Fixed Point Iteration Computation Of Nominal Mean Motion And Semi Major Axis Of Artificial Satellite Orbiting An Oblate Earth

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Abstract— **In this paper, fixed point iteration method for the computation of the semi major axis and the nominal mean motion of an artificial satellite orbiting an oblate earth is presented. The algorithm and the flowchart, along with relevant analytical expressions are presented. FPI computation of the semi major axis and the nominal mean motion for the case study orbit was performed with required estimation error, $\varepsilon \leq 1\text{x}\, 10^{-6}$. The results of the FPI iteration show that the algorithm converged at the second (2nd) cycle with an estimation error of -8.74134E-08 km which is less than the specified required estimation error of $1\text{x}\, 10^{-6}$ km. At the convergence cycle, the semi major axis of the orbit is 27338.632605314 km and the nominal mean motion (n_0) is 0.000139670 rad/s. Meanwhile, the mean motion (n) is 0.000139683 rad/s. and the difference between n and n_0 (that is, n - n_0) is 1.2345640E-08 rad/s. Also, the results show that the orbit semi major axis without the impact of oblate earth is 27332.341001289 km whereas the orbit semi major axis with the impact of oblate earth is 27338.632605314 which gives a difference of (27332.341001289-27338.632605314) of -6.291604025 km . Accordingly, with the oblate earth, there is an increase in the semi major axis of the orbit when compared to the case of non-oblate earth. Additional effect is that the mean motion of the satellite orbiting an oblate earth is higher or faster than the nominal mean motion of the satellite.**

Keywords—Fixed Point Iteration, Perturbed Orbit, Artificial Satellite, Oblate Earth, Nominal Mean Motion, Anomalistic Period, Orbital Semi Major Axis, Seeded Secant Iteration, Anomalistic Period, Iterative Root Finding Scheme

I. INTRODUCTION

Today, the earth is orbited by numerous artificial satellites deployed for diverse applications ranging from telecommunication, remote sensing, astronomical purposes, weather forecasting, climatic studies, radar applications, military applications and many others [1,2,3,4,5,6,7,8,9,10,11]. The motion of each of the satellites in their orbits is governed by planetary motion laws postulated by Kepler [12,13,14,15,16,17,18]. The orbital motion laws by Kepler are based on ideal case where the earth is assumed to be a perfect sphere with homogeneous density [19,20,21,22,23]. However, the earth is neither spherical nor homogeneous in mass or density. Rather, the shape of the earth has bulge at the equator and flattening at the poles [24,25,26,27,28,29,30,31]. As such, the shape of the earth is better modeled as oblate spheroid [32,33,34,35,36]. The oblateness of the earth introduces some complex motion of the satellites in their orbits. Such orbits with the complex motions are referred to as perturbed orbit.

Analysis of the mean motion of satellites on perturbed orbit is quite complex; it requires iterative approach to determine the nominal mean motion or the semi major axis of the perturbed orbit when the anomalistic orbit period is given. Consequently, in this paper, fixed point iteration (FPI) method [37,38,39,40] is adapted for the computation of the semi major axis and the nominal mean motion of an artificial satellite orbiting an oblate earth. The FPI flowchart is presented along with the algorithm and requisite analytical expressions associated with the FPI algorithm. Furthermore, sample perturbed orbit parameters are used for numerical examples.

II. METHODOLOGY

A satellite orbit with earth geocentric gravitational constant, μ and semi major axis, a , then the nominal mean motion (n_o), is given as;

$$n_o = \sqrt{\frac{\mu}{a^3}} \quad (1)$$

Also, for an artificial satellite orbiting an oblate earth with anomalistic period , P, the mean motion, n is given as;

$$\text{n} = \frac{2\pi}{\text{P}} \quad (2)$$

The mean motion n is related to the nominal mean motion, (n_o) as follows;

$$n = n_o\left[1 + \frac{K_1(1-1.5\sin(i)^2)}{a^2(1-e^2)^{1.5}}\right] \quad (3)$$

Then, the semi major axis (a) can be expressed in terms of n and n_o as follows;

$$n = \left(\sqrt{\frac{\mu}{a^3}}\right)\left[1 + \frac{K_1(1-1.5\sin(i)^2)}{a^2(1-e^2)^{1.5}}\right] \quad (4)$$

$$a = \left(\frac{\mu}{n^2}\left[1 + \frac{K_1(1-1.5\sin(i)^2)}{a^2(1-e^2)^{1.5}}\right]^2\right)^{1/3} \quad (5)$$

