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Positivity-preserving And Monotonicity-preserving Using Rational Fractal Interpolation Function

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***Abstract*—In this paper, the development process of rational fractal interpolation is simply explained. The traditional interpolation methods, such as polynomial interpolation, have great limitations, especially for highly irregular data or irregular derivatives. The emergence of rational fractal interpolation provides an effective geometric modeling method to solve this kind of problems. In this paper, a classic interpolation method based on the previous construction is introduced. In this paper, the rational fractal spline function constructed by rational cubic spline is introduced simply, and it is deformed to some extent. Then, based on this, the author explores its positivity preservation, obtains better conditions for positivity preservation, and monotonicity preservation.**

Keywords—fractal; Iterated function system; rational fractal interpolation; Positivity-preserving；spline

I. INTRODUCTION

In many practical cases, such as data visualization, information science, computer graphics, data is generated by complex functions or scientific phenomena, usually need to generate a smooth function, interpolate a set of specified data and retain some geometric attributes, we generally call it shape preserving function, the usual interpolation spline method, such as the classic cubic spline interpolation, usually ignores shape preserving function In recent 30 years, rational cubic spline has become a hot spot in industrial design and scientific data visualization because of its less oscillation and better properties than ordinary polynomial interpolation. Barnsley put forward fractal interpolation for the first time in literature [1], and continued to improve the fractal theory in the following years [2-3]. Since then, the theory of fractal interpolation has been preliminarily improved. With its development, more and more scholars have participated in the research of fractal theory, and rapidly developed, which has been widely used in many fields [4-5], but with the development of theory and application in practice Although the above fractal interpolation is very useful and easy to calculate, it also has its limitations. In order to meet the actual needs, a new fractal interpolation structure came into being, which is rational fractal interpolation function. In recent years, we have explored the properties of various rational fractal functions. For example, SQ Deng et al. In reference [6] carried out the shape control conditions of a class of rational splines In reference [7], the condition of monotonicity of a class of continuous rational fractal interpolation is obtained. In reference [8], the monotonicity data visualization of bivariate rational splines is further explored.

In recent years, the study of positive preserving is also very hot, and has certain practical significance in reality. For example, in the literature [9-11], the research on positive preserving of all kinds of systems, equations and functions is carried out, and the research on positive preserving of rational fractal interpolation function emerges at the historic moment. For example, in the literature [12], the research on positive preserving of a kind of rational fractal interpolation is discussed, and in the literature [13], a kind of continuous In reference[14], the corresponding conditions for preserving the positivity of a rational fractal interpolation with constrained parameters are given. In reference [15], a class of cubic rational trigonometric fractal functions are analyzed for preserving the positivity. In the above literature, various rational fractal functions are studied for preserving the positivity, and most of them are for ordinary rational fractal The advantages of fractal function are explained

In this paper, based on the rational spline fractal interpolation function constructed in reference [16], the properties of the function are further explored, and the corresponding conclusions are obtained in terms of positivity preserving and. monotonicity. preserving

II. THE CONSTRUCTION OF RATIONAL FRACTAL INTERPOLATION

A. *Fractal interpolation function(FIF)*

Interpolation function is proposed by m.f.barnsley on the basis of iterative function system(IFS). The principle is to construct corresponding IFS for a set of given interpolation points, so that the attractor of IFS is the function diagram passing through the set of interpolation points

For $r \in N$, let N_r denote the subset N 在 $\{1,2, \ldots, r\}$ of N.Let a set of data points satisfying $x_1<x_2<x_3...<x_N, N>2$,Set $I=[x_1, x_N], I_i=[x_i, x_{i+1}]$,

$i \in N_{n-1}$,Suppose $L_i: I \to I_i$ be contraction homeomorphisms such that

$$L_i(x_1) = x_i, L_i(x_n) = x_{i+1},$$
$$|L_i(c_1) - L_i(c_2)| \le l|c_1 - c_2|;$$

Here, $0 \le l \le 1$,and $X = I \times R$ Let continuous mapping $F_i: X \to R$ satisfying

$$F_i(x_1, y_1) = y_i, F_i(x_N, y_N) = y_{i+1},$$
$$|F_i(x, y) - F_i(x, y^*)| \le r_i|y - y^*|$$

here $(x, y), (x, y^*) \in X$,

define $w_i: X \to I_i \times R \subseteq X, w_i(x, y) = (L_i(x), F_i(x, y))$

Then the IFS generates a unique attractor, which is a continuous function image of $f: I \to R$ satisfying $f(x_i) = y_i$. This function is called FIF. In most studies, FIFs is given by ifs in the following form

