Successive Approximation For Solution of Two-Parameter Stochastic Differential Equations

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Abstract—In this paper we discuss the successive approximations for solutions of two-parameter stochastic differential equations under non-Lipschitz conditions. Some results in [1] have been improved.

Keywords—Two-parameter stochastic differential equation; Lipschitz condition; Successive approximation.

I. INTRODUCTION

Let $R_2^3 = [0, +\infty) \times [0, +\infty) \times [0, +\infty)$, $\partial R_2^3 = R_2 \times \{0\} \cup \{0\} \times R_2 \cup \{0\} \times R_2$ for $Z = (s, t) \in R_2^3$, $R_2 = [0, s] \times [0, t]$. But for fixed $T \in R_2$, we promise $R_\tau = [0, T] \times [0, T]$. In $R_2^3$, we introduce a partial order $\leq$ by writing $(s, t) \leq (s', t')$ when $s \leq s', t \leq t'$.

Let $(Q, F, \mathcal{P}, F_t)$ be a probability space with a family of sub-$\sigma$-algebras $F_t$, $z \in R_2^3$ satisfying the usual conditions $F_0 = F_1 = F_2$ in [2], and $B_z \in R_2^3$ be a two-parameter Brownian motion with $\partial B = 0$ (that is $B(z) = 0$, $z \in \partial B_\tau$) on $(Q, F, \mathcal{P}, F_t)$ consider the non-Markovian stochastic differential system in plane.

\[\begin{array}{l}
\frac{dX(z)}{dt} = \alpha(z, X(z))dz + \beta(z, X(z))dB(z) \\
X(z) = \varphi(z)
\end{array}\]  \hspace{1cm} (I)

Under the assumption that the coefficients $\alpha$ and $\beta$ satisfy a Lipschitz condition and a growth condition,[1] established that equation (I) has unique solution, moreover the solution can be approximated successively. It is well known that there are some examples where the coefficients do not satisfy the Lipschitz condition. Yet we can prove the existence of solutions [3],[4]. But in those cases, it is not known whether the solutions can be approximated successively or not.

In the present paper we propose a condition weaker than Lipschitz’s which guaranteed the successive approximation of solutions. The method in this paper is different from [1].

II. MAIN RESULTS

Through out this paper, we denote $m(R_\tau) = s \cdot t$ for $z = (s, t) \in R_2^3$, which is Lebesgue measure of $R_\tau$.

Theorem let $\varphi(z)$, $z \in R_2^3$ be a continuous $F_\tau$ adapted two-parameter stochastic process on $(Q, F, \mathcal{P}, F_t)$, for every $T > 0$, $E\sup_{z \in R_\tau} |\varphi(z)|^2 < \infty$.

$\alpha, \beta$ are continuous on $R_2^3 \times R$ and satisfy the following conditions.

\[\begin{array}{l}
|\alpha(z, x) - \alpha(z, y)|^2 \leq \psi_1 \rho(|x - y|^2) \\
|\beta(z, x) - \beta(z, y)|^2 \leq \psi_2 \kappa(|x - y|^2)
\end{array}\]  \hspace{1cm} (C_1)

Where $\psi_1, \psi_2, \kappa, \rho$ are defined on $[0, +\infty)$, continuous non-decreasing and concave function with $\rho(0) = \kappa(0) = 0$, and

\[\frac{du}{\rho(u) + \kappa(u)} = \infty\]  \hspace{1cm} (C_2)

$\psi_1(z), \psi_2(z)$ are locally bounded functions on $R_2^3$.

Then there exists unique solution $X(z)$, $z \in R_2^3$ of equation (I) and for every $T > 0$

\[\lim_{k \to \infty} E\sup_{z \in R_\tau} |X^{(k)}(z) - X(z)|^2 = 0\]  \hspace{1cm} (2.1)

Where

\[X^{(0)}(z) = \varphi(z)\]

\[X^{(k+1)}(z) = \varphi(z) + \int_{[0, z]} \alpha(\xi, X^{(k)}(\xi))d\xi + \int_{[0, z]} \beta(\xi, X^{(k)}(\xi))dB(\xi)\]  \hspace{1cm} (2.2)

In order to prove this theorem we need the following lemma.
Lemma Let $X^{(m)}(z)$, $m = 0, 1, 2 \ldots$ be the successive approximations defined by (2.2), under the conditions of the theorem, for every $T > 0$
\[ E \sup_{\xi \in \mathcal{E}} \left| X^{(m)}(\xi) \right|^2 \leq C(T) \quad m = 0, 1, 2, \ldots \quad (2.3) \]

Where $C(T)$ is a positive constant depending on $T$ only.

