The Extension Form Of Hardy's Inequality

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***Abstract*—Hardy's inequality is an important inequality in analysis. This paper uses the traditional convolution transformation to prove a mixed norm Hardy's inequality.**

Keywords*—*Hardy Inequality; mixed norm; convolution transformation

1 Introduction:

Hardy's inequality is an inequality full of vitality. It was first proposed by the British mathematician Hardy in 1927 when he was studying basic mathematics. Since the advent of Hardy's inequality, it has been nearly a hundred years, and people have achieved many excellent results in the forward deformation of Hardy's inequality, the determination of the best constant and the application of integrals. Hardy's inequality has been widely used in the fields of differential equations and harmonic analysis. [1-8], This article summarizes and discusses the proofs of Hardy's inequality with mixed norms. Through the decomposition and reorganization of the structure of Hardy's inequality, the relevant characteristics and application methods of Hardy's inequality are clarified, and the application scope of Hardy's inequality is further broadened. It is Hardy's inequality. The research and derivation provide a new idea.

II main conclusion

Theorem 1: Assume $q \geq p > 1, \frac{1}{q} = \frac{1}{p} + \frac{1}{r} - 1$, $r = \frac{1}{1 + \frac{1}{q} - \frac{1}{p}}$,we can get

$$\left[\int_0^{+\infty} t^{\frac{(s+1)q}{p}-1} F^q(t)dt\right]^{\frac{1}{q}} \leq \left[\frac{p}{(s+1-p)r}\right]^{\frac{1}{r}} \left[\int_0^{+\infty} t^s f^p(t)dt\right]^{\frac{1}{p}}$$

Theorem 2: Assume $p > 1$, $a \in (0, +\infty]$, $r < 1$,If $\int_0^a x^{p-r} f^p(x)dx < +\infty$, then

$$\int_0^a \frac{\left[\int_x^a f(t)dt\right]^p}{x^r} dx \leq \left(\frac{p}{1-r}\right)^p \int_0^a t^{p-r} f^p(t)dt \text{ ,} \left(\frac{p}{1-r}\right)^p$$ is the best coefficient.

III Proof of main conclusion

Proof of Theorem 1

Step 1 Introduction of parameters

We take $F(x)=\frac{1}{x}\int_x^{+\infty} f(t)dt, x>0$。By Holder inequality

$$\int_x^{+\infty} f(t)dt=\int_x^{+\infty} t^{-\frac{s}{p}}t^{\frac{s}{p}} f(t)dt \le (\int_x^{+\infty} t^{-\frac{qs}{p}}dt)^{\frac{1}{q}}\left[\int_x^{+\infty} t^s f^p(t)dt\right]^{\frac{1}{p}},$$

To make $\int_x^{+\infty} t^{-\frac{qs}{p}}dt<+\infty$, Conditions must also be met $\frac{q}{p}s>1$, then $s>\frac{p}{q}=p-1$。

Assume $r\ge 1$, $\frac{1}{\alpha}=\frac{1}{p}+\frac{1}{r}-1>0, s>p-1$, To make $\int_0^{+\infty} x^s f^p(x)dx<+\infty$。

In this way, $F(x)$ is meaningful to all $x>0$

Step 2 Substitution of variables

$$\int_0^{+\infty} t^s f^p(t)dt \overset{\substack{t=e^x\\ x=\ln t}}{=} \int_{-\infty}^{+\infty} e^{sx} f^p(e^x)e^x dx=\int_{-\infty}^{+\infty} e^{(s+1)x} f^p(e^x)dx=\int_{-\infty}^{+\infty}\left[e^{\frac{(s+1)x}{p}} f(e^x)\right]^p dx,$$

so

$$\left[\int_0^{+\infty} t^s f^p(t)dt\right]^{\frac{1}{p}}=\left\{\int_{-\infty}^{+\infty}\left[e^{\frac{(s+1)x}{p}} f(e^x)\right]^p dx\right\}^{\frac{1}{p}}=\left\|e^{\frac{(s+1)x}{p}} f(e^x)\right\|_p$$

take $\varphi(x)=e^{\frac{(s+1)x}{p}} f(e^x)$, then $f(e^x)=e^{-\frac{(s+1)x}{p}}\varphi(x)$

$$\left[\int_0^{+\infty} t^s f^p(t)dt\right]^{\frac{1}{p}}=\|\varphi\|_p$$

Step 3 Introduce convolution

$F(x)=\frac{1}{x}\int_x^{+\infty} f(t)dt, x>0$, so,for any $-\infty<x<+\infty$, we can get

