.1415924116
137	218	3.1415926750
159	253	3.1415926390
296	471	3.1415926557
1047	1666	3.1415926532

The technique of such linear combinations is applied to other methods given in the presentation below. In general, the described approach can be used to find interesting approximations to the number π.

II. DÖRRIE'S MODIFICATION

The BPS method works as follows: new values a′, b′ are produced from the old values a, b – which are the values from the previous step. It's an iterative process and is relatively easy to implement on a computer. Starting with a = 2√3 and b = 3, the values for circumscribed and inscribed regular 6-gons, we can produce the sequence of intervals [b, a], b < a. The successive intervals contain the number π. The process of the BPS algorithm is described by the formula

$$a' = \frac{ab}{a+b},\ b' = \sqrt{ba'}.$$

The formulae generate the sequence of the intervals $[a', b'] \subset [a, b]$ and $b < \pi < a$. The convergence of this method is rather slow comparing to others.

In his book, the German mathematician Heinrich Dörrie, in problem No. 38, presented a method to improve Archimedes' estimations [5]. He defined two new series B and A, which give a better approximation for the length of the circumference (|C|) of the circle with diameter d=1 (r=1/2).

For given values b and a (obtained from the BPS algorithm) a simple transformation is executed.

$$B = \frac{3ab}{2a+b}, \qquad A = \sqrt[3]{ab^2}.$$

Dörrie demonstrated that the following inequalities hold b < B < C < A < a. The sequence B increases to C, and the sequence A decreases to C. Consequently, the values of B and A bound the interval that contains the number pi. The interval [b, a] generated in each step of the BPS algorithm contains the interval [B, A]. For an initial regular hexagon with d=1 we have a= 2√3, b = 3, and after applying Dörrie's method: B = 3.14023, A =3.14734. This is already an accuracy achieved by the BPS method with a 96-gon.

Here the linear transformation is applied to the sequences b and a produced by the BPS algorithm and its modified version developed by Dörrie, thus we have two approximations as the number pi is transcendental.

$$\pi = x * a + (1 - x) * b$$

and

$$\pi = x * A + (1 - x) * B.$$

The above equations are solved and the parameter x is approximated using the continued fraction u. Table 2 summarizes the results for two equations. The table gives values for the parameter x and the corresponding continued fraction. As the number of sides of the regular polygons increases for the BPS algorithm, x tends to 1/3. This is exactly the same linear weight as was discovered by Snell and later proved by Huygens [6,7]. Similarly, based on the Taylor series it was shown that the linear combination of the form B-(A-B)/5 (thus x=0.2) increases the accuracy [8,9]. For example, for k=3 (regular triangles) this transformation gives pi= 3.148827 (with B = 3.117691 and A = 3.273370) where the BPS algorithm produces the interval [2.598076, 5.196152] and the Snell-Huygens transformation gives pi= 3.464101.

TABLE 2. The continued fractions u for the parameter x for regular polygons with n=6,12, 24,48,96,192 sides.

Archimedes	
Parameter x	**Continued fraction u**
0.305090	0;3 3 1 1 1 1 56 1
0.326429	0;3 15 1 3 6 1 1 1
0.331617	0;3 64 2 1 1 5 3 1
0.332905	0;3 258 1 13 3 1 10 7
0.333226	0;3 1037 13 15 2 2 5 23
0.333307	0;3 4149 1 1 1 31 1 3
Dörrie	
Parameter x	**Continued fraction u**
0.190678027	0;5 4 10 1 293 2 2 3
0.197786862	0;5 17 1 6 1 13 10 1
0.199453729	0;5 73 42 20 14526 1 1 2
0.199863866	0;5 293 1 1 1 2 4 1
0.199965995	0;5 1176 8 1 2 1 4 25
0.199991481	0;5 4695 3 4 2 2 1 36160

The parameter x converges to1/3 and 1/5, respectively.

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TABLE 3. The approximation generated by Archimedes and Dörrie algorithms + transformation.