Proof First, since $\rho$, $\kappa$ non-negative concave function with $\rho(0) = \kappa(0) = 0$. $\alpha$, $\beta$ are continuous in $(z, x)$. We know that condition $(C_1)$ implies the growth condition $(\rho, \kappa)$ are dominated by a linear function
\[ |\alpha(z, x)|^2 + |\beta(z, x)|^2 \leq T_1(1 + |x|^2) \quad z \in R_T \quad (2.4) \]

Where $T_1$ is a constant depending on $T$.

By the definition of $X^{(m)}(z)$, for any $z \in R_T$
\[ E \sup_{\xi \in \mathcal{F}} \left| X^{(m)}(\xi) \right|^2 \leq 3E \sup_{\xi \in \mathcal{F}} \left| \varphi(\xi) \right|^2 + 3E \sup_{\xi \in \mathcal{F}} \left[ \int_{\mathcal{R}} \alpha(\eta, X^{(m-1)}(\eta))d\eta \right]^2 
+ 3E \sup_{\xi \in \mathcal{F}} \left[ \int_{\mathcal{R}} \beta(\eta, X^{(m-1)}(\eta))d\eta \right]^2 \quad (2.5) \]

Utilizing Cauchy inequality and Doob-Carori inequality [2] the right side of (2.5)
\[ \leq 3E \sup_{\xi \in \mathcal{F}} \left| \varphi(\xi) \right|^2 + m(R_T)E \left[ \int_{\mathcal{R}} \alpha(\eta, X^{(m-1)}(\eta))d\eta \right]^2 d\eta \]
\[ + 4E \left[ \int_{\mathcal{R}} \beta(\eta, X^{(m-1)}(\eta))d\eta \right]^2 dB(\eta) \leq 3E \left( \sup_{\xi \in \mathcal{F}} \left| \varphi(\xi) \right| \right)^2 
+ T_1 \int_{\mathcal{R}} E \left[ \alpha^2(\eta, X^{(m-1)}(\eta)) \right] d\eta 
+ T_2 \int_{\mathcal{F}} \int_{\mathcal{R}} E \left[ \beta^2(\eta, X^{(m-1)}(\eta)) \right] d\eta \]
\[ \leq 3E \sup_{\xi \in \mathcal{F}} \left| \varphi(\xi) \right|^2 + T_1 \cdot T_2 \int_{\mathcal{R}} \left[ 1 + \left| X^{(m-1)}(\eta) \right|^2 \right] d\eta \]
\[ \leq 3E \sup_{\xi \in \mathcal{F}} \left| \varphi(\xi) \right|^2 + T_1 \int_{\mathcal{F}} \left[ 1 + E \sup_{\xi \in \mathcal{F}} \left| X^{(m-1)}(\xi) \right|^2 \right] d\eta \]

Where $T_2 = \max \left\{ 48, 3T^2 \right\}$, $T_1 = T_1 \cdot T_2$

Now we denote
\[ h(T) = 3E \sup_{\xi \in \mathcal{F}} \left| \varphi(\xi) \right|^2 \quad A^{(m)}(z) = E \sup_{\xi \in \mathcal{F}} \left| X^{(m)}(\xi) \right|^2 \]

Then
\[ A^{(m)}(z) \leq h(T) + T_3 \int_{R_T} \left[ 1 + A^{(m-1)}(\eta) \right] d\eta \]
\[ m = 0, 1, 2, \ldots \quad z \in R_T \quad (2.6) \]

We shall show that
\[ A^{(m)}(z) \leq h(T) + (1 + h(T)) \sum_{k=1}^{m} \frac{T_3^k}{(k!)^2} \left[ m(R_T) \right]^k \]
\[ m = 0, 1, 2, \ldots \quad z \in R_T \quad (2.7) \]

holds by induction.

By the definition of $A^{(m)}(z)$, (2.7) holds for $m = 0$. Assume that (2.7) holds for $m$, then we have for $m+1$ and $z = (s, t) \in R_T$
\[ A^{(m+1)}(z) \leq h(T) + T_3 \int_{R_T} \left[ 1 + A^{(m)}(\eta) \right] d\eta \]
\[ \leq h(T) + T_3 \int_{R_T} \left[ 1 + h(T) + (1 + h(T)) \sum_{k=1}^{m} \frac{T_3^k}{(k!)^2} \left[ m(R_T) \right]^k \right] d\eta \]
\[ = h(T) + (1 + h(T))T_3 m(R_T) + T_3 (1 + h(T)) \sum_{k=1}^{m} \frac{T_3^k}{(k!)^2} \left[ m(R_T) \right]^k \]
\[ = h(T) + (1 + h(T))T_3 m(R_T) + (1 + h(T)) \sum_{k=1}^{m} \frac{T_3^k}{(k!)^2} \left[ m(R_T) \right]^k \]
\[ = h(T) + (1 + h(T)) \sum_{k=1}^{m+1} \frac{T_3^k}{(k!)^2} \left[ m(R_T) \right]^k \]
\[ = h(T) + (1 + h(T)) \sum_{k=1}^{m+1} \frac{T_3^k}{(k!)^2} \left[ m(R_T) \right]^k \]

Thus (2.7) holds for all $m$.