$$F(e^x)=\frac{1}{e^x}\int_{e^x}^{+\infty} f(t)dt \overset{\substack{t=e^u\\ u=\ln t}}{=} \frac{1}{e^x}\int_x^{+\infty} e^u f(e^u)du=\int_x^{+\infty} e^{u-x} f(e^u)du$$
$$=\int_x^{+\infty} e^{u-x}e^{-\frac{(s+1)u}{p}}\varphi(u)du=\int_x^{+\infty} e^{(\frac{1}{q}-\frac{s}{p})u-x}\varphi(u)du=\int_x^{+\infty} e^{(\frac{1}{q}-\frac{s}{p})(u-x)+(\frac{1}{q}-\frac{s}{p}-1)x}\varphi(u)du$$
$$=e^{(-\frac{s}{p}-\frac{1}{p})x}\int_x^{+\infty} e^{(\frac{s}{p}-\frac{1}{q})(x-u)}\varphi(u)du=e^{-\frac{s+1}{p}x}\int_x^{+\infty} e^{(\frac{s}{p}-\frac{1}{q})(x-u)}\varphi(u)du$$

then

$$e^{\frac{s+1}{p}x}F(e^x)=\int_x^{+\infty}e^{(\frac{s}{p}-\frac{1}{q})(x-u)}\varphi(u)du$$

$s>\frac{p}{q}$，故$\frac{s}{p}-\frac{1}{q}>0$。令$\chi(x)=\begin{cases}e^{(\frac{s}{p}-\frac{1}{q})x}, x\le 0\\ 0, \quad x>0\end{cases}$，则

$e^{\frac{s+1}{p}x}F(e^x)=\int_x^{+\infty}e^{(\frac{s}{p}-\frac{1}{q})(x-u)}\varphi(u)du=\int_x^{+\infty}\chi(x-u)\varphi(u)du=\int_{-\infty}^{+\infty}\chi(x-u)\varphi(u)du=(\chi*\varphi)(x)$ 故

$e^{\frac{s+1}{p}x}F(e^x)=(\chi*\varphi)(x)$。

$$\|\chi\|_l=\left[\int_{-\infty}^{+\infty}\chi^l(x)dx\right]^{\frac{1}{l}}=\left[\int_{-\infty}^{0}e^{(\frac{s}{p}-\frac{1}{q})lx}dx\right]^{\frac{1}{l}}=\left[\frac{1}{(\frac{s}{p}-\frac{1}{q})l}\right]^{\frac{1}{l}},$$

$$=\left[\frac{1}{(\frac{s}{p}-(1-\frac{1}{p}))l}\right]^{\frac{1}{l}}=\left[\frac{1}{(\frac{s+1}{p}-1)l}\right]^{\frac{1}{l}}=\left[\frac{p}{(s+1-p)l}\right]^{\frac{1}{l}}$$

$$\|\varphi\|_p=\left[\int_0^{+\infty}t^s f^p(t)dt\right]^{\frac{1}{p}}$$

$\frac{1}{\alpha}=\frac{1}{p}+\frac{1}{r}-1>0$， $p>1$， $r\ge 1$， $\chi\in L_r$， $f\in L_p$， By Young inequality

$$\|\chi*\varphi\|_\alpha\le\|\chi\|_r\|\varphi\|_p=\left[\frac{p}{(s+1-p)r}\right]^{\frac{1}{r}}\left[\int_0^{+\infty}t^s f^p(t)dt\right]^{\frac{1}{p}}$$

On the other hand

$$\|(\chi*\varphi)(x)\|_\alpha=\left\|e^{\frac{s+1}{p}x}F(e^x)\right\|_\alpha=\left[\int_{-\infty}^{+\infty}e^{\frac{(s+1)\alpha}{p}x}F^\alpha(e^x)dx\right]^{\frac{1}{\alpha}},$$

$$\overset{\substack{e^x=t\\x=\ln t}}{=}\left[\int_0^{+\infty}t^{\frac{(s+1)\alpha}{p}}F^\alpha(t)\frac{1}{t}dt\right]^{\frac{1}{\alpha}}=\left[\int_0^{+\infty}t^{\frac{(s+1)\alpha}{p}-1}F^\alpha(t)dt\right]^{\frac{1}{\alpha}}$$

so

$$\left[\int_0^{+\infty}t^{\frac{(s+1)\alpha}{p}-1}F^\alpha(t)dt\right]^{\frac{1}{\alpha}}\le\left[\frac{p}{(s+1-p)r}\right]^{\frac{1}{r}}\left[\int_0^{+\infty}t^s f^p(t)dt\right]^{\frac{1}{p}}$$

Now,we assume $q \ge p > 1$ .take $\frac{1}{q} = \frac{1}{p} + \frac{1}{r} - 1$， $r = \frac{1}{1 + \frac{1}{q} - \frac{1}{p}}$ 。

$$1 + \frac{1}{q} - \frac{1}{p} = (1 - \frac{1}{p}) + \frac{1}{q} > 0， \quad 1 + \frac{1}{q} - \frac{1}{p} = 1 - (\frac{1}{p} - \frac{1}{q}) \le 1，$$