Numerator	Denominator	Archimedes' pi
Archimedes		
0	1	3.00000000000000
1	3	3.15470053837925
3	10	3.13923048454133
4	13	3.14280049696546
0	1	3.10582854123025
1	3	3.14234913054466
15	46	3.14155520469000
16	49	3.14160381239538
0	1	3.13262861328124
1	3	3.14163905621999
64	193	3.14159236998715
129	389	3.14159273003522
0	1	3.13935020304687
1	3	3.14159554040839
258	775	3.14159264319889
259	778	3.14159265437065
0	1	3.14103195089051
1	3	3.14159283380880
1037	3112	3.14159265357650
13482	40459	3.14159265358986
0	1	3.14145247228546
1	3	3.14159266485025
4149	12448	3.14159265358799
4150	12451	3.14159265359071
Dörrie		
Numerator	**Denominator**	**Dörrie's pi**
0	1	3.14023734336617
1	5	3.14165891274593
4	21	3.14159121896594
41	215	3.14159279323989
0	1	3.14150999364292
1	5	3.14159357851421
17	86	3.14159260659710
18	91	3.14159265999914
0	1	3.14158751885795
1	5	3.14159266765297
73	366	3.14159265358522
3067	15377	3.14159265358980
0	1	3.14159233315964
1	5	3.14159265380805
293	1466	3.14159265358933
294	1471	3.14159265359007
0	1	3.14159263357057
1	5	3.14159265359320
1176	5881	3.14159265358979
9409	47053	3.14159265358979
0	1	3.14159265233871
1	5	3.14159265358985
4695	23476	3.14159265358979
14086	70433	3.14159265358979

In the table only the 4 first terms of the continued fractions are shown. They are grouped by the results generated for 6, 12, 24, 48, 96 and 192 sided regular polygons.

III. PLATO AND THE NUMBER PI

There is an assertion that Plato used the relation √2+√3 (= 3.14626437…) as an approximation for the number π [10]. For this fact we really do not have any documented evidence or supporting source materials. It is rather only a common sense and widely accepted assumption that the philosopher Plato knew of this representation for the number pi.

Consider the unit circle, i.e., the circle with radius one (r=1, d=0.5). In the considered scenario, two regular polygons are constructed: a regular octagon inscribed and a regular hexagon circumscribed, both realized on this unit circle. The area of this octagon is 2√2. It is 8 times the area of the isosceles triangle with two sides of the length one. The area of the circumscribed hexagon is 6 times the area of the equilateral triangle with side √3/3 and this area is 2√3. The approximation √2+√3 of the number pi is the arithmetic average of the two areas 2√2 and 2√3.

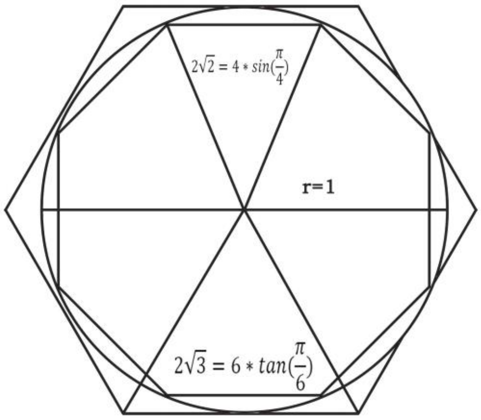


Fig. 3. A regular octagon and hexagon. The area of the unit circle (=π) is approximated as √2+√3 (an average).

Using the same technique of a linear combination of these two estimates (2√2 and 2√3), we obtain the continued fraction u=[0;1,1,33,1,1,14,19,1,3] for the value x=0.5073492…. The results presented in Table 4 suggest using the ratio 69/136. Such a choice gives an elegant new approximation to the number pi. Using this formula, we have an approximation which has 6 correct digits [3].

$$\pi \approx \sqrt{2}+\sqrt{3}+ (\sqrt{2}-\sqrt{3})/68=3.141590293\ldots$$

TABLE 4. Rational approximation to the number x and the obtained pi based on 2√2 and 2√3.

Numerator	Denominator	Approximation of pi
0	1	3.4641016150
1	1	2.8284271250
1	2	3.1462643700
34	67	3.1415205305
35	69	3.1416580331
69	136	3.1415902928
1001	1973	3.1415926618
19088	37623	3.1415926533
20089	3959600	3.1415926537
79355	1564110	3.1415926536

IV. RATIONAL APPROXIMATION

Continued fractions allow generating successive best rational estimations of the number pi. These approximations are the best among possible rational approximations relative to the size of their denominators. The accuracy can be increased by using fractions with larger numerators and denominators. This usually requires more digits in the approximation than there are correct significant figures achieved in the result. Using the continued fraction representations of the number pi we have the following sequence:
3/1, 22/7, 333/106 (= 3.1415 0943396226…), 355/113 (=3.141592 92035398…), 103993/33102, 104348/33215, 208341/66317, 312689/99532, 833719/265381, 1146408/364913, 4272943/1360120, 5419351/1725033 (=3.141592653 58982…).
Some of these approximations were used for hundreds of years and among them are 3, 22/7, 333/106, and 355/113. Here we propose to apply the weighted combination of two of them, 333/106 and 355/113, with the goal of increasing the accuracy of estimating the number pi. The fraction 355/113 gives more exact digits of π than the number of digits used to approximate it (i.e., 7 vs. 6). The accuracy can be improved by using other fractions with larger numerators and denominators. This needs more digits in the approximation than correct significant figures achieved in the result. The following relation is considered to define the approximation for the number pi.

