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Remarks For A Hyperbolic Wave Equation Of Kirchhoff Type In Unbounded Domains

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***Abstract*—We discuss the asymptotic behavior of solutions for the nonlocal quasilinear hyperbolic problem of Kirchhoff type $u_{tt}-\phi(x)\|\nabla u(t)\|^2\Delta u+\delta u_t=|u|^3u,\ x\in R^N$, $t\geq 0$, with initial conditions $u(x,0)=u_0(x)$ and $u_t(x,0)=u_1(x)$, in the case where $N\geq 3, \delta>0$ and $(\phi(x))^{-1}=g(x)$ is a positive function lying in $L^{N/2}(R^N)\bigcap L^\infty(R^N)$. It is proved that when the initial energy $E(u_0,u_1)$ which corresponds to the problem, is non-negative and small, there exists a unique global solution in time in the space $X_0=:D(A)\times D^{1,2}(R^N)$. When the initial energy $E(u_0,u_1)$ is negative, the solution blows-up in finite time. For the proofs, a combination of the modified potential well method and the concavity method is used. Also, the existence of an absorbing set in the space $X_1=:D^{1,2}(R^N)\times L_g^2(R^N)$ is proved and that the dynamical system generated by the problem possess an invariant compact set $\mathcal{A}$ in the same space.**

Finally, for the generalized Kirchhoff's String problem with no dissipation

$$u_{tt}=-\|A^{1/2}u\|_H^2\,Au+f(u), x\in R^N, t\geq 0,$$

with the same hypotheses as above, we study the stability of the trivial solution $u\equiv 0$. It is proved that if $f'(0)>0$, then the solution is unstable for the initial Kirchhoff's system, while if $f'(0)<0$ the solution is asymptotically stable.

Keywords— Quasilinear Hyperbolic Equations, Global Solution, Dissipation, Potential Well, Cocavity method, Unbounded Domains, Kirchhoff Strings, Generalized Sobolev Spaces, Weighted L^p spaces.

I. Introduction-Preliminaries

We study the following quasilinear hyperbolic initial value problem

(1.1) $u_{tt}-\phi(x)\|\nabla u(t)\|^2\Delta u+\delta u_t-|u|^3u=0$,

(1.2) $u(x,0)=u_0(x), \quad u_t(x,0)=u_1(x)$

$x\in R^N, t\geq 0$, with initial conditions u_0, u_1 in appropriate function spaces, $N\geq 3$, and $\delta\geq 0$. Throughout the paper we assume that the function ϕ and $g: R^N\to R$ satisfy the following condition:

(G) $\phi(x)>0$, for all $x\in R^N$ and $(\phi(x))^{-1}=g(x)\in L^{N/2}(R^N)\bigcap L^\infty(R^N)$.

This class will include functions of the form $\phi(x)\sim c_0+\varepsilon|x|^a$, $\varepsilon>0$ and $a>0$, resembling phenomena of slowly varying wave speed around the constant speed c_0.

G. Kirchhoff in 1883 proposed the so called Kirchhoff string model in the study of oscillations of stretched strings and plates

$$ph\frac{\partial^2 u}{\partial t^2}+\delta\frac{\partial u}{\partial t}=\left\{p_0+\frac{Eh}{2L}\int_0^L(\frac{\partial u}{\partial x})^2\right\}\frac{\partial^2 u}{\partial x^2}+f \quad \text{for } 0<x<L,\ t\geq 0,$$

where $u = u(x, t)$ is the lateral displacement at the space coordinate x and the time t, E the Young modulus, p the mass density, h the cross-section area, L the length, p_0 the initial axial tension, δ the resistance modulus and f the external force (see [7]). When $p_0=0$ the equation is considered to be of *degenerate type,* otherwise it is of nondegenerate type.

In the case of bounded domain, T. Kobayashi [8] constructed a unique weak solution by a Faedo-Galerkin method for a quasilinear wave equation with strong dissipation (see also [1, 10]). K. Nishihara [11], has derived a decay estimate from below of the potential of solutions. Also R. Ikehata [4], has shown that for sufficiently small initial data, global existence can be obtained, even when the influence of the source terms is stronger than that of the damping terms. Finally K. Ono [12] for $\delta\geq 0$, has proved global existence and blow up results for a degenerate non-linear wave equation of Kirchhoff type with strong dissipation.