$\{X: w_i(x, y) \equiv (L_i(x) = a_i x + b_i, F_i(x, y) = \alpha_i y + q_i(x)\}$

Here, $q_i(x): I \to R$ is a suitable continuous function satisfying the above description

B. The rational Fractal interpolation function

Proposition 1 Let $(x_i, y_i): i \in N_N$ be a given interpolation data with strictly increasing abscissae. Here $L_i(x) = \alpha_i x + b_i$ satifying $L_i(x_1) = x_i, L_i(x_n) = x_{i+1}$, and $F_i(x, y) = \alpha_i y + q_i(x)$

$q_i(x) = \dfrac{P_i(x)}{Q_i(x)}, P_i(x), Q_i(x)$ are suitably chosen polynomials. $Q_i(x) \ne 0$, Suppose for some integer $r \ge 0$, $|\alpha_i| = a_i^{\,r}, i \in N_{N-1}$ Let $F_{i,m}(x, y) = \dfrac{\alpha_i y + q_i^{(m)}(x)}{a_i^{\,m}}$

$$y_{i,m} = \frac{q_i^{(m)}(x_1)}{a_1^{\,m} - \alpha_1}. \quad y_{N,m} = \frac{q_{N-1}^{(m)}(x_N)}{a_{N-1}^{\,m} - \alpha_{N-1}}.$$

If $F_{i-1,\ m}(x_N, y_{N,m}) = F_{i,\ m}(x_1, y_{1,m})$ i=2,3,…N-1 m=1,2,…r,then $\{X: (L_i(x), F_i(x, y)): i \in N_{N-1}\}$ determines a rational FIF $g \in C^r[x_1, x_N]$, $g^{(m)}$ is determined by the IFS $\{X: (L_i(x), F_{i,\ m}(x, y)): i \in N_{N-1}\}$,for m=1,2,…r.

Let $\Delta = \{(x_i, y_i, d_i): i \in N_N\}$ be a given set of data points,and $x_1 < x_2 < ... < x_N$, f_i denote the function value and d_i denote the derivative value at x_i , consider the IFS as in the front part with q_i as

$$q_i(x) = q_i^*(\theta) = \frac{P_i(\theta)}{Q_i(\theta)} = \frac{A_i(1-\theta)^3 + B_i\theta(1-\theta)^2 + C_i\theta^2(1-\theta) + D_i\theta^3}{(1-\theta)^2 v_i + 2(1-\theta)u_i v_i + u_i\theta^2}$$

Here $\theta = \dfrac{x - x_1}{x_N - x_1}$, u_i, v_i are the free shape parameters. According to the previous section, we can get the following equation

$$g(L_i(x)) = F_i(x, g(x)) = \alpha_i g(x) + q_i(x)$$

Because of the C^1 -continuity, we can obtain the following equation on the scale factor by applying the condition

$$g^{(1)}(L_i(x)) = F_{i,1}(x, g^{(1)}(x)) = \frac{\alpha_i g^{(1)}(x) + q_i^{(1)}(x)}{\alpha_i}$$

According to the above equation, we can get

$$g(L_i(x_1)) = \alpha_i g(x_1) + \frac{P_i(0)}{Q_i(0)}$$

So $A_i = y_i - \alpha_i y_1$,

Similarly, we obtain $D_i = y_{i+1} - \alpha_i y_N$

$$g^{(1)}(L_i(x_1)) = \frac{\alpha_i g^{(1)}(x_1)}{a_i} + \frac{Q_i(0)P_i^{(1)}(0) - Q_i^{(1)}(0)P_i(0)}{a_i Q_i(0)^2(x_N - x_1)}$$

So

$$B_i = 2(u_i v_i + v_i)y_i + v_i h_i d_i - \alpha_i(2(u_i v_i + v_i)y_1 + v_i(x_N - x_1)d_1)$$

Similarly

$$C_i = 2(u_i v_i + u_i)y_{i+1} - u_i h_i d_i - \alpha_i(2(u_i v_i + u_i)y_N + u_i(x_N - x_1)d_N)$$

