Global Regularity for Very Weak Solutions to Boundary Value Problem of Homogeneous $A$-Harmonic Equation

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Abstract—The very weak solution to elliptic boundary value problems is considered. A global regularity result is derived for very weak solutions under some controllable and coercivity conditions, by using the Hodge decomposition theorem and the methods in Sobolev spaces.

Keywords—Hodge decomposition theorem; $A$-harmonic equation; global regularity

I. INTRODUCTION

Let $\Omega$ be a bounded regular open set of $\mathbb{R}^n (n \geq 2)$. We consider the boundary value problem for the second order degenerate elliptic equation

$$
\begin{cases}
\text{div} A(x, \nabla u) = 0, & \text{in } \Omega \\
u = u_0, & \text{on } \partial \Omega
\end{cases}
$$

(1.1)

where $A(x, \xi) : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory function satisfying the coercivity and growth conditions: for almost all $x \in \Omega$, all $\xi \in \mathbb{R}^n$,

(H1) $|A(x, \xi)| \leq \beta |\xi|^{p-1}$,

(H2) $\langle A(x, \xi), \xi \rangle \geq \alpha |\xi|^p$,

where $1 < p \leq \infty$, $0 < \alpha \leq \beta < \infty$, $u_0 \in W_0^{1,p} (\Omega)$ is a boundary value function.

Definition 1.1 A function $u \in u_0 + W_0^{1,p} (\Omega)$ satisfies in $\Omega$.

$$
\int_{\Omega} \langle A(x, \nabla u), \nabla \phi \rangle \, dx = 0
$$

holds true for any $\phi \in W_0^{1,\frac{p}{r}} (\Omega)$ with compact support in $\Omega$.

A crucial fact is that $r$ can be smaller than the natural exponent $p$. For variational extremals the global higher integrability of the derivative $\nabla u$ has been studied by Granlund S in the case $p = n$. For this it seems necessary to impose a regularity condition for $\partial \Omega$.

We say that $\partial \Omega$ is $r$-Poincaré thick, if there is $0 < C < \infty$ such that for all open cube $Q_\delta \subset \mathbb{R}^n$ with side length $R > 0$, it holds

$$
\left( \int_{Q_{\frac{\delta}{2}}} |u|^r \, dx \right)^{\frac{1}{r}} \leq C \left( \int_{Q_{\frac{\delta}{2}}} |\nabla u|^r \, dx \right)^{\frac{1}{r}}
$$

(1.2)

whenever $u \in W^{1,r} (Q_{\frac{\delta}{2}})$, $u \equiv 0$ a.e. on $(\mathbb{R}^n \setminus \Omega) \setminus Q_{\frac{\delta}{2}}$, and $Q_{\frac{\delta}{2}} \setminus \Omega \subset \subset \Omega$. Here, and in the following, $Q(R) \subset \subset Q(\lambda R)$ with the same center as $Q(R)$ and with side length $\lambda R$. See [2].

The following is the main conclusion of this paper.

Theorem 1.2 Suppose that a bounded regular domain $\Omega$ has a $r$-Poincaré thick boundary and that $r \geq \frac{n}{n-1}$, operator $A$ satisfy conditions (H1)-(H2). If $u_0 \in W_0^{1,r} (\Omega)$ is the boundary value function, $u \in W_0^{1,r} (\Omega)$ is the very weak solution of Dirichlet problem (1.1), then there exists $R_0 > 0$ and $r_1, r_2$, satisfying

$$
r_1 = r_1 (n, p, K, R_0, \alpha, \beta, \Omega),
$$

such that $\forall r \in [r_1, r_2)$, $u \in W_0^{1,\frac{n}{r}} (\Omega)$, then $u$ is the weak solution in the classical meaning.

II. PRELIMINARY LEMMAS

Let $\Omega$ be a bounded regular domain, $x_0 \in \Omega$, $0 < R < \text{dist}(x_0, \partial \Omega)$, $Q_{\frac{\delta}{2}} (x_0) \subset \Omega$, here $Q_{\frac{\delta}{2}} (x_0)$ is a cube with side length of $R$ and a center of $x_0$.

Lemma 2.1 Let $1 < p < n$, $0 < q \leq \frac{np}{n-p}$ if $u \in W_0^{1,r} (B_R (x_0))$, then
Where $b, \theta, q, n$ are positive constants only depending on $b, \theta, q, n$.

III. PROOF OF THEOREM 1.2

Proof. Let $x_0 \in \Omega$, $Q_r(x_0)$ is a cube with side length of $\rho$ and a center of $x_0$, $\varepsilon$ is a sufficiently small positive number, $r = p - \varepsilon$. Since $\Omega$ is bounded, we can choose a cube $Q_0 = Q_{x_0} \subset \Omega$ such that $Q \subset Q_0$. Next let $Q_{2r} \subset Q_0$. There are two possibilities: (1) $Q_{2r} \subset \Omega$; (2) $Q_{2r} \cap \Omega^c \neq \emptyset$.

