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# Simple measurement of DOA and exponential decay rate for a class of uncertain nonlinear discrete-time systems

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***Abstract*—In this paper, the exponential stability for a class of uncertain nonlinear discrete-time systems is investigated. Based on the time-domain approach, a simple criterion is developed to ensure the exponential stability of such systems. An estimate of the exponential decay rate of such stable systems is also derived. Besides, the domain of attraction (DOA) of such stable systems can be accurately calculated. Finally, several numerical simulations are offered to verify the feasibility and correctness of the obtained results.**

***Keywords*—Discrete-time system; uncertain systems, exponential stability; domain of attraction**

## I. Introduction

In recent years, various issues related to discrete systems, such as stability sub-system and controller design, have been investigated and discussed by experts and scholars. In addition, various analysis methods and design methodologies have been developed for discrete systems; see, for example, [1]-[7] and the references therein.

On the other hand, due to the inaccuracy of the system model or the difficulty of estimating the system parameters, it is necessary to use nonlinear systems as analysis models. At this time, the analysis and design of the uncertain nonlinear system become more critical.

A better local stability analysis provides a criterion to ensure stability, as well as the scope of DOA. As we know, the DOA calculation of a locally stable system has always been a problem that researchers urgently need to overcome; see, for example, [8]-[11] and the references therein.

In this paper, the exponential stability for a class of uncertain nonlinear discrete-time systems is studied. Based on the time-domain approach, a simple stability criterion is developed to guarantee the exponential stability of such uncertain systems. An estimate of the exponential decay rate of such stable systems is also derived. Furthermore, the domain of attraction of such stable systems can be precisely calculated. Finally, some numerical simulations are given to demonstrate the feasibility and correctness of the obtained results.

## II. Problem formulation and main results

Nomenclature

| | |
|---|---|
| $\Re^{n\times 1}$ | the $n$-dimensional real space |
| $\|x\|$ | the Euclidean norm of the vector $x \in \Re^{n\times 1}$ |

Consider the following uncertain nonlinear discrete-time systems:

$$x(k+1) = \Delta f(x(k)), \quad \forall\, k \in Z^+, \tag{1a}$$

$$x(0) = \phi, \tag{1b}$$

where $x \in \Re^{n\times 1}$ is state vector, $\Delta f$ is uncertain vector-valued function, and $\phi$ represents the initial condition. Without loss of generality, we assume that $x = 0$ is the equilibrium point of uncertain nonlinear system (1).

The domain of attraction and local exponential stability of the exponential decay rate of the system (1) are defined as follows.

**Definition 1:**

The origin of the system (1) is exponentially stable if there exist two positive numbers $\alpha$ and $\gamma$, with $0 < \alpha < 1$, such that

$$\|x(k)\| \le \|x(0)\| \cdot \alpha^k, \quad \forall\, k \in Z^+ \text{ and } \|\phi\| \le \gamma .$$

In this case, the positive number $\alpha$ and the set of $\left\{x \in \Re^{n\times 1} \,\middle|\, \|x\| \le \gamma\right\}$ are often called the exponential decay rate and domain of attraction, respectively.

Now, we make the following assumption.

(A1) There exist $p \in N$ and nonnegative constants $a_1, a_2, \cdots, a_p$ such that $\|\Delta f(x)\| \le \sum_{i=1}^{p} a_i \|x\|^i$, with $\sum_{i=2}^{p} a_i^2 > 0$.

Now we present a simple criterion to guarantee the exponential stability of the uncertain nonlinear discrete-time systems of (1).

**Theorem 1:** The origin of the uncertain system (1) with (A1) is exponentially stable provided that

$$a_1 < 1. \tag{2}$$

In this situation, the domain of attraction and exponential decay rate are $\left\{x \in \Re^{n\times 1} \middle| \|x\| \le \gamma := \gamma^* - \varepsilon\right\}$ and $\alpha = g(\gamma)+1$, respectively, where $\gamma^*$ is the unique positive zero of the following function

$$g(x) := -1 + \sum_{i=1}^{p-1} a_i x^{i-1}, \quad x \ge 0, \tag{3}$$

and $\varepsilon$ is any positive number with $\varepsilon < \gamma^*$.

**Proof:** Clearly, the polynomial $g(x)$ is increasing function for every $x \ge 0$, in view of (A1). In addition, by Descartes' rule of signs [19] in (3) with (2), it is obvious that the polynomial equation $g(x)=0$ has a unique positive root, denoted as $\gamma^*$. Moreover, it is easy to see that

$$-1 < g(0) < 0,\ g(x) < 0,\ \forall\, x \in [0, \gamma^*), \tag{4a}$$

$$g(\gamma^*) = 0,\ g(x) > 0,\ \forall\, x \in (\gamma^*, \infty). \tag{4b}$$

This implies that $0 \le a_1 = g(0)+1 \le g(\gamma)+1 < g(\gamma^*)+1 = 1$. From (1) and (4) with (A1), it is easy to see that

$$\begin{aligned} \|x(k+1)\| &= \|\Delta f(x(k))\| \\ &\le \sum_{i=1}^{p} a_i \|x(k)\|^i \\ &\le \left[\sum_{i=1}^{p} a_i \|x(k)\|^{i-1}\right] \cdot \|x(k)\| \\ &\le [g(\|x(k)\|)+1] \cdot \|x(k)\| \\ &\le [g(\gamma)+1] \cdot \|x(k)\|, \quad \forall\, \|x(k)\| \le \gamma. \end{aligned}$$

This implies

$$\begin{aligned} &\|x(1)\| \le [g(\gamma)+1] \cdot \|x(0)\|, \quad \forall\, \|x(0)\| \le \gamma; \\ &\|x(2)\| \le [g(\gamma)+1] \cdot \|x(1)\| \le [g(\gamma)+1]^2 \cdot \|x(0)\|, \\ &\qquad \forall\, \|x(0)\| \le \gamma; \\ &\|x(3)\| \le [g(\gamma)+1] \cdot \|x(2)\| \le [g(\gamma)+1]^3 \cdot \|x(0)\|, \\ &\qquad \forall\, \|x(0)\| \le \gamma; \\ &\qquad \vdots \\ &\qquad \vdots \end{aligned}$$

Consequently, we conclude that

$\|x(k)\| \le [g(\gamma)+1]^k \cdot \|x(0)\|,\ \forall\, k \in Z^+$ and $\|x(0)\| \le \gamma$.

