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Comparative Evaluation Of Initial Value Options For Numerical Iterative Solution To Eccentric Anomalies In Kepler's Equation For Orbital Motion

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Abstract— **In this paper, comparative evaluation of initial value options for numerical iterative solution to eccentric anomaly (E) in Kepler's equation for orbital motion is presented. Two existing analytical expressions for the initial value of E are considered and compared with two new analytical expressions for the initial value of E proposed in this paper. Particularly, the initial value of E for each of the four options is separately applied in Newton Raphson iteration for computing E and then, the convergence cycles are compared for the four different initial value options. The two existing initial value options are tagged Eo1 and Eo2 while the two proposed initial value options are tagged Eo3 and Eo4. Specifically, the comparison is performed for different configurations of the parameters in the Kepler's equation for E, namely, the first eccentricity (e) and the mean anomaly (M). The results showed that for M =7 ° and e = 0.999, Eo1 has convergence cycle greater than 13, Eo2 has convergence cycle of 5, Eo3 has convergence cycle of 4 and Eo4 has convergence cycle of 3. Also, for M =7 ° and e = 0.09 , Eo1 has convergence cycle of 3 while the rest of the three options have convergence cycle of 2. Again, for M =0.7 ° and e = 0.09 all the four options have convergence cycle of 2. Finally, for M =0.7 ° and e = 0.99 , Eo1 and Eo2 have convergence cycle of 8 while Eo3 has convergence cycle of 6 and Eo4 has convergence cycle of 5. In all, the proposed Eo4 initial value option is the best option with the lowest convergence cycle in all cases. Again, the Eo4 and Eo3 initial value options performed better that the existing two initial value options, Eo1and Eo2.**

Keywords— Kepler's Equation, Eccentricity, Satellite Communications, Newton Raphson Iteration , Astronomy, Mean Anomaly , Eccentric Anomaly ***Introduction***

I. INTRODUCTION

In satellite orbital motion studies, the location of satellite on a Keplerian orbit requires either the eccentric anomaly (E) or the true anomaly (v) [1,2,3,4]. Fortunately, Kepler's equation for orbital motion provided analytical expression that relates the mean anomaly (M) and the eccentric anomaly [5,6,7,8,9,10,11,12,14]. Hence, Kepler's equation can be used to determine the eccentric anomaly (E) and hence the location of a satellite in its orbit. However, the Kepler's equation in terms of eccentric anomaly (E) is transcendental in nature [12,14,15,16,17,18,19,20]. As such, it requires iterative solution to determine the value of E.

Over the years, researchers have applied several iterative methods to solve for the eccentric anomaly [7,8,9,14,2122,23,24]. However, there has been consistent effort to find ways to minimize the number of iterations required to arrive at acceptable solution to the Kepler's equation for eccentric anomaly. In most cases, the choice of the initial value of E has been found to be the key factor. Several authors have published different initial value analytical expression options for E for application in the Kepler's equation [7,8,9,14,2122,23,24]. In this paper, two additional analytical expressions for the initial value of E are presented and then compared with two other existing initial value options that are widely used in different published works. The essence of this paper is to demonstrate the higher efficiency of the proposed initial value options for E that are presented in this paper.

Notably, the convergence cycle is the key parameter used in comparing the performance of the various initial values of E presented in this paper. Particularly, the four different initial value options of E are applied in Newton Raphson iteration solution to E and the convergence cycles of the Newton Raphson iteration are compared. In all, the initial value expressions for E that are proposed in this paper will be very useful for researchers on satellite orbital motion studies.

2. METHODOLOGY

The Kepler's e orbit equation relating the mean anomaly (M) and the eccentric anomaly (E) is given as [12,14,15,16,17,18,19,20,2122,23,24];

$$E = M + e(\sin(E)) \quad (1)$$

$$f(E) = E - M - e(\sin(E)) \quad (2)$$

$$\delta f(E) = 1 - e(\cos(E)) \quad (3)$$

In the Kepler's orbit equation, circular and elliptic orbit has its eccentricity (e) value in the range; $0 \le e \le 1$. Let the initial value of E be E_n , then the next expected actual value of E is E_{n+1} and it is given as;

$$E_{n+1} = E_n - \frac{E_n - M - e(\sin(E_n))}{1 - e(\cos(E_n))} \text{ for n = 0,1,2,3...N} \quad (4)$$

For each E_{n+1} , the convergence to the actual value of E is tested using the specified tolerance error, $\boldsymbol{\varepsilon}$, such that the iteration stops when $|E_{n+1} - E_n| \le |\varepsilon|$. The Newton Raphson's algorithm for the solution to E is given as follows;

Step 1::

Input the values of M, e and $\boldsymbol{\varepsilon}$

Step 2::

Step 2.1::

Initialize the counter , n =0

Step 2.2::

Compute the initial value of eccentric anomaly, E_n

Step 3::

Compute $f(E_n) = E_n - M - e(\sin(E_n))$

Step 4::

Compute $\delta f(E_n) = 1 - e(\cos(E_n))$

Step 5::

Increment the counter , **n = n + 1**

Step 6::

Compute $E_n = E_{n-1} - \frac{E_{n-1} - M - e(\sin(E_{n-1}))}{1 - e(\cos(E_{n-1}))}$

Step 7::

If $(|E_n - E_{n-1}| > |\varepsilon|$ then Goto Step 3 EndIf

Step 8::

Output E_n

Step 9::

End

III. INITIAL VALUE EXPRESSIONS OPTIONS FOR ITERATIVE SOLUTION OF E.

There are different analytical expression options for the initial value of E. In this section, two existing initial value options are presented along with two other initial value expressions proposed in this paper.