Let $f(a_k)$ and $g(a_k)$ denote functions of a in cycle k which are given as follows;

$$a_{k+1} = g(a_k) = \left(\frac{\mu}{n^2}\left[1+\frac{K_1(1-1.5\sin(i)^2)}{(a_k)^2(1-e^2)^{1.5}}\right]^2\right)^{1/3} \quad (6)$$

$$f(a_k) = a_k - \left(\frac{\mu}{n^2}\left[1+\frac{K_1(1-1.5\sin(i)^2)}{(a_k)^2(1-e^2)^{1.5}}\right]^2\right)^{1/3} = a_k - g(a_k) \quad (7)$$

The algorithm and the flowchart (Figure 1) for the fixed point iteration (FPI) used in the computation of the semi major axis and the nominal mean motion of artificial satellite orbiting an oblate earth are given as follows;

Step 1: The initial parameters;

Step 1.1: Counter for the cycle, k = 0

Step 1.2: Set the tolerance error, $\varepsilon = 0.001$

Step 1.3: $a_k =$

$$a_o = (\mu/n)^{1/3} = \left(\mu/\frac{2\pi}{P}\right)^{1/3} \quad (8)$$

Step 2 Compute the next root, a_{k+1}

$$a_{k+1} = g(a_k) = \left(\frac{\mu}{n^2}\left[1+\frac{K_1(1-1.5\sin(i)^2)}{(a_k)^2(1-e^2)^{1.5}}\right]^2\right)^{1/3} \quad (9)$$

Step 3:

$$\mathbf{f}(a_k) = a_k - \left(\frac{\mu}{n^2}\left[1+\frac{K_1(1-1.5\sin(i)^2)}{(a_k)^2(1-e^2)^{1.5}}\right]^2\right)^{1/3} = a_k - \mathbf{g}(a_k) \quad (10)$$

Step 4:

Step 4.1:

Step 4.1.1:

If $|f(a_k)| \leq \varepsilon$ then

Step 4.1.2:

$$n_o = \sqrt{\frac{\mu}{a^3}} = \sqrt{\frac{\mu}{(a_{k+1})^3}}$$

Step 4.2:

If however, $|f(a_k)| > \epsilon$, then

Step 4.2.1:

$a_k = a_{k+1}$

Step 4.2.2:

$k = k + 1$

Step 4.2.3:

Repeat step 2 to step 4

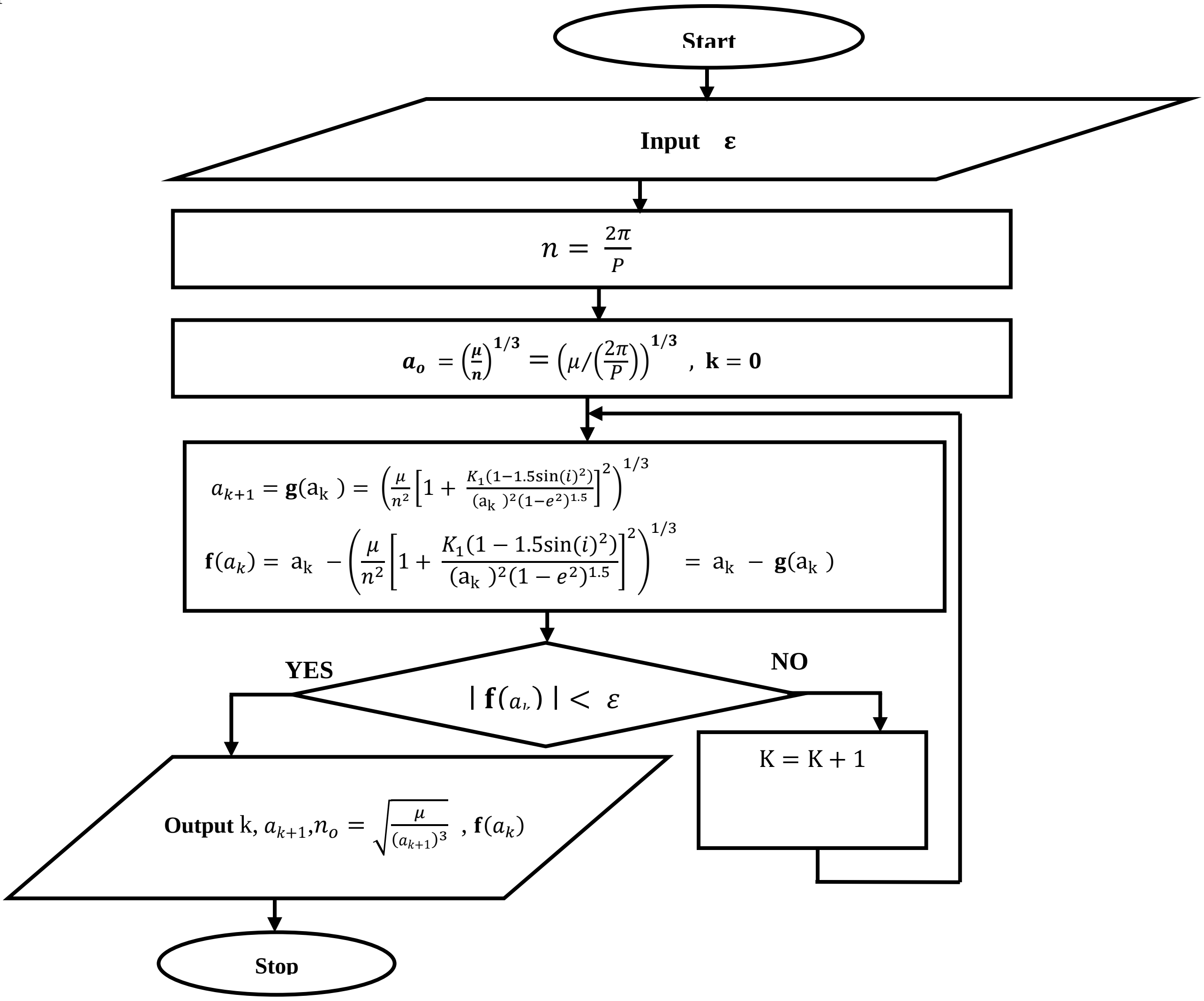

Figure 1: The flowchart for the Fixed Point Iteration (FPI) computation of the semi major axis and the nominal mean motion of artificial satellite orbiting an oblate earth

III RESULTS AND DISCUSSION

A sample numerical computation was performed for the orbit of a hypothetical artificial satellite orbiting an oblate earth, where the orbit parameters are as follows:

i. Eccentricity (e) = 0.0025;
ii. Earth Geocentric Gravitational Constant (μ) = 3.986005 x10^{14} m^3/s^2
iii. Inclination Angle (i) = 0 degree
iv. Anomalistic Period (P) = 12.5 hours
v. Constant (K1) = 66,063.1704 km^2

FPI computation of the semi major axis and the nominal mean motion for the case study orbit was performed with required estimation error, $\boldsymbol{\varepsilon} \leq \mathbf{1x\,10^{-6}}$. The results of the FPI iteration show that the algorithm converged at the second (2nd) cycle with an estimation error of -8.74134E-08 km which is less than the specified required estimation error of 1x 10^{-6} km. At the convergence cycle, the semi major axis of the orbit is 27338.632605314 km and the nominal mean motion (n_0) is 0.000139670 rad/s. Meanwhile, the mean motion (n) is 0.000139683 rad/s. and the difference between n and n_0 (that is, n - n_0) is 1.2345640E-08 rad/s. Also, the results show that the orbit semi major axis without the impact of oblate earth is 27332.341001289 km whereas the orbit semi major axis with the impact of oblate earth is 27338.632605314 which gives a difference of (27332.341001289-27338.632605314) of -6.291604025 km . Accordingly, with the oblate earth, the semi major axis is larger and hence, in order to maintain the given anomalistic period , the mean motion (n) has to be larger than the nominal mean motion (n_0). In this case, the difference, n - n_0 is 1.2345640E-08 rad/s. That means, the case study artificial satellite orbiting an oblate earth has men motion that is faster (by a value of 1.2345640E-08 rad/s) than what it would have been if the earth is not oblate.