In the case of unbounded domain, P. D'Ancona and S. Spagnolo [2] have shown the global existence of a unique C^∞ solution for the non-degenerate type with small C_0^∞ data. N. Karahalios and N. Stavrakakis (see [5],[6]), have proved global existence and blow-up results for some semilinear wave equations with variable wave speed on all R^N. T Mizumachi (see [9]), studied the asymptotic behavior of solutions to the Kirchhoff equation with a viscous damping term with no external force. In our previous work (see [13]), we prove global existence and blow-up results of an equation of Kirchhoff type in all of R^N. Also, in [14] we prove the existence of compact invariant sets for the same equation. Finally, in [15] we study the stability of the trivial solution u = 0 for the generalized Kirchhoff's string equation, using the central manifold theory.

As we will see, the space setting for the initial conditions and the solutions of our problem is the product space $X_0 =: D(A) \times D^{1,2}(R^N)$.

By $D^{1,2}(R^N)$ we define the closure of the $C_0^\infty(R^N)$ functions with respect to the *energy norm* $\|u\|_{D^{1,2}} =: \int_{R^N} |\nabla u|^2 \, dx$. It is known that

$$D^{1,2}(R^N) = \left\{ u \in L^{\frac{2N}{N-2}}(R^N) : \nabla u \in (L^2(R^N))^N \right\}$$

The weighted Lebesque space $L_g^2(R^N)$ is the closure $C_0^\infty(R^N)$ functions with respect to the inner product $(u,v)_{L_g^2(R^N)} =: \int_{R^N} guvdx$ (see [3]). We also have that the operator $A = -\phi\Delta$ is self-adjoint and therefore graph-closed. Its domain $D(A)$, is a Hilbert space with respect to the norm $\|Au\|_{L_g^2} = \left\{ \int_{R^N} \phi |\Delta u|^2 \, dx \right\}^{1/2}$. So, we construct the following evolution quartet, with compact and dense embeddings:

$$D(A) \subset D^{1,2}(R^N) \subset L_g^2(R^N) \subset D^{-1,2}(R^N).$$

For the positive selfadjoint operator $A = -\phi\Delta$, we may define the fractional powers in the following way. For every $s > 0, A^s$ is an unbounded selfadjoint operator in $L_g^2(R^N)$ with its domain $D(A^s)$ to be a dense subset in $L_g^2(R^N)$. The operator A^s is strictly positive and injective. Also $D(A^s)$, endowned with the scalar product $(u,v)_{D(A^s)} = (u,v)_{L_g^2} + (A^s u, A^s v)_{L_g^2}$, becomes a Hilbert space. We write as usual $V_{2s} = D(A^s)$ and we have the following identifications $D(A^{-1/2}) = D^{-1,2}(R^N)$, $D(A^0) = L_g^2$, $D(A^{1/2}) = D^{1,2}(R^N)$. Moreover the mapping $A^{s/2} : V_x \to V_{x-s}$ is an isomorphism. Furthermore, we have that the injection $D(A^{s_1}) \subset D(A^{s_2})$ is compact and dense, for every $s_1, s_2 \in R, s_1 > s_2$.

In order to clarify the kind of solutions we are going to obtain for our problem, we give the definition of the weak solution for the problem.

Definition 1.1 *A* weak solution *of the problem* (*1.1*)-(*1.2*) *is a function* u *such that*

(i) $u \in L^2[0,T; D(A)], u_t \in L^2[0,T; D^{1,2}],$ $u_{tt} \in L^2[0,T; L_g^2],$

(ii) for all $v \in C_0^\infty\left([0,T] \times (R^N)\right)$, satisfies the generalized formula

(1.3) $$\int_0^T (u_{tt}(\tau), v(\tau))_{L_g^2} \, d\tau + \int_0^T \left(\|\nabla u(\tau)\|^2 \int_{R^N} \nabla u(\tau) \nabla v(\tau) dx \right) d\tau + \delta \int_0^T \left(u_t(\tau), v(\tau)\right)_{L_g^2} d\tau - \int_0^T f(u(\tau), v(\tau))_{L_g^2} \, d\tau = 0$$

where $f(s) = |s|^3 s$, and (iii) satisfies the initial conditions

$$u(x,0) = u_0(x) \in D(A), u_t(x,0) = u_1(x) \in D^{1,2}(R^N).$$

In the following section we briefly discuss the results concerning the asymptotic behavior of solutions for the problem (1.1)-(1.2). Among the global existence and blow-up results we also prove existence of a compact functional invariant set. We would like to mention that up to our knowledge, this is the first result concerning existence of functional invariant sets for mathematical models of Kirchoff's strings type.

