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Development Of Closed-Form Approximation Of The Eccentric Anomaly For Circular And Elliptical Keplerian Orbit

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***Abstract*—In this paper, the development of closed-form approximation of the eccentric anomaly for circular and elliptical Keplerian orbit is presented. The derivation of the closed-form solution to the transcendental equation in respect of the eccentric anomaly, E of Keplerian orbit given in terms of the M (the mean anomaly) and e (the first eccentricity) is presented. The closed-form solution is a composite function that approximates the eccentric anomaly in the following two cases; $0.5 \leq e \leq 1$ and $0.001 \leq e \leq 0.5$ or M < 1°. A sample numerical computation for M= 35° =1.396263402 radian and e = 0.99 yielded a value of E=1.600430567 radians or 91.69791688°. The closed-form approximation has an error of 0.382069122° or 0.006668 radian which amounts to a percentage error of 0.416660634%. The results of the computation of E for M =30° and 0.58 ≤ e ≤ 1; for M =30° and 0.01 ≤ e ≤0 and for M =30° and 0.001 ≤ e ≤ 0.01 showed good approximation of the eccentric anomaly with percentage error that is less than ± 1%. In all, the closed-form solution can be exercised in all other combinations of M and e. It is believed that the percentage error performance will not deviate much from what is presented in this paper. Also, in the worst case, the value of E obtained can serve as a good approximation for the initial value that can be used in any iterative solution for the eccentric anomaly, E . In that case, the initial value of E obtained using this closed-form approximation will greatly reduce the convergence cycle of the iteration algorithm.**

Keywords— Eccentric Anomaly, Keplerian Orbit, First Eccentricity, Mean Anomaly, Closed-Form Solution, Iteration Algorithm

I. INTRODUCTION

In planetary studies, Kepler developed some laws that are very useful in the determination of essential parameters used to characterize the orbits and the orbital motions of satellites [1,2,3,4,5,6,7]. One of such laws gave rise to transcendental expression that relates the eccentric anomaly (E) to the first eccentricity (e) and the mean anomaly (M) [8,9,10,11,12,13]. Over the year, researchers have tried to provide solution to the Kepler's transcendental equation and most of the published solution approaches focused on the derivation of initial value of E for iterative solution [14,15,16,17,18,19,20,21]. In that case, the expression derived in those approaches only serve as the initial value of E which will then be employed in an iterative solution to determine the required or actual value of E.

Conversely, in this paper, a composite function is derived which is a closed-form approximation of E in the Kepler's transcendental equation for the eccentric anomaly (E). The closed-form solution is for circular and elliptic Keplerian orbit in which the value of first eccentricity, e is between 0 and 1. The detailed mathematical analysis used in the development of the closed-form approximation is presented. Also, some numerical examples are presented along with performance analysis of the estimation error.

II. METHODOLOGY

The first eccentricity, e of circular and elliptic Keplerian orbit is between 0 and 1. Also, the eccentric anomaly, E of Keplerian orbit is given in terms of the M (the mean anomaly) and e(the first eccentricity) as follows;

$$E = M + e(\sin(E)) \quad (1)$$

The popular approach employed in solving the Keplerian equation for E is iterative method. In this paper, a closed-form approximation for the determination of E is presented. The solution applies to circular and elliptic Keplerian orbit where $0 \leq e \leq 1$.

The fundamental idea behind the closed-form solution is that for any given value of e and M, the value of E is such that $M \leq E \leq M + e$, where M and e are in radians and for the circular and elliptical orbits, $0 \leq e \leq 1$. Hence, generally, E can be expressed as;

$$E = M + e(\sin(M + \varphi)) \quad \text{where } 0 \leq \varphi \leq e \quad (2)$$

In order to determine φ, the following expression is used;

$$M + \varphi = M + e(\sin(M + \varphi)) \quad \text{where } 0 \leq \varphi \leq e \quad (3)$$

Hence,

$$\varphi = e(\sin(M + \varphi)) \quad \text{where } 0 \leq \varphi \leq e \quad (4)$$

$$\frac{\varphi}{e} = \sin(M+\varphi) \quad \text{where } 0 \le \varphi \le e \tag{5}$$