Table 1 The result of the FPI computation of the semi major axis and the nominal mean motion for the case study orbit of an artificial satellite orbiting an oblate earth (the required estimation error, $\boldsymbol{\varepsilon} \leq \mathbf{1x\,10^{-6}}$)

	Semi major axis	**g(a_k) (km)**	**Estimation Error**	**Nominal mean motion (no)**	**Mean motion (n)**	**Difference between n and n_o**
Cycle	a_k **(km)**	a_{k+1} **(km)**	**f(a_k) (km)**	n_o **(rad/s)**	**n (rad/s)**	**n-n_o (rad/s)**
0	**27332.341001289**	**27338.633347090**	**-6.29235E+00**	**0.000139718**	**0.000139683**	**-3.5882947E-08**
1	**27338.633347090**	**27338.632605314**	**7.41776E-04**	**0.000139670**	**0.000139683**	**1.2351325E-08**
2	**27338.632605314**	**27338.632605402**	**-8.74134E-08**	**0.000139670**	**0.000139683**	**1.2345640E-08**
3	**27338.632605402**	**27338.632605402**	**0.00000E+00**	**0.000139670**	**0.000139683**	**1.2345641E-08**
4	**27338.632605402**	**27338.632605402**	**0.00000E+00**	**0.000139670**	**0.000139683**	**1.2345641E-08**
5	**27338.632605402**	**27338.632605402**	**0.00000E+00**	**0.000139670**	**0.000139683**	**1.2345641E-08**
6	**27338.632605402**	**27338.632605402**	**0.00000E+00**	**0.000139670**	**0.000139683**	**1.2345641E-08**

IV. CONCLUSION

Fixed Point Iteration (FPI) computation of the semi major axis and the nominal mean motion of an artificial satellite orbiting an oblate earth is presented. The algorithm and the flowchart, along with relevant analytical expressions are presented. A hypothetical artificial satellite orbiting an oblate earth was used as a case study. The results showed that the algorithm converged at the 2^{nd} iteration. Also, the oblate earth causes an increase in the semi major axis of the orbit when compared to the case of non-oblate earth. Additional effect is that the mean motion of the satellite orbiting an oblate earth is higher or faster than the nominal mean motion of the satellite.

REFERENCES

1. Tremaine, S., & Yavetz, T. D. (2014). Why do Earth satellites stay up?. *American Journal of Physics*, *82*(8), 769-777.
2. Arroyo, V., Cordero, A., & Torregrosa, J. R. (2011). Approximation of artificial satellites' preliminary orbits: the efficiency challenge. *Mathematical and Computer Modelling*, *54*(7-8), 1802-1807.
3. Carvalho, J. D. S., De Moraes, R. V., & Prado, A. F. B. A. (2010). Some orbital characteristics of lunar artificial satellites. *Celestial Mechanics and Dynamical Astronomy*, *108*(4), 371-388.
4. Cordeau, J. F., & Laporte, G. (2005). Maximizing the value of an earth observation satellite orbit. *Journal of the Operational Research Society*, *56*(8), 962-968.
5. Cakaj, S., Kamo, B., Lala, A., & Rakipi, A. (2014). The coverage analysis for low earth orbiting satellites at low elevation. *International Journal of Advanced Computer Science and Applications*, *5*(6).
6. Montenbruck, O., Gill, E., & Lutze, F. (2002). Satellite orbits: models, methods, and applications. *Appl. Mech. Rev.*, *55*(2), B27-B28.
7. Liu, X., Baoyin, H., & Ma, X. (2010). Five special types of orbits around Mars. *Journal of guidance, control, and dynamics*, *33*(4), 1294-1301.
8. Bianchessi, N., Cordeau, J. F., Desrosiers, J., Laporte, G., & Raymond, V. (2007). A heuristic for the multi-satellite, multi-orbit and

multi-user management of Earth observation satellites. *European Journal of Operational Research*, *177*(2), 750-762.