II. Global Existence, Blow Up Results and Invariant Sets

In this section we give global existence and blow-up results for the problem (1.1)-(1.2) in the space X_0. We also prove existence of an attractor like set. For the proofs we refer on [15], [16]. In order to obtain a local existence result for the problem (1.1)-(1.2), we need information concerning the solvability of the corresponding non-homogeneous linearized problem around the function v, where $(v, v_t) \in C(0,T; D(A) \times D^{1,2})$, is given restricted in the sphere B_R:

(2.1)

$u_{tt} - \phi(x) \| \nabla v \|^2 \Delta u + \delta u_t = |v|^3 v$,
$(x,t) \in B_R \times (0,T)$,
$u(x,0) = u_0(x), u_t(x,0) = u_1(x), x \in B_R$,
$u(x,t) = 0, (x,t) \in \partial B_R \times (0,T),\ v \in C(0,T; D(A))$
and $v_t \in C(0,T; D^{1,2})$

Proposition 2.1 Assume that $u_0 \in D(A), u_1 \in D^{1,2}(R^N)$ and $0 \le N \le 10/3$, then the linear wave equation (2.1) has a unique solution such that $u \in C(0,T;D(A)), u_t \in C(0,T;D^{1,2})$.

Proof. The proof follows the lines of [6, Proposition 3.1]. The Galerkin method is used, based on the information taken from the eigenvalue problem.

Next, we have the following theorem (for the proof see also [16]).

Theorem 2.2 If $(u_0, u_1) \in D(A) \times D^{1,2}$ and satisfy the non-degenerate condition $\| \nabla u_0 \|^2 > 0$, then there exists $T > 0$, such that the problem (1.1)-(1.2) admits a unique local weak solution u satisfying:
$u \in C(0,T;D(A)), u_t \in C(0,T;D^{1,2})$. Moreover, at least one of the following statements holds true, either

(i) $T = +\infty$, or

(ii) $e(u(t)) =: \| u_t \|^2_{D^{1,2}} + \| u \|^2_{D(A)} \to \infty$, as $t \to T_-$.

The next theorem deals with the global existence, blow-up results and the energy decay property of the problem.
First we define as the energy of the problem (1.1)-(1.2) the quantity

$$(2.2)\ E(t) =: E(u(t), u_t(t)) =: \| u(t) \|^2_{L^2_g} + \frac{1}{2} \| u(t) \|^4_{D^{1,2}} - \frac{2}{5} \| u(t) \|^5_{L^5_g}$$

Also, we introduce the potential of the problem (1.1)-(1.2), as

(2.3) $J(u) =: \frac{1}{2} \| u(t) \|^4_{D^{1,2}} - \frac{2}{5} \| u(t) \|^5_{L^5_g}$.

So, we get the following relation

(2.4) $E(t) = \| u_t(t) \|^2_{L^2_g} + J(u)$.

Finally, we introduce a modified version of the modified potential well used in [6] (see also [13]), by

(2.5)

$$W =: \left\{ u \in D(A); K(u) = \| u \|^4_{D^{1,2}} - \| u \|^5_{L^5_g} > 0 \right\} \cup \{0\}.$$

Theorem 2.3 Assume that $N = 3, u_0 \in W (\subset D(A))$ and $u_1 \in D^{1,2}$. Also suppose that the following inequality holds

(2.6) $E(u_0, u_1) \le \left(\frac{1}{C_0 \mu_0^{p_1}} \right)^{1/p_2}$ and $p_2 > 0$. Then **a)** for $p_1 =: \frac{1}{2}$ and $p_2 =: \frac{1}{8}$, there exists a unique global solution $u \in W$ of the problem (1.1)-(1.2) satisfying $u \in C([0,+\infty); D(A))$ and $u_t \in C([0,+\infty); D^{1,2})$.