$$x * \frac{355}{113} + (1-x) * \frac{333}{106} = \pi.$$

The solution of this equation is the parameter x=0.996804698539321…. The continued fraction for this solution is u = [0;1 311 1 23 1 2 2 1]. In Table 5 are presented the corresponding convergents and pi.

$$\frac{355}{113} + \frac{\left(\frac{333}{106} - \frac{355}{113}\right)}{313} = 3.141592653\ 62430\cdots.$$

TABLE 5. Rational approximation to x and the obtained pi (π)

Numerator	Denominator	Approximation of pi
0	1	3.14150943396226
1	1	3.14159292035398
311	312	3.14159265276939
312	313	3.14159265362430
7487	7511	3.14159265358879
7799	7824	3.14159265359021
23085	23159	3.14159265358974
53969	54142	3.14159265358981
77054	77301	3.14159265358979

Thus, the linear combination of two fractions of the form

$$\frac{355}{113} + \frac{\left(\frac{333}{106} - \frac{355}{113}\right)}{313} = \mathbf{3.141592653}\ 62430\cdots$$

increases the accuracy of the approximation of π.

V. CONCLUSION

In this paper was justified the association between the number pi and the square root of 2 and 3. A formula was proposed to use these values and obtain a higher accuracy for π. A linear combination was proposed to combine rational approximations such as 333/106 and 355/113 to improve the accuracy. The presented technique is useful to find effective approximation to the number pi. Other approaches are possible, as ones presented by the author, where the onscribed regular polygons are applied [11-15]. In Appendix a program in R is presented. This program realizes Dörrie's approximation and a linear combination with the bounds A and B, where $\pi \in [A, B]$.

REFERENCES

[1] TL. Heath. The works of Archimedes. Cambridge Univ. Press, Cambridge, 1897.
[2] WR. Knorr. Archimedes and the measurement of the circle: A new Interpretation. Arch. Hist. Exact Sci. 15 (1976), 115–140.
[3] M. Szyszkowicz. The Number pi (π) from a Square Root of 2 and 3. East European Scientific Journal 6(70), (2021). PDF is available at the location https://cyberleninka.ru/article/n/the-number-pi-from-a-square-root-of-2-and-3/viewer; DOI: 10.31618/ESSA.2782-1994.2021.1.70.68.
[4] M. Szyszkowicz. Squaring the Circle by Using Proportion. The Mathematical Gazette, (2019): 103 (556), pp. 156 – 158. DOI: https://doi.org/10.1017/mag.2019.28.
[5] H. Dörrie. 100 Great Problems of Elementary Mathematics: Their History and Solutions. New York: Dover, pp.184-188, 1965.
[6] W. Snell. Cyclometricus. Leiden, 1621.
[7] Ch. Huygens. De circuli magnitudine inventa. Elzevier, Leiden, 1654.
[8] M. Szyszkowicz. Approximations of pi and Squaring the Circle. J. Multidiscip. Eng. Sci. Technol. (JMEST, ISSN: 3159-00400), (2015). 2(1), pp: 330-332.
[9] M. Szyszkowicz. Algorithm 1 and Algorithm 2 for Determining the Number pi (π). Asian Journal of Research in Computer Science. 2019. 2(4), 1–7. https://doi.org/10.9734/ajrcos/2018/v2i430079.
[10] KRA. Popper, EH. Gombrich, The Open Society and Its Enemies. (2014). Princeton and Oxford: Princeton University Press. ISBN 978-0-691-15813-6. Vol. 1, pp: 252–253.
[11] M. Szyszkowicz. Approximations of the Number π Using Onscribed Regular Polygons. British Journal of Mathematics & Computer Science. 14(5): 1-7, (2016), Article no. BJMCS. 23947. DOI: 10.9734/BJMCS/2016/23947.
[12] M. Szyszkowicz. Ahmes' Method to Squaring the Circle. European Journal of Mathematics and Computer Science. 4 (1), (2017). ISSN 2059-9951.
[13] M. Szyszkowicz. Ancient Egyptian Quadrature Executed Using a Set Square. J. Multidiscip. Eng. Sci. Technol. (JMEST). 3 (10) (2016). ISSN: 2458-9403.