According to law of trigonometry,

$$\sin(A + B) = \sin A \cos B + \cos A \sin B \tag{6}$$

Hence,

$$\sin(M+\varphi) = \sin(M)\cos(\varphi) + \cos(M)\sin(\varphi) \tag{7}$$

Now, since, $0 \le \varphi \le e \le 1$, and the angles are all in radian, then, $\cos(\varphi)$ and $\cos(\varphi)$ can be approximated by a linear function of the form;

$$\cos(\varphi) = A(\varphi) + B \tag{8}$$

$$\sin(\varphi) = C(\varphi) + D \tag{9}$$

Then,

$$\sin(M+\varphi) = \sin(M)\big((A)(\varphi) + B\big) + \cos(M)\big((C)(\varphi) + D\big) \tag{10}$$

$$\sin(M+\varphi) = (\varphi)(A)\sin(M) + (B)\sin(M) + (\varphi)(C)\cos(M) + (D)\cos(M) \tag{11}$$

Therefore,

$$\frac{\varphi}{e} = (\varphi)(A)\sin(M) + (B)\sin(M) + (\varphi)(C)\cos(M) + (D)\cos(M) \tag{12}$$

Hence,

$$\frac{\varphi}{e} - (\varphi)(A)\sin(M) - (\varphi)(C)\cos(M) = (B)\sin(M) + (D)\cos(M) \tag{13}$$

$$\varphi = \frac{(B)\sin(M) + (D)\cos(M)}{\left(\frac{1}{e}\right) - (A)\sin(M) - (C)\cos(M)} \tag{14}$$

$$\varphi = \frac{(B)\sin(M) + (D)\cos(M)}{\left(\frac{1}{e}\right) - (A)\sin(M) - (C)\cos(M)} \tag{15}$$

$$E_2 = M + (e)\sin[M + e\{\sin(M+\varphi)\}] \tag{16}$$

The values of A and B were obtained from trend line fitted to the graph of cos(x) versus x for $0 \le x \le 1$, where x is in radians. Also, the values of C and D were obtained from trend line fitted to the graph of sin (x) versus x for $0 \le x \le e \le 1$, where x is in radians. In order to improve on the accuracy of the solution, the values of x were considered in the following two (2) range of values;

i. $0.5 \le x \le e \le 1$
ii. $0.001 \le x \le e \le 0.5$ or M < 1°

The values of A, B , C and D obtained for the two range of values of x are given in Table 1.

1 The values of coefficient A, B , C and D obtained for the two range of values of x

	Range of value for e	A	B	C	D
1	$0.5 \le e \le 1$	-0.584013113	1.173439404	0.809460441	0.077357763
2	$0.01 \le e \le 0.5$ Or M < 1.1°	-0.248393819	1.019165175	0.961260155	0.004043021

III. APPLICATION OF THE CLOSED-FORM SOLUTION FOR SOLVING THE KEPLERIAN EQUATION FOR ECCENTRIC ANOMALY.

The closed-form solution can be employed as follows;

Step 1: Given M and e, then, using the data in Table 1 determine the value of A, B, C and D based on the value of e

For instance, if e = 0.99, then from Table 1, the values of A, B, C and D are given as A = -0.584013113; B = 1.173439404; C =0.809460441; D = 0.077357763.

Step 2: Determine φ using Eq 13, where for instance, M= 35° =1.396263402 radian and e = 0.99 , then,

$$\varphi = \frac{(1.173439404)\sin(0.573576436) + (0.07735776)\cos(0.819152044)}{\left(\frac{1}{0.99}\right) - (-0.584013113)\sin(0.573576436) - (0.809460441)\cos(0.819152044)} = \frac{0.736424961}{0.682005996} = 1.079792503$$

Step 3: Determine E using Eq 2 [$E = M + e(\sin(M+\varphi))$] which gives,

$$E = 0.610865238 + 0.99(\sin(0.610865238 + 1.079792503))$$

$E = 0.610865238 + \sin(1.690657741)$ =0.610865238 + 0.982896965 = 1.593762203 radians = 91.32°

The solution by secant method yielded a value of E=1.600430567 radians = 91.69791688°. Hence, the closed form approximation has an error of 0.382069122° or 0.006668 radian which amounts to a percentage error of 0.416660634%.