b) Moreover, this solution obeys the following energy estimates

$$(2.7) \| u_t \|^2_{L^2_g} + d_*^{-1} \| \nabla u \|^4 \le E(u, u_t) \le \left\{ E(u_0, u_1)^{-1/2} + d_0^{-1} [t-1]^+ \right\},^{-2}$$

Where $d_* = 10$ and $d_0 \ge 1$, that is

(2.8) $\| \nabla u \|^4 \le C_*(1+t)^{-1}$,

where C_* is some constant depending on $\| u_0 \|^4_{D^{1,2}}$ and $\| u_1 \|_{L^2_g}$.

c) Suppose that $N \ge 3$ and the initial energy $E(u_0, u_1)$ is negative. Then there exists a time T, where

(2.9)

$$0 < T \le 3^{-2} (-E(u_0,u_1))^{-1} \left[\left\{ \left(2\delta \| u_0 \|^2_{L^2_g} - 3(u_0,u_1)_{L^2_g} \right)^2 + 9(-E(u_0,u_1)) \| u_0 \|^2_{L^2_g} \right\}^{1/2} + 2\delta \| u_0 \|^2_{L^2_g} - 3(u_0,u_1)_{L^2_g} \right],$$

such that the (unique) solution of the problem (1.1)-(1.2) blows-up at T, i.e,

(2.10) $\lim_{t \to T_-} \| u(t) \|^2_{L^2_g} = +\infty$.

The existence of an absorbing set in X_0 is given below. See also [14].

Lemma 2.4 Assume that $\rho_1 > 4 \cdot 3^{-1/2} R^2 c_3^2, N \ge 3$ and $\| \nabla u_0 \| > 0$ .Then the unique local solution defined by Theorem 2.1 exists globally in time.

Remark 2.5 (Global Solutions) From the last Lemma 2.4, we may observe that solutions of the problem (1.1)-(1.2),(given by Theorem 2.2), belong to the space $C_b(R_+, X_0)$, i.e., we have achieved global solutions for the given problem. Let us remark that, in the Theorem 2.3, using a modified potential well technique, we have proved global existence results under the condition $N = 3$ and the initial energy

$E(0)$ been non-negative and small. On the other hand, in Lemma 2.4, we could achieve global results for different type of nonlinearities, i.e. for any $N \geq 3$ and independently of the sign of the initial energy $E(0)$.

Lemma 2.4 has an immediate consequence:

Remark 2.6 A nonlinear semigroup $S(t): X_0 \to X_0, t \geq 0,$ may be associated to the problem (1.1)-(1.2) such that for $\psi = \{u_0, u_1\} \in X_0, S(t)\psi = \{u(t), u_t(t)\}$ is the weak solution of the problem (1.1)-(1.2). Moreover the ball $B_0 =: B_{X_0}(0, \overline{R}_*)$ for any $\overline{R}_* > R_*$, where R_* is defined by Lemma 2.4, is **an absorbing set** for the semigroup $S(t)$ in the energy space $X_0 \subset X_1$, compactly.

In the rest of the paper we show that the ω-limit set of the absorbing set B_0 is a compact invariant set. To this end, we need to decompose the semigroup $S(t)$, in the form $S(t) = S_1(t) + S_2(t)$, where for a suitable bounded set $B \subset X_0$, the semigroups $S_1(t), S_2(t)$ satisfy the following properties:

(S1) $S_1(t)$ is uniformly compact for t large, i.e., $\cup_{t \geq t_0} S_1(t)B$ is relatively compact in X_1.

(S2) $\sup_{k \in B} \| S_2(t)k \|_{X_1} \to 0$, as $t \to \infty$.

As a consequence of the above properties we have the following result

Theorem 2.7 Let ϕ satisfy hypothesis (G). Then the semigroup $S(t)$ associated with the problem (1.1)-(1.2) possesses a functional invariant set $A = \omega(B_0)$, which is compact in the weak topology of X_1.

Remark 2.8 We have that X_0 is compactly embedded in X_1, so the set $\cup_{t \geq t_0} S_1(t)B$ is compact with respect to the strong topology in X_1. For the functional invariant compact set $A = \omega(B_0)$, we observe that $(u_0, u_1) \in A$, if $|\nabla u_0| > 0$. So, A is **an attractor like set**.

Remark 2.9 The above set, $A = \omega(B_0)$ is a positively invariant set in the space X_0, because we have that $S(t)A \subset A$, from the definition of the absorbing set. This set is not invariant in the space X_0 because the semigroup S(t) is weakly continuous in X_0, (see the following Lemma), but it is not continuous in X_0.