Option 1 initial value of E

Generally, if M and e are given, the initial value of E is simply taken as M i(n radians). Hence, in this paper, this option of the initial value is denoted as E_{01} where;

$$E_{01} = M \quad (5)$$

Option 2 initial value of E

Based on [25], if M and e are given, the initial value of E, denoted in this paper as E_{02} is given as;

$$E_{02} = M + \frac{e(\sin(M))}{1 - \sin(M+e) + \sin(M)} \quad (6)$$

Option 3 : Proposed initial value of E

In this paper, two other initial value options are presented. Based on the $E_{01} = M$, an enhanced initial value , denoted as E_{03} is defined as follows;

$$E_{03} = M + e[\sin(M + e\{\sin(M + e)\})] \quad (7)$$

Option 4 : Proposed initial value of E

In this paper, an empirically derived initial value expression for E is given as follows;

$$E_{04} = M + e[\sin(M + e\{\sin(M + \varphi)\})] \quad (8)$$

Where φ is defined with respect to the parameter values in Table 1.

$$\varphi = \frac{(B)\sin(M) + (D)\cos(M)}{\left(\frac{1}{e}\right) - (A)\sin(M) - (C)\cos(M)} \quad (9)$$

Table 1 The values of coefficient A, B , C and D obtained for the three range of values of x

	Range of value of e	A	B	C	D
1	$0.5 e \le 1$	-0.584013113	1.173439404	0.809460441	0.077357763
2	$0.01 \le e \le 0.5$ Or M < 1.1°	-0.248393819	1.019165175	0.961260155	0.004043021

IV. RESULTS AND DISCUSSION

Newton Raphson iteration solution to eccentric anomaly (E) was conducted based on the four (4) initial value expressions for E with tolerance error in the order of 10^{-7}. The results of the Newton Raphson iteration solution to eccentric anomaly (E) for M =7 °; e = 0.999 and Eo =Eo1 =7° are given in Table 1 and Table 5. The results showed that the algorithm converged to acceptable tolerance error at cycle greater than the 13th cycle. Also, the results of the Newton Raphson iteration solution to eccentric anomaly for M =7 ° ; e = 0.999 and Eo =Eo2 = 38.52700657° are given in Table 2 and Table 5. The results showed that the algorithm converged to acceptable tolerance error at the 5th cycle. Again, the results of the Newton Raphson iteration solution to eccentric anomaly for M =7 ° ; e = 0.999 and Eo =Eo3 = 38.52700657° are given in Table 3 and Table 5. The results showed that the algorithm converged to acceptable tolerance error at the 4th cycle. Furthermore, the results of the Newton Raphson iteration solution to eccentric anomaly for M =7 °; e = 0.999 and Eo =Eo4 = 38.52700657° are given in Table 4 and Table 5. The results showed that the algorithm converged to acceptable tolerance error at the 3rd cycle.

Further comparison are made for different combinations of the values of e and M. The comparison of the convergence cycle for the four initial value expressions for M =7 ° and e = 0.09 is given in Table 6. The results showed that Eo1 has convergence cycle of 3 while the rest of the three options have convergence cycle of 2.

Also, the comparison of the convergence cycle for the four initial value expressions for M =0.7 ° and e = 0.09 is given in Table 7. The results showed that all the four options have convergence cycle of 2.

Again, the comparison of the convergence cycle for the four initial value expressions for M =0.7 ° and e = 0.99 is given in Table 8. The results showed that Eo1 and Eo2 have convergence cycle of 8 while Eo3 has convergence cycle of 6 and Eo4 has convergence cycle of 5.

Furthermore, the comparison of the convergence cycle for the four initial value expressions for M =0.7 ° and e = 0.999 is given in Table 9. The results showed that Eo1 has convergence cycle of 15, Eo2 has convergence cycle of 8, Eo3 has convergence cycle of 6 and Eo4 has convergence cycle of 5.

In all, the proposed Eo4 initial value option is the best option with the lowest convergence cycle in all cases. Again, the Eo4 and Eo3 initial value options performed better that the existing two initial value options, Eo1and Eo2.