9. dos Santos Carvalho, J. P., de Moraes, R. V., & de Almeida Prado, A. F. B. (2013). Dynamics of Artificial Satellites around Europa. *Mathematical Problems in Engineering*.
10. Granvik, M., Vaubaillon, J., & Jedicke, R. (2012). The population of natural Earth satellites. *Icarus*, *218*(1), 262-277.
11. Byrne, C. (2014). Kepler's Laws of Planetary Motion.
12. Yajima, Y. (2013). Geometrical Approach to Kepler's Laws of Planetary Motion. *Creative Education*, *4*(8), 6-8.
13. Golden, L. M. (2013). Experiment 12 Kepler's Laws of Planetary Motion. In *Laboratory Experiments in Physics for Modern Astronomy* (pp. 291-317). Springer, New York, NY.
14. Benacka, J. (2014). On planetary motion—a way to solve the problem and a spreadsheet simulation. *European Journal of Physics*, *35*(4), 045016.
15. Stephenson, B. (2012). *Kepler's physical astronomy* (Vol. 13). Springer Science & Business Media.
16. Kuehn, K. (2014). *A Student's Guide Through the Great Physics Texts: Volume II: Space, Time and Motion*. Springer.
17. Urone, P. P., & Hinrichs, R. (2012). Satellites and Kepler's Laws: An Argument for Simplicity. *College Physics*.
18. Capderou, M. (2014). *Handbook of satellite orbits: From kepler to GPS*. Springer Science & Business.
19. Kaula, W. M. (2013). *Theory of satellite geodesy: applications of satellites to geodesy*. Courier Corporation.
20. Strauch, D. (2009). Central Potentials and the Kepler Problem. In *Classical Mechanics* (pp. 157-182). Springer, Berlin, Heidelberg.
21. Gingerich, O. (2011). Kepler and the Laws of Nature. *Perspectives on Science & Christian Faith*, *63*(1).
22. Kamal, A. A. (2011). Gravitation. In *1000 Solved Problems in Classical Physics* (pp. 189-233). Springer, Berlin, Heidelberg.
23. Sandwell, D., & Schubert, G. (2010). A contraction model for the flattening and equatorial ridge of Iapetus. *Icarus*, *210*(2), 817-822.
24. Burnett, D. G. (2008). Flat Earth and Amazons. *Nature*, *454*(7207), 942-943.
25. Johan, N. B., & Sani, N. B. (2013). Derivation of the Geoid-Ellipsoid (N) Separation Valve within campus.
26. Strahler, A. H. (2008). *Modern physical geography*. John Wiley & Sons.
27. Harris, P. L., & Koenig, M. A. (2006). Trust in testimony: How children learn about science and religion. *Child development*, *77*(3), 505-524..
28. Simoson, A. J. (2010). The Man Who Flattened the Earth by Mary Terrall. *The Mathematical Intelligencer*, *32*(4), 69-72.
29. Gomarasca, M. A. (2009). *Basics of geomatics*. Springer Science & Business Media.
30. Swinerd, G. (2009). Real Orbits. *How Spacecraft Fly: Spaceflight Without Formulae*, 1-20.
31. Srivastava, V. K., Ashutosh, A., Roopa, M. V., Ramakrishna, B. N., Pitchaimani, M., & Chandrasekhar, B. S. (2014). Spherical and oblate Earth conical shadow models for LEO satellites: Applications and comparisons with real time data and STK to IRS satellites. *Aerospace Science and Technology*, *33*(1), 135-144.
32. Zhang, K., Chan, K. H., & Liao, X. (2014). On precessing flow in an oblate spheroid of arbitrary eccentricity.
33. Bénard, P. (2014). An oblate‐spheroid geopotential approximation for global meteorology. *Quarterly Journal of the Royal Meteorological Society*, *140*(678), 170-184.
34. Bénard, P. (2014). An oblate‐spheroid geopotential approximation for global meteorology. *Quarterly Journal of the Royal Meteorological Society*, *140*(678), 170-184.
35. Busch, M. W., Ostro, S. J., Benner, L. A., Brozovic, M., Giorgini, J. D., Jao, J. S., ... & Brisken, W. (2011). Radar observations and the shape of near-Earth asteroid 2008 EV5. *Icarus*, *212*(2), 649-660.
36. Walker, H. F., & Ni, P. (2011). Anderson acceleration for fixed-point iterations. *SIAM Journal on Numerical Analysis*, *49*(4), 1715-1735.
37. Hale, E. T., Yin, W., & Zhang, Y. (2008). Fixed-point continuation for \ell_1-minimization: Methodology and convergence. *SIAM Journal on Optimization*, *19*(3), 1107-1130.
38. Kernane, T., & Raizah, Z. A. (2009). Fixed point iteration for estimating the parameters of extreme value distributions. *Communications in Statistics-Simulation and Computation*, *38*(10), 2161-2170.
39. Argyros, I. K., & Hilout, S. (2013). *Computational methods in nonlinear analysis: efficient algorithms, fixed point theory and applications*. World Scientific.