So $r = \frac{1}{1 + \frac{1}{q} - \frac{1}{p}} \ge 1$ 。 So,if $s > p - 1$， $\int_0^{+\infty} t^s f^p(t)dt < +\infty$， then

$$\left[\int_0^{+\infty} t^{\frac{(s+1)q}{p} - 1} F^q(t)dt\right]^{\frac{1}{q}} \le \left[\frac{p}{(s+1-p)r}\right]^{\frac{1}{r}} \left[\int_0^{+\infty} t^s f^p(t)dt\right]^{\frac{1}{p}}$$

Proof of Theorem 2

For any $x \in (0, a)$， by Holder inequality,

$$\begin{aligned}\int_x^a f(t)dt &= \int_x^a f(t)t^\alpha \frac{1}{t^\alpha} dt \le \left[\int_x^a f^p(t)t^{p\alpha}dt\right]^{\frac{1}{p}} (\int_x^a \frac{dt}{t^{q\alpha}})^{\frac{1}{q}} \\ &= \left[\int_x^a f^p(t)t^{p\alpha}dt\right]^{\frac{1}{p}} (\frac{x^{1-q\alpha} - a^{1-q\alpha}}{q\alpha - 1})^{1-\frac{1}{p}} \le \left[\int_x^a f^p(t)t^{p\alpha}dt\right]^{\frac{1}{p}} (\frac{x^{1-q\alpha}}{q\alpha - 1})^{1-\frac{1}{p}}\end{aligned},$$

Here, α is to be determined, meet $q\alpha > 1$， then $\alpha > \frac{1}{q} = 1 - \frac{1}{p}$， so

$$\left[\int_x^a f(t)dt\right]^p \le \frac{1}{(q\alpha - 1)^{p-1}} x^{(1-q\alpha)(p-1)} \int_x^a f^p(t)t^{p\alpha}dt = \frac{x^{p-1-p\alpha}}{(q\alpha - 1)^{p-1}} \int_x^a f^p(t)t^{p\alpha}dt \ ,$$

$$\frac{\left[\int_x^a f(t)dt\right]^p}{x^r} \le \frac{x^{p-1-p\alpha-r}}{(q\alpha - 1)^{p-1}} \int_x^a f^p(t)t^{p\alpha}dt \ ,$$

So

$$\begin{aligned}\int_0^a \frac{\left[\int_x^a f(t)dt\right]^p}{x^r} dx &\le \frac{1}{(q\alpha - 1)^{p-1}} \int_0^a x^{p-1-p\alpha-r} dx \int_x^a f^p(t)t^{p\alpha}dt \\ &= \frac{1}{(q\alpha - 1)^{p-1}} \int_0^a f^p(t)t^{p\alpha}dt \int_0^t x^{p-1-p\alpha-r}dx \\ &= \frac{1}{(q\alpha - 1)^{p-1}(p - p\alpha - r)} \int_0^a f^p(t)t^{p\alpha}t^{p-p\alpha-r}dt \\ &= \frac{1}{(q\alpha - 1)^{p-1}(p - p\alpha - r)} \int_0^a t^{p-r} f^p(t)dt\end{aligned},$$

Here $1-\frac{1}{p}<\alpha<1-\frac{r}{p}$。

Take $f(\alpha)=(q\alpha-1)^{p-1}(p-p\alpha-r), \alpha\in\left[1-\frac{1}{p}, 1-\frac{r}{p}\right]$,

$$\begin{aligned} f'(\alpha)&=(p-1)q(q\alpha-1)^{p-2}(p-p\alpha-r)-p(q\alpha-1)^{p-1}\\ &=p(q\alpha-1)^{p-2}(p-p\alpha-r)-p(q\alpha-1)^{p-1}\\ &=p(q\alpha-1)^{p-2}\left[p-r+1-(p+q)\alpha\right]\end{aligned}$$

Take $f'(\alpha)=0$, we can get

$$\alpha=\frac{p-r+1}{p+q}=\frac{p-r+1}{p+\frac{p}{p-1}}=\frac{p-r+1}{\frac{p^2}{p-1}}=\frac{(p+1-r)(p-1)}{p^2}$$

so $f(\alpha)$ get the maximum value in $\alpha=\frac{(p+1-r)(p-1)}{p^2}$

$$f\left(\frac{(p+1-r)(p-1)}{p^2}\right)=\left(q\frac{(p+1-r)(p-1)}{p^2}-1\right)^{p-1}\left(p-p\frac{(p+1-r)(p-1)}{p^2}-r\right)$$
$$=\left(\frac{p+1-r}{p}-1\right)^{p-1}\left(p-\frac{(p+1-r)(p-1)}{p}-r\right)=\left(\frac{1-r}{p}\right)^p$$