[14] M. Szyszkowicz. Rectification of a Circle Using Modified Kochański's and Pupovac's Method. International Scientific Survey Journal. 2 (5), (2019).
[15] M. Szyszkowicz. Krótka historia ekierki Binga. A brief history of Bing's set square. East European Scientific Journal. 12(2). pp: 57-60. (2016).
PDF at: https://cyberleninka.ru/article/n/kr-tka-historia-ekierki-binga/viewer.
Webpage: https://mszyszkowicz.wixsite.com/szysz

APPENDIX

```
#Program realizes Dörrie's method
#+ combination pi=x*A+(1-x)*B
# Author: Mieczysław Szyszkowicz
library(contfrac); options(digits=15)
#m=4; b=2*sqrt(2); a=4 #Use a square or others.
m=6; b=3; a=2*sqrt(3)  #Use a hexagon
for (k in 1:6){
cn=c(k-1,m); print(cn)
arch = c(b,a) #Archimedes' results
# print(arch)
# Dörrie's transformation
# The bounds: b<B<pi<A<a
B=(3*a*b)/(2*a + b)
A=(a*b*b)^(1/3)
Dor = c(B,A) # Dörrie's results
print(Dor);
#Next Archimedes:
a=2*a*b/(a+b)
b=sqrt(a*b)
##############################
#Determine x: PI=x*A+(1-x)*B
#pi= x*A+(1-x)*B
x =(pi-B)/(A-B); print(x); # x= 0.2
fracx = as_cf(x, n = 9)
print(fracx) #continued fraction
fraction = convergents(fracx)
print(fraction$A) #Numerators
print(fraction$B) #Denominators
for (k in 1:9) { #Use x=fraction
N=fraction$A[k]; D=fraction$B[k]
w=N/D; aprPi=w*A+(1-w)*B
result =c(N,D,aprPi)
print(result) } # Results
################################
# Results with the estimated x
aprPix=x*A+(1-x)*B
resx=c("x=",x,aprPix);
# Combination with x
print (resx) }# The end #########################
```

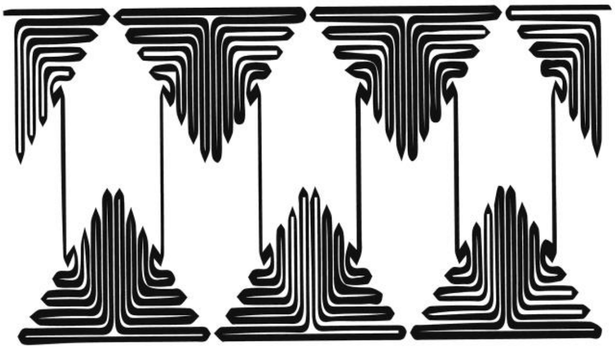
A part of the results from the program are presented. Here we start with a regular hexagon: 0 6, 0 – first step.

The results for a regular hexagon:

```
0 6
3.14023734336617 3.14734519026494
0.190678027105049
 0 5 4 10 1 293 2 2 3
 0 1 4 41 45 13226 26497 66220 225157
 1 5 21 215 236 69363 138962 347287 1180823
0.00000000000000    1.00000000000000
3.14023734336617
1.00000000000000    5.00000000000000
3.14165891274593
 4.00000000000000    21.00000000000000
3.14159121896594
 41.00000000000000    215.00000000000000
3.14159279323989
 45.00000000000000    236.00000000000000
3.14159265315619
13226.0000000000000    69363.0000000000000
3.1415926535904
2.64970000000000e+04    1.38962000000000e+05
3.14159265358966e+00
6.62200000000000e+04    3.47287000000000e+05
3.14159265358981e+00
2.25157000000000e+05    1.18082300000000e+06
3.14159265358979e+00
"x=" "0.190678027105049" "3.14159265358979"
##############################################
```

The results for a regular 96-gon: (4,6), 4 -fourth step

```
4 6
 3.14159263357057 3.14159273368372
 0.199965995487192
 0    5 1176    8    1    2    1    4   25
 0       1      1176     9409    10585    30579    41164
195235 4922039
 1       5       5881     47053     52934    152921    205855
976341  24614380
 0.00000000000000    1.00000000000000
3.14159263357057
 1.0000000000000    5.0000000000000
3.1415926535932
 1176.00000000000000    5881.00000000000000
3.14159265358979
  9409.00000000000000    47053.00000000000000
3.14159265358979
 10585.00000000000000    52934.00000000000000
3.14159265358979
 3.05790000000000e+04    1.52921000000000e+05
3.14159265358979e+00
 4.11640000000000e+04    2.05855000000000e+05
3.14159265358979e+00
 1.95235000000000e+05    9.76341000000000e+05
3.14159265358979e+00
 4.92203900000000e+06    2.46143800000000e+07
3.14159265358979e+00
 "x="    "0.199965995487192"
"3.14159265358979"
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```



A graphic based on work of W. Szpakowski.
The construction of the bike paths: methods/software.
https://szyszkowiczs.wixsite.com/tylkorower