Furthermore, the results of the computation of E for M =30° and $0.58 \le e \le 1$ are shown in Table 1. Again, the results of the computation of E for M =30° and $0.01 \le e \le 0.58$ are shown in Table 2 while the results of the computation of E for M =30° and $0.001 \le e \le 0.01$ are shown in Table 3. The results showed a good approximation of the eccentric anomaly with percentage error that is less than ± 1%.

In all, the close-form solution can be exercised in all other combinations of M and e . It is believed that the percentage error performance will not deviate much from what is presented in this paper. Also, in the worst case, the value of E obtained can serve as a close approximation for the initial value that can be used in the iterative solution. In that case, the value of E obtained as the initial value for the iteration will greatly reduce the convergence cycle number of the iteration algorithm.

Table 1 The results of the computation of E for M =30° and 0.58 ≤ e ≤ 1

For 0.58 ≤ e ≤ 1								
M (Degree)	M (radian)	e	e(sin(M+φ))	E	E (degree)	Ea	e	e%
30	0.523599	0.58	0.497304094	1.020903	58.49343	1.016868	-0.00403	-0.39675
30	0.523599	0.6	0.520870669	1.044469	59.84369	1.041495	-0.00297	-0.28562
30	0.523599	0.7	0.642274076	1.165873	66.79959	1.167416	0.001544	0.132225
30	0.523599	0.8	0.766558931	1.290158	73.92059	1.292908	0.002751	0.212748
30	0.523599	0.9	0.888215817	1.411815	80.89102	1.412321	0.000506	0.035849
30	0.523599	1	0.999727611	1.523326	87.28017	1.522429	-0.0009	-0.05892

Table 2 The results of the computation of E for M =30° and 0.01 ≤ e ≤0.58

For 0.01 ≤ e ≤0.58								
M (Degree)	M (radian)	e	e(sin(M+φ))	E	E (degree)	Ea	e	e%
30	0.523599	0.1	0.054704	0.578302	33.13428	0.578255	-4.7E-05	-0.00815
30	0.523599	0.2	0.119944	0.643543	36.87228	0.643617	7.47E-05	0.011599
30	0.523599	0.3	0.197602	0.721201	41.32177	0.721826	0.000625	0.086526
30	0.523599	0.4	0.289705	0.813303	46.59885	0.814571	0.001268	0.155674
30	0.523599	0.5	0.398046	0.921644	52.80633	0.922007	0.000362	0.039292
30	0.523599	0.58	0.496837	1.020435	58.46665	1.016868	-0.00357	-0.35079

Table 3 The results of the computation of E for M =30° and 0.001 ≤ e ≤ 0.01

For 0.001 ≤ e ≤ 0.01								
M (Degree)	M (radian)	e	e(sin(M+φ))	E	E (degree)	Ea	e	e%
30	0.523599	0.001	0.0005	0.524099	30.02867	0.524099	-1.1E-08	-2.1E-06
30	0.523599	0.003	0.001504	0.525103	30.08617	0.525103	-1E-07	-1.9E-05
30	0.523599	0.005	0.002511	0.52611	30.14388	0.52611	-2.8E-07	-5.2E-05
30	0.523599	0.007	0.003522	0.527121	30.20179	0.52712	-5.3E-07	-0.0001
30	0.523599	0.008	0.004029	0.527627	30.23082	0.527627	-6.9E-07	-0.00013
30	0.523599	0.01	0.005045	0.528643	30.28904	0.528642	-1.1E-06	-0.0002

IV. CONCLUSION

A closed-form solution for calculating the eccentric anomaly of circular and elliptical Keplerian orbit is presented. A linear function approximation of cosine and sin function for angles in the range of 0 to 1 radian was also employed in deriving the closed-form solution. The closed-form solution is a composite function that approximates the eccentric anomaly based on the value of mean anomaly and the first eccentricity of the orbit.

In all, the values of eccentric anomaly obtained from the closed-form solution were compared with the actual values of eccentric anomaly and the results show that the percentage error is less than 1 %. As such, the closed-form solution is a good approximation of the eccentric anomaly and can be used instead of the tedious iterative solution approach.

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