This completes the proof. □

## III. Numerical Simulations

Consider the following uncertain discrete-time system:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \Delta f(x(k)) = \begin{bmatrix} \Delta b_1 x_2(k) + \Delta b_2 x_1(k) x_2(k) + \Delta b_3 x_2^3(k) \\ \Delta b_4 x_1(k) + \Delta b_5 x_1^3(k) + \Delta b_6 x_1^2(k) x_2^2(k) \end{bmatrix} \tag{5}$$

where $x(k) := \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \in \Re^{2\times 1}$ and $\Delta b_i$'s, $\forall\, i \in \underline{6}$, are the uncertain parameters with

$$|\Delta b_1| \le 0.2,\quad |\Delta b_2| \le 0.2,\quad |\Delta b_3| \le 0.06, \tag{6a}$$

$$|\Delta b_4| \le 0.2,\quad |\Delta b_5| \le 0.06,\quad |\Delta b_6| \le 0.1, \tag{6b}$$

Comparison of (5) with (1) and (A1), one has $a_4 = 0.05$, $a_3 = 0.06$, $a_2 = 0.1$, and $a_1 = 0.2$. By (3), we obtain

$$g(x) = 0.05x^3 + 0.06x^2 + 0.1x - 0.8.$$

Meanwhile, the unique positive solution of $g(x)=0$ is given by $\gamma^* = 1.9567$. Consequently, by Theorem 1 with the choice $\varepsilon = 0.0567$, the uncertain system (5) with (6) is locally exponentially stable with the guaranteed DOA of $\left\{x \in \Re^{n\times 1} \middle| \|x\| \le \gamma := \gamma^* - \varepsilon = 1.9\right\}$. Besides, the guaranteed exponential decay rate is given by $\alpha = g(1.9)+1 = 0.9495$. Some state trajectories of the uncertain system (5) are depicted in Figures 1-3. It is known from the aforementioned simulation figures that when the initial values are within DOA, the uncertain system (5) with (6) is indeed an exponentially stable system.

## Conclusions

In this paper, the exponential stability for a class of uncertain nonlinear discrete-time systems has been explored. Based on the time-domain approach, a simple criterion has been developed to ensure the exponential stability of such systems. An estimate of the exponential decay rate of such stable systems has also been derived. Meanwhile, the DOA of such stable systems can be accurately calculated. Finally, several numerical simulations have been presented to verify the feasibility and correctness of the obtained results.

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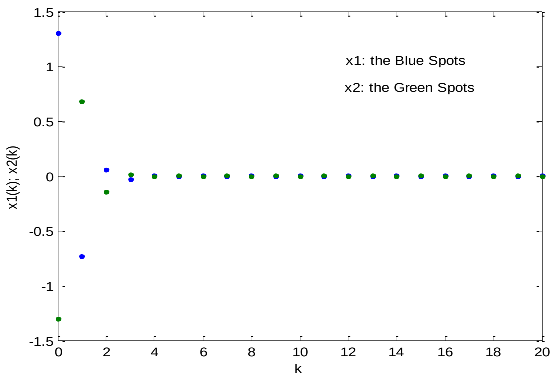


**Figure 1:** Typical state trajectories of system (5)-(6) with $x(0)=[1.3 \quad -1.3]^T$.

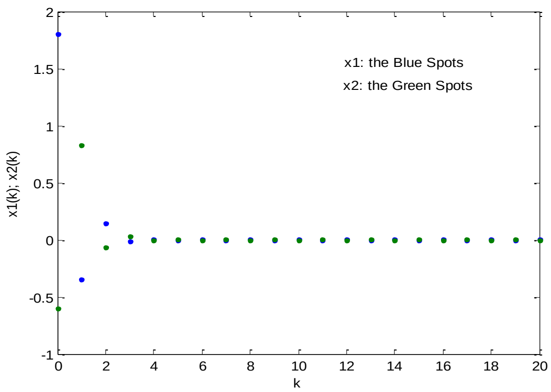


**Figure 2:** Typical state trajectories of system (5)-(6) with $x(0)=[1.8 \quad -0.6]^T$.

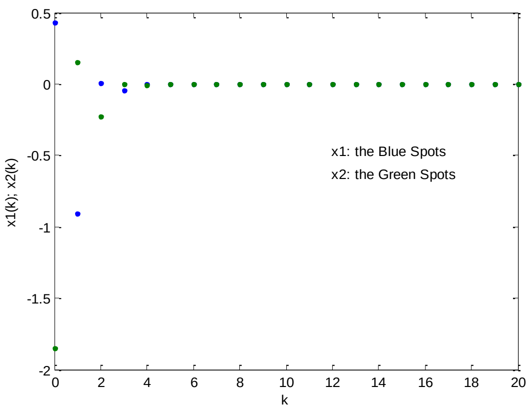


**Figure 3:** Typical state trajectories of system (5)-(6) with $x(0)=[0.43 \quad -1.85]^T$.

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