Then we get the following rational FIF

$$g(L_i(x)) = \alpha_i g(x) + \frac{P_i(\theta)}{Q_i(\theta)}$$

$$P_i(\theta) = (y_i - \alpha_i y_1)(1-\theta)^3 + [2(u_i v_i + v_i)y_i + v_i h_i d_i - \alpha_i(2(u_i v_i + v_i)y_1 + v_i(x_N - x_1)d_1)]\theta(1-\theta)^2 + [2(u_i v_i + u_i)y_{i+1} - u_i h_i d_i - \alpha_i(2(u_i v_i + u_i)y_N + u_i(x_N - x_1)d_N)]\theta^2(1-\theta) + (y_{i+1} - \alpha_i y_N)\theta^3$$

$$Q_i(\theta) = (1-\theta)^2 v_i + 2(1-\theta)u_i v_i + u_i\theta^2$$

III. POSITIVITY PRESERVING

Theorem 1. Suppose $\{(x_i, y_i): i \in N_N\}$ is a set of positive data, G is the associated rational spline FIF as described in the previous section, then the following conditions about the scale factor and shape parameter on each subinterval are sufficient conditions for G to keep positive

$$0 \le \alpha_i < \min\left\{a_i, \frac{y_i}{y_1}, \frac{y_{i+1}}{y_N}\right\}$$

$$u_i > \max\left\{0, \frac{\alpha_i(x_N - x_1)d_1 - h_i d_i}{2(y_i - \alpha_i y_i)} - \frac{1}{2}\right\},$$

$$v_i > \max\left\{0, \frac{h_i d_{i+1} - \alpha_i(x_N - x_1)d_N}{2(y_{i+1} - \alpha_i y_N)} - \frac{1}{2}\right\},$$

Proof: According to the existing conditions, the positivity of the denominator is obvious, so whether it is greater than 0 depends entirely on the $P_i(\theta)$.Substituting $\theta = \frac{v}{v+1}$,so we can get

$$P_i(\theta) = P_i(v) = D_1 v^3 + C_1 v^2 + B_1 v + A_1$$

According to the conclusion about the positivity of polynomials of the third degree in the literature[17]

$P_i(v) \ge 0$ if and only if $(A_1, B_1, C_1, D_1) \in W_1 \cup W_2$

$$W_1 = (A_1, B_1, C_1, D_1): A_1 > 0, B_1 > 0, C_1 > 0, D_1 > 0$$

$$W_2 = (A_1, B_1, C_1, D_1): A_1 > 0, D_1 > 0,$$
$$4A_1C_1^3 + 4D_1B_1^3 + 27A_1^2D_1^2 - 18A_1B_1C_1D_1 - B_1^2C_1^2 > 0$$

Due to the complexity of the calculation, our goal is to obtain a set of sufficient conditions to meet the positivity, and we use relatively effective and reasonable determined parameters, that is, if $A_1 > 0, B_1 > 0, C_1 > 0, D_1 > 0$ is true, $g(L_i($ is positive

It can be rewritten as the following equation

$$A_1 = y_i - \alpha_i y_1 > 0 \quad D_1 = y_{i+1} - \alpha_i y_N > 0$$

$$B_1 = 2(u_i v_i + v_i)y_i + v_i h_i d_i - \alpha_i(2(u_i v_i + v_i)y_1 +$$
$$v_i(x_N - x_1)d_1)$$

$$C_1 = 2(u_i v_i + u_i)y_{i+1} - u_i h_i d_i - \alpha_i(2(u_i v_i + u_i)y_N +$$
$$u_i(x_N - x_1)d_N)$$

According to the first two inequalities, we can get

$$\alpha_i < \frac{y_i}{y_1}, \alpha_i < \frac{y_{i+1}}{y_N}$$

The last two inequalities can be rewritten as

$$\begin{matrix} (2u_i + 1)(y_i - \alpha_i y_i) < \alpha_i(x_N - x_1)d_1 - h_i d_i \\ (2v_i + 1)(y_{i+1} - \alpha_i y_N) < h_i d_{i+1} - \alpha_i(x_N - x_1)d_N \end{matrix} :$$

So we can get

$$u_i > \frac{\alpha_i(x_N - x_1)d_1 - h_i d_i}{2(y_i - \alpha_i y_i)} - \frac{1}{2},$$

$$v_i > \frac{h_i d_{i+1} - \alpha_i(x_N - x_1)d_N}{2(y_{i+1} - \alpha_i y_N)} - \frac{1}{2},$$

Then

$$u_i > \max\left\{0, \frac{\alpha_i(x_N - x_1)d_1 - h_i d_i}{2(y_i - \alpha_i y_i)} - \frac{1}{2}\right\},$$

$$v_i > \max\left\{0, \frac{h_i d_{i+1} - \alpha_i(x_N - x_1)d_N}{2(y_{i+1} - \alpha_i y_N)} - \frac{1}{2}\right\},$$