Now we have
\[ A^{(m)}(z) \leq h(T) + (1 + h(T)) \sum_{k=1}^{m} \frac{T_3^k}{(k!)^2} \left[ m(R_T) \right]^k \]
\[ \leq (1 + h(T)) \sum_{k=0}^{m} \frac{T_3^k}{(k!)^2} T^{2k} \leq (1 + h(T)) \sum_{k=0}^{m} \frac{(T_3 T^2)^k}{k!} \leq C(T) \]

where $C(T) = (1 + h(T)) e^{T_3 T^2}$

Take $z = (T, T)$ then the lemma is obtained.
The proof of the theorem.

Let $L_1(T), L_2(T)$ be the bounds of $\psi_1, \psi_2$ on $R$. For $z \in R$, by the definition of $X^{(n)}(z)$ and lemma, we have

$$E[\sup_{\xi \in R_z} |X^{(n)}(\xi) - X^{(m)}(\xi)|^2] \leq 2I_1 + 2I_2$$

Where

$$I_1 = E \sup_{\xi \in R_z} \left\{ \int_{R_z} [\alpha(\eta, X^{(n-1)}(\eta)) - \alpha(\eta, X^{(m-1)}(\eta))]d\eta \right\}^2$$

$$\leq E(\sup_{\xi \in R_z} [\alpha(\eta, X^{(n-1)}(\eta)) - \alpha(\eta, X^{(m-1)}(\eta))])^2 d\eta$$

$$\leq T^2 E \int_{R_z} L_1(T) \rho(\xi | X^{(n-1)}(\xi) - X^{(m-1)}(\xi)|^2 d\eta$$

$$\leq T^2 L_1(T) E \rho(\sup_{\xi \in R_z} [\alpha(\eta, X^{(n-1)}(\eta)) - \alpha(\eta, X^{(m-1)}(\eta))])^2 d\eta$$

$$I_2 = E \sup_{\xi \in R_z} \left\{ \int_{R_z} [\beta(\eta, X^{(n-1)}(\eta)) - \beta(\eta, X^{(m-1)}(\eta))]d\eta \right\}^2$$

$$\leq 16 \int_{R_z} E \left\{ \beta(\eta, X^{(n-1)}(\eta)) - \beta(\eta, X^{(m-1)}(\eta)) \right\}^2 d\eta$$

$$\leq 16L_2(T) \int_{R_z} \kappa(E \sup_{\xi \in R_z} [\beta(\eta, X^{(n-1)}(\eta)) - \beta(\eta, X^{(m-1)}(\eta))]^2 d\eta$$

Let

$$H(u) = \rho(u) + \kappa(u)$$

Then $H$ is non-decreasing and continuous. Utilizing Fatou lemma and Lemma in this paper, for a fixed natural number $k$

$$\sup_{m,n,k} E \sup_{\xi \in R_z} |X^{(n)}(\xi) - X^{(m)}(\xi)|^2$$

$$\leq T_4 \int_{R_z} H(\sup_{m,n,k} E \sup_{\xi \in R_z} [X^{(n-1)}(\xi) - X^{(m-1)}(\xi)]^2 d\eta$$

Where

$$T_4 = \max \left\{ T^2 L_1(T), 16L_2(T) \right\}$$

Denote

$$V_k(z) = \sup_{m,n,k} E \sup_{\xi \in R_z} |X^{(n)}(\xi) - X^{(m)}(\xi)|^2$$

then $V_k(z)$ is non-increasing about $k$.

Thus there exists function $V(z)$ defined on $R_z$ such that

$$V_k(z) \rightarrow V(z) \quad (as \ k \rightarrow \infty, \ z \in R)$$

By (2.8), it is clear that

$$V_k(z) \leq T_4 \int_{R_z} H(V_{k-1}(\eta))d\eta \quad k \geq 1$$

Let $k \rightarrow \infty$, using the dominated convergence theorem we have

$$V(z) \leq T_4 \int_{R_z} H(V(\eta))d\eta \quad k \geq 1$$

By the condition $(C_3)$ and Lemma 2.3 in [5],

$$V(z) = 0, \ z \in R, \ then \ V_k(z) \rightarrow 0, \ (as \ k \rightarrow \infty, \ z \in R)$$

Which is

$$\lim \sup_{k \rightarrow n} \sup_{\xi \in R_z} |X^{(n)}(\xi) - X^{(m)}(\xi)|^2 = 0 \quad z \in R$$