So

$$\int_0^a\frac{\left[\int_x^a f(t)dt\right]^p}{x^r}dx\le\left(\frac{p}{1-r}\right)^p\int_0^a t^{p-r}f^p(t)dt$$

We take $f(x)=x^\alpha, x\in(0,a]$。

$$\int_0^a x^{p-r}f^p(x)dx=\int_0^a x^{p+p\alpha-r}dx=\frac{a^{p+p\alpha-r+1}}{p+p\alpha-r+1}$$

$$\int_x^a f(t)dt=\int_x^a t^\alpha dt=\frac{a^{1+\alpha}-x^{1+\alpha}}{1+\alpha}=\frac{x^{1+\alpha}-a^{1+\alpha}}{-1-\alpha},$$

So

$$\frac{\left[\int_x^a f(t)dt\right]^p}{x^r}=\frac{(x^{1+\alpha}-a^{1+\alpha})^p}{(-1-\alpha)^p x^r},$$

So

$$\int_0^a \frac{\left[\int_x^a f(t)dt\right]^p}{x^r}dx=\int_0^a \frac{(x^{1+\alpha}-a^{1+\alpha})^p}{(-1-\alpha)^p x^r}dx=\frac{1}{(-1-\alpha)^p}\int_0^a \frac{\left[x^{1+\alpha}(1-\frac{a^{1+\alpha}}{x^{1+\alpha}})\right]^p}{x^r}dx$$

$$=\frac{1}{(-1-\alpha)^p}\int_0^a\left[1-(\frac{x}{a})^{-1-\alpha}\right]^p x^{p+p\alpha-r}dx \overset{\substack{y=(\frac{x}{a})^{-1-\alpha}\\ x=ay^{-\frac{1}{1+\alpha}}}}{=} \frac{1}{(-1-\alpha)^p}\int_0^1(1-y)^p(ay^{-\frac{1}{1+\alpha}})^{p+p\alpha-r}d(ay^{-\frac{1}{1+\alpha}})$$

$$=\frac{a^{p+p\alpha-r+1}}{(-1-\alpha)^{p+1}}\int_0^1(1-y)^p y^{-\frac{p+p\alpha-r+1}{1+\alpha}-1}dy=\frac{a^{p+p\alpha-r+1}}{(-1-\alpha)^{p+1}}B(p+1,\frac{r-1}{1+\alpha}-p)$$

$$\frac{\int_0^a \frac{\left[\int_x^a f(t)dt\right]^p}{x^r}dx}{\int_0^a x^{p-r}f^p(x)dx}=\frac{\frac{a^{p+p\alpha-r+1}}{(-1-\alpha)^{p+1}}B(p+1,\frac{r-1}{1+\alpha}-p)}{\frac{a^{p+p\alpha-r+1}}{p+p\alpha-r+1}}$$

$$=\frac{p+p\alpha-r+1}{(-1-\alpha)^{p+1}}B(p+1,\frac{r-1}{1+\alpha}-p)=\frac{1}{(-1-\alpha)^p}(\frac{r-1}{1+\alpha}-p)B(p+1,\frac{r-1}{1+\alpha}-p)$$

$$=\frac{1}{(-1-\alpha)^p}(\frac{r-1}{1+\alpha}-p)\frac{\Gamma(p+1)\Gamma(\frac{r-1}{1+\alpha}-p)}{\Gamma(1+\frac{r-1}{1+\alpha})}=\frac{1}{(-1-\alpha)^p}\frac{\Gamma(p+1)\Gamma(\frac{r-1}{1+\alpha}-p+1)}{\Gamma(1+\frac{r-1}{1+\alpha})}$$

$$\lim_{\alpha\to(\frac{r-1}{p}-1)^+}\frac{\int_0^a \frac{\left[\int_x^a f(t)dt\right]^p}{x^r}dx}{\int_0^a x^{p-r}f^p(x)dx}=\lim_{\alpha\to(\frac{r-1}{p}-1)^+}\frac{1}{(-1-\alpha)^p}\frac{\Gamma(p+1)\Gamma(\frac{r-1}{1+\alpha}-p+1)}{\Gamma(1+\frac{r-1}{1+\alpha})}$$

$$=\frac{1}{(-1-(\frac{r-1}{p}-1))^p}\frac{\Gamma(p+1)\Gamma(\frac{r-1}{1+\frac{r-1}{p}-1}-p+1)}{\Gamma(1+\frac{r-1}{1+\frac{r-1}{p}-1})}=(\frac{p}{1-r})^p\frac{\Gamma(p+1)\Gamma(1)}{\Gamma(p+1)}=(\frac{p}{1-r})^p$$

Therefore, $(\frac{p}{1-r})^p$ is the best constant.

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