In practical problems, derivatives are often difficult to get, so this paper uses the following artithmetic mean method[18]:

$$d_1 = \Delta_1 - \frac{h_1}{h_1 + h_2}(\Delta_2 - \Delta_1)$$

$$d_i = \frac{1}{h_{i-1} + h_i}(h_{i-1}\Delta_i - h_i\Delta_{i-1})$$

$$d_n = \Delta_{n-1} - \frac{h_{n-1}}{h_{n-1} + h_{n-2}}(\Delta_{n-1} - \Delta_{n-2})$$

This completes the proof

IV. Monotonicity preserving

Theorem 2. For the above rational FIF, let $x_1 < x_2 < x_3 \ldots < x_N, N > 2$ be monotonically increasing data, and the above rational FIF remain monotone if

$$u_i > 0, v_i > \max(u_i(d_{i+1} - \frac{2(y_{i+1} - y_i)}{h_i}), \frac{-u_i(y_{i+1} - y_i)}{2(y_{i+1} - y_i) - d_i h_i}, 0)$$

Proof

Suppose $x_1 < x_2 < x_3 \ldots < x_N, N > 2$ be monotonically increasing data thent he above rational FIF remain monotone if and only if $g^{(1)}(L_i(x)) > 0$,so we can know that:

$$g^{(1)}(L_i(x)) = \frac{\alpha_i}{a_i} g^{(1)}(x) + H(x)$$

Here,

$$H(x) = \frac{1}{h_i Q_i^2(x)}[(1-\theta)M_{1,i} + \theta(1-\theta)^3 M_{2,i} +$$
$$\theta^2(1-\theta)^2 M_{3,i} + \theta^3(1-\theta)M_{4,i} + \theta^4 M_{5,i}]$$

It can be seen from literature [] that monotonic data only need to satisfy the following conditions

$$0 \le \alpha_i < a_i, M_{1,i} > 0, M_{2,i} > 0, M_{3,i} > 0, M_{4,i} > 0, M_{5,i} > 0$$

Because it is too complicated to calculate the necessary and sufficient conditions for monotonicity maintenance, this paper only discusses the necessary conditions for monotonicity maintenance under $\alpha_i = 0$, then the following results can be obtained

$$M_{1,i} = v_i^2 d_i, M_{2,i} = 2v_i\{(v_i + 2u_i)\frac{y_{i+1} - y_i}{h_i} - u_i d_{i+1}\},$$

$$M_{3,i} = M_{2,i} + M_{4,i} - (M_{1,i} + M_{5,i}) + 2u_i v_i (d_i + d_{i+1}),$$

$$M_{4,i} = 2u_i\{(2v_i + u_i)\frac{y_{i+1} - y_i}{h_i} - v_i d_i\}, M_{5,i} = u_i^2 d_{i+1}$$

$M_{1,i} > 0, M_{5,i} > 0$ is obvious

When $M_{2,i} > 0, M_{4,i} > 0$, Inequality $M_{3,i} > 0$ is true

So we can get

$$2v_i\{(v_i + 2u_i)\frac{y_{i+1} - y_i}{h_i} - u_i d_{i+1}\} > 0,$$

$$2u_i\{(2v_i + u_i)\frac{y_{i+1} - y_i}{h_i} - v_i d_i\} > 0$$

Then

$$v_i > u_i(d_{i+1} - \frac{2(y_{i+1} - y_i)}{h_i})$$

$$v_i > \frac{-u_i(y_{i+1} - y_i)}{2(y_{i+1} - y_i) - d_i h_i}$$

So

$$u_i > 0, v_i > \max(u_i(d_{i+1} - \frac{2(y_{i+1} - y_i)}{h_i}), \frac{-u_i(y_{i+1} - y_i)}{2(y_{i+1} - y_i) - d_i h_i}, 0)$$

This completes the proof

V. CONCLUSION

Fractal interpolation use the principle of self-affine to obtain various curve that ups and downs. While the hidden variable fractal interpolation curve is non-self-affine. It involves more free variables, we could obtain different curves by changing those free parameters. Therefore, the fractal interpolation function of hidden variable is more flexible and more accurate, which provides a strong theoretical basis for simulating many objects and phenomena in nature. The realization of MATLAB also makes us deeply feel the beauty of mathematics.

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