Take $z = (T, T),$ we obtain

$$E \sup_{\xi \in R_z} |X^{(n)}(\xi) - X^{(m)}(\xi)|^2 \rightarrow 0 \quad (as \ m, n \rightarrow \infty)$$

Then there exists a square integrable process $X(z)$ such that [1]

$$E \sup_{\xi \in R_z} |X^{(n)}(\xi) - X^{(m)}(\xi)|^2 \rightarrow 0 \quad (as \ n \rightarrow \infty)$$

If we put

$$Y_n = \sup_{z \in R_t} |X^{(n)}(z) - X(z)|^2$$

Then we have

$$EY_n \rightarrow 0 \quad (as \ n \rightarrow \infty)$$

It follows that there exists $\{n_k\} \subset \{n\}$ such that

$$Y_{n_k} \rightarrow 0 \quad a.s \ (as \ k \rightarrow \infty)$$

Therefore $X^{(n_k)}(z) \rightarrow X(z)$ uniformly in $z \in R$, as $k \rightarrow \infty$. Since $X^{(n)}(z)$ is continuous and $F_\xi$ adapted, then so is $x(z)$.

Now we rove that $X(z)$ is the solution of equation (l). It is sufficient to show that for $z \in R$ we have

$$E \left[ \psi(z) + \int_{R_z} \alpha(\xi, X(\xi))d\xi + \int_{R_z} \beta(\xi, X(\xi))dB(\xi) \right] - X(z) = 0$$

The left side of (2.16)

$$\leq 2E \left[ X^{(n-1)}(z) - X(z) \right]^2 + 2E \left[ X^{(n)}(z) - \psi(z) - \int_{R_z} \alpha(\xi, X(\xi))d\xi - \int_{R_z} \beta(\xi, X(\xi))dB(\xi) \right]^2$$

$$= 2J_1^{(n)} + 2J_2^{(n)}$$

where

$$J_1^{(n)} = \sup_{m,n,k} E \sup_{\xi \in R_z} |X^{(n)}(\xi) - X^{(m)}(\xi)|^2$$

$$J_2^{(n)} = \sup_{m,n,k} E \sup_{\xi \in R_z} |X^{(n)}(\xi) - X^{(m)}(\xi)|^2$$

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From (2.15), $J_{1}^{(n)} \to 0 \ (as \ n \to \infty)$

\[
J_{1}^{(n)} = E \left[ \int_{\mathbb{R}} \left[ \alpha(\xi, X(\xi)) - \alpha(\xi, X^{(n)}(\xi)) \right] d\xi \right] + E \left[ \int_{\mathbb{R}} \left[ \beta(\xi, X(\xi)) - \beta(\xi, X^{(n)}(\xi)) \right] dB(\xi) \right]
\leq 2T^{2} L_{1}(T) \int_{\mathbb{R}} \rho(E \left| X(\xi) - X^{(n)}(\xi) \right|^{2} d\xi)
+ 2L_{2}(T) \int_{\mathbb{R}} \kappa(E \left| X(\xi) - X^{(n)}(\xi) \right|^{2} d\xi)
\]

By Fatou lemma $J_{1+2}^{(n)} \to 0 \ (as \ n \to \infty)$. Therefore $X(\xi)$ is a solution of equation (I).

Finally we prove the uniqueness for (I)

Let $X'(\xi), X''(\xi)$ be two solutions of (I), and $T$ be an arbitrary positive number, $\xi \in R_T$ then

\[
E \left| X'(\xi) - X''(\xi) \right|^{2} \leq 2E \left[ \int_{\mathbb{R}} \left[ \alpha(\xi, X'(\xi)) - \alpha(\xi, X''(\xi)) \right] d\xi \right]^{2}
+ 2E \left[ \int_{\mathbb{R}} \left[ \beta(\xi, X'(\xi)) - \beta(\xi, X''(\xi)) \right] d\xi \right]^{2}
\leq 2T^{2} \int_{\mathbb{R}} E \left| \alpha(\xi, X'(\xi)) - \alpha(\xi, X''(\xi)) \right|^{2} d\xi
+ 2 \int_{\mathbb{R}} E \left| \beta(\xi, X'(\xi)) - \alpha(\xi, X''(\xi)) \right|^{2} d\xi
\]

\[
\leq T \int_{\mathbb{R}} \left[ \rho(E \left| X'(\xi) - X''(\xi) \right|^{2} d\xi \right] + \kappa(E \left| X'(\xi) - X''(\xi) \right|^{2} d\xi)
\]

where

\[
T_{3} = \max \left\{ 2T^{2} L_{1}(T), 2L_{2}(T) \right\}
\]

By Lemma 2.3 in [5]

\[
E \left| X'(\xi) - X''(\xi) \right|^{2} = 0, \quad \xi \in R_T
\]

Since $T$ is arbitrary we obtain $X'(\xi) = X''(\xi) \quad a.s$

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