At the end, we prove the following Lemma.

Lemma 2.10 For every $t \in R$, the mapping S(t) is weakly continuous from X_0 into X_0.

Proof Let $\{u^n\}$ be a weakly convergent sequence in X_0 and u its (weak) limit. We fix $t \in R$; we have that the sequence $\{S(t)u^n\}$ is bounded in X_0. We extract a subsequence $\{S(t)u^{n'}\}$ that converges weakly to $v \in X_0$. On the other hand, the compactness of the injection of X_0 into X_1 insures that $\{u^n\}$ converges strongly to u in X_1. Hence, $\{S(t)u^n\}$ converges strongly to $S(t)u$ in X_1 and then $v = S(t)u$ .Therefore, the whole sequence $\{S(t)u^n\}$ weakly converges to $S(t)u$ in X_0 and the lemma is proved.

At last, in the following section we study the stability of the initial solution $u = 0$ for the generalized Kirchhoff equation with no dissipation.

III. Stability Results

We consider the generalized quasilinear Kirchhoff's String problem with no dissipation

$$u_{tt} = -\| A^{1/2}u \|_H^2 \; Au + f(u), x \in R^N, t \geq 0,$$

under the same initial conditions as above and H is a Hilbert space. First, we prove existence of solution for our problem, under small initial data (for the proof we follow the lines of [15]).

Theorem3.1. (Local Existence) Let $f(u)$ a C^1- function such that $|f(u)| \leq k_1 |u|^{a+1}$, $|f'(u)| \leq k_2 |u|^a$, $0 \leq a \leq 4/N(-$, $N \geq 3$. Consider that $(u_0, u_1) \in D(A) \times V$ and satisfy the non-degenerate condition

(3.1) $\| A^{1/2}u_0 \| > 0.$

Then there exists $T_0 > 0$ such that our problem admits a unique local weak solution u satisfying $u \in C(0, T; V)$ and $u_t \in C(0, T; H)$.

Proof The proof follows the lines of [13, Theorem 3.2]. In this case, because of the compact embedding

$X_0 \subset X_1 =: V \times H$, we obtain for the associated norms that $e_1(u(t)) \le e(u(t))$, where $e_1(u(t)) = \| u \|_V^2 + \| u' \|_H^2$ and $e(u(t)) = \| u \|_{D(A)}^2 \times \| u' \|_V^2$. Following the same steps as in Theorem 3.2 we take the inequality $e_1(u(t)) \le e(u(t)) \le R^2$, where R is a positive parameter. So, u is a solution such that

$u \in L^\infty(0,T;V)$, $u' \in L^\infty(0,T;H)$.

The continuity properties, are also proved with the methods indicated in [18, sections II.3 and II.4]. Finally, the uniqueness of the solution can also be taken from [18, Proposition 4.1, p. 215].

Now, we have that the linearized equation of the system around the solution $u = 0$ is

(3.2) $\overline{u_t} + A^* \overline{u} = 0,$

where

(3.3) $\overline{u_t} = (w, v)^T$ and $A^* = \begin{bmatrix} 0 & -f'(0) \\ -1 & 0 \end{bmatrix}$.

So, in order to study the stability of the solution, we study the spectrum of the operator A^*. The characteristic polynomial of A^* is $\begin{vmatrix} \mu_j & f'(0) \\ 1 & \mu_j \end{vmatrix} = 0,$ or equivalently $\mu_j^2 - f'(0) = 0.$

Then according to the sign of $f'(0)$, we have the following cases (see also [19], Theorem 5.1.1 and Theorem 5.1.3):

I) Let $f'(0) > 0$, then we have that 0 is **unstable** for the initial Kirchhoff's system, because we have two real eigenvalues of different sign $\mu_j \pm = \pm (f'(0))^{1/2}$ and we can easily see that the continuous spectrum of the operator A^* is empty.

II) Let $f'(0) < 0$. This implies that the operator A^* admits two complex eigenvalues. Thus we obtain that the solution $u = 0$ is **asymptotically stable** for the initial Kirchhoff's system.

III) Let $f'(0) = 0$. In this case we have that the initial solution is **stable** using the fact that the continuous spectrum of the operator A^* is equal to zero.

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