Table 1 Results of the Newton Raphson iteration solution to eccentric anomaly for M =7 °; e = 0.999 and Eo =Eo1 =7°

Cycle	Xi	x = E radian	Error in radian	E_in degree
1	0.122173047639611	14.536308441504100	1.3493662E+01	832.87
2	14.536308441504100	4.816322642723040	5.6877587E+00	275.95
3	4.816322642723040	-1.529093414458350	-6.5313503E-01	-87.61
4	-1.529093414458350	-0.847573673862184	-2.2081954E-01	-48.56
5	-0.847573673862184	-0.195915163638087	-1.2361860E-01	-11.23
6	-0.195915163638087	5.950918708566250	6.1546060E+00	340.96
7	5.950918708566250	-104.664118867952000	-1.0562234E+02	-5996.81
8	-104.664118867952000	-36.381313805268300	-3.7470685E+01	-2084.50
9	-36.381313805268300	13.583437732934300	1.2611546E+01	778.27
10	13.583437732934300	-12.986033563543600	-1.2701161E+01	-744.04
11	-12.986033563543600	131.860700875498000	1.3182453E+02	7555.06
12	131.860700875498000	-27865.892292707200000	-2.7866049E+04	-1596598.02
13	-27865.892292707200000	17431955.825668200000000	1.7431956E+07	998777497.47

Table 2 Results of the Newton Raphson iteration solution to eccentric anomaly for M =7 ° ; e = 0.999 and Eo =Eo2 = 38.52700657°

Cycle	Xi	x = E radian	Error in radian	E_in degree
1	0.672423115651716	1.0020393991491100000	3.8137796E-02	57.41
2	1.002039399149110	0.9194817462886810000	2.8164415E-03	52.68
3	0.919481746288681	0.9123401935492860000	2.0223407E-05	52.27
4	0.912340193549286	0.9122881672950390000	1.0693433E-09	52.27
5	0.912288167295039	0.9122881645437810000	0.0000000E+00	52.27

Table 3 Results of the Newton Raphson iteration for solution to eccentric anomaly for M =7 °; e = 0.999 and Eo =Eo3 = 38.52700657°

Cycle	Xi	x = E radian	Error in radian	E_in degree
1	0.974412139449801	0.9158698975614130000	1.3972004E-03	52.48
2	0.915869897561413	0.9123011335337930000	5.0407797E-06	52.27
3	0.912301133533793	0.9122881647147340000	6.6445183E-11	52.27
4	0.912288164714734	0.9122881645437810000	0.0000000E+00	52.27

Table 4 Results of the Newton Raphson iteration for solution to eccentric anomaly for M =7 °; e = 0.999 and Eo =Eo4 = 38.52700657°

Cycle	Xi	x = E radian	Error in radian	E_in degree
1	0.922346108393473	0.9123894402910420000	3.9367333E-05	52.28
2	0.912389440291042	0.9122881749674050000	4.0513949E-09	52.27
3	0.912288174967405	0.9122881645437810000	0.0000000E+00	52.27

Table 5 Comparison of the convergence cycle for the four initial value expressions for M =7 ° and e = 0.999

Initial Value option	M	e	Eo	Ea	n
Eo1	7	0.999	7	52	>> 13
Eo2	7	0.999	38.52700657	52.27026153	5
Eo3	7	0.999	55.8297031	52.27026153	4
Eo4	7	0.999	52.84653926	52.27026153	3

Table 6 Comparison of the convergence cycle for the four initial value expressions for M =7 ° and e = 0.09

Initial Value option	M	e	Eo	Ea	n
Eo1	7	0.09	7	7.690026	3
Eo2	7	0.09	7.689613	7.690026	2
Eo3	7	0.09	7.725318	7.690026	2
Eo4	7	0.09	7.694186	7.690026	2

Table 7 Comparison of the convergence cycle for the four initial value expressions for M =0.7 ° and e = 0.09

Initial Value option	M	e	Eo	Ea	n
Eo1	0.7	0.09	0.7	0.769228	2
Eo2	0.7	0.09	0.769216	0.769228	2
Eo3	0.7	0.09	0.810348	0.769228	2
Eo4	0.7	0.09	0.769422	0.769228	2

Table 8 Comparison of the convergence cycle for the four initial value expressions for M =0.7 ° and e = 0.99

Initial Value option	M	e	Eo	Ea	n
Eo1	0.7	0.99	0.7	21.36188	8
Eo2	0.7	0.99	4.787187	21.37795	8
Eo3	0.7	0.99	43.18186	21.36052	6
Eo4	0.7	0.99	25.15964	21.35957	5

Table 9 Comparison of the convergence cycle for the four initial value expressions for M =0.7 ° and e = 0.999

Initial Value option	M	e	Eo	Ea	n
Eo1	0.7	0.999	0.7	23.78045	15
Eo2	0.7	0.99	4.787187	21.37795	8
Eo3	0.7	0.99	43.18186	21.36052	6
Eo4	0.7	0.99	25.15964	21.35957	5

V. CONCLUSION

The initial value options for solving Kepler's equation for eccentric anomaly (E) are studied. Two existing analytical expressions for the initial value of E are considered and compared with two new analytical expressions for the initial value of E proposed in this paper. The initial value options are used in the Newton Raphson iteration scheme to sole for E in the Kepler's equation and the convergence cycle of the iteration scheme for the different initial value options are compared for different configuration of the parameters of the Kepler's equation. The results showed that the proposed two initial value options performed better than the existing two initial value options.

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