In the case (1), for $Q_{2r} \subset \Omega$, fix a cutoff function $\eta \in C^\infty_c(Q_{\frac{3r}{2}})$ such that $0 \leq \eta \leq 1, |\nabla \eta| \leq \frac{C}{R}$, and $\eta = 1$ on $x \in Q_{r}$. Let $u \in W^{1,r}(\Omega)$ be a very weak solution of problem (1.1). Consider the following Hodge decomposition

$$|\nabla (\eta u)|^r \nabla (\eta u) = \nabla \phi + H,$$  

(3.1)

Here $\phi \in W^{1,r}_0(Q_{\frac{3r}{2}})$, $H \in L^1(Q_{\frac{3r}{2}})$ is a (divergence free) matrix-field, satisfying

$$\|H\|_{L^1} \leq C\|\nabla (\eta u)\|_{L^1}^r,$$  

(3.2)

$$PHP_{r-1}^r \leq C\varepsilon PV(\eta u)\eta_{r-1}^r.$$  

(3.3)

Let

$$E(\eta, u) = |\nabla (\eta u)|^r \nabla (\eta u) - |\nabla \eta|^r \eta \nabla u,$$  

(3.4)

by Lemma 2.4 we have

$$|E(\eta, u)| \leq 2e^{\frac{1+\varepsilon}{1-\varepsilon}} |\nabla \eta|^r \eta \nabla u.$$  

(3.5)

A useful technique in the following calculation is to use $\phi$ in Hodge decomposition (3.1) as the test function in Definition 1.1. Then

$$\int_{Q_{\frac{3r}{2}}} \left( A(x, \nabla u), E(\eta, u) \right) dx$$  

$$+ \int_{Q_{\frac{3r}{2}}} \left( A(x, \nabla u), |\eta \nabla u|^r \eta \nabla u \right) dx$$  

(3.6)

$$= \int_{Q_{\frac{3r}{2}}} \left( A(x, \nabla u), H \right) dx,$$  

that is

$$\int_{Q_{\frac{3r}{2}}} \left( A(x, \nabla u), \eta \nabla \eta \eta \nabla u \right) dx$$  

$$= \int_{Q_{\frac{3r}{2}}} \left( A(x, \nabla u), H \right) dx - \int_{Q_{\frac{3r}{2}}} \left( A(x, \nabla u), E(\eta, u) \right) dx$$  

(3.7)

$$= I_1 + I_2.$$  

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Let’s first estimate the left side of formula (3.7). By the hypothesis condition (H2) and the definition of \( \eta \),

\[
\int_{Q_{\frac{3}{2}}} \left\langle A(x, \nabla u), \eta \nabla u \right\rangle dx \\
= \int_{Q_{\frac{3}{2}}} | \nabla u |^{\varepsilon} \left( A(x, \nabla u), \nabla u \right) dx \\
\geq \alpha \int_{Q_{\frac{3}{2}}} | \nabla u |^{\varepsilon} | \nabla u |^{\varepsilon} dx \tag{3.8}
\]

The estimate of \( I_1 \) is given below. By the hypothesis (H1), the Hölder inequality and (3.3), we can get the result

\[
|I_1| = \left| \int_{Q_{\frac{3}{2}}} \left\langle A(x, \nabla u), H \right\rangle dx \right| \leq \int_{Q_{\frac{3}{2}}} | A(x, \nabla u) | |H| dx \\
\leq \beta \int_{Q_{\frac{3}{2}}} | \nabla u |^{\varepsilon} |H| dx \tag{3.9}
\]

Notice that \( u \) plus a constant vector does not affect \( \nabla u \) and the \( A \)-harmonic equation in (1.1) in our case, so let’s assume that the average integral of \( u \) on \( Q_{\frac{3}{2}} \) is zero, and then by using Lemma 2.1,

\[
| \nabla (\eta u) |^{\varepsilon} = | u \nabla \eta + \eta \nabla u |^{\varepsilon} \\
\leq (| u \nabla \eta |^{\varepsilon} + | \eta \nabla u |^{\varepsilon}) \\
\leq \left( \frac{C}{r} | u |^{\varepsilon} + | \eta \nabla u |^{\varepsilon} \right) \\
\leq \left( \frac{C}{r} | \nabla u |^{\varepsilon} + | \nabla u |^{\varepsilon} \right) \\
\leq C \| \nabla u \|^{\varepsilon}, \tag{3.10}
\]

then we have

\[
|I_1| \leq C \beta \varepsilon \| \nabla u \|^{\varepsilon}. \tag{3.11}
\]

The estimate of \( I_2 \) is given below. By the hypothesis (H1), (3.5) and the definition of \( \eta \), we have

\[
|I_2| = \left| \int_{Q_{\frac{3}{2}}} \left\langle A(x, \nabla u), E(\eta, u) \right\rangle dx \right| \\
\leq \beta \int_{Q_{\frac{3}{2}}} | \nabla u |^{\varepsilon} | E(\eta, u) | dx \tag{3.12}
\]

By Young’s inequality, for any \( \theta > 0 \),

\[
|I_2| \leq \beta C \theta \int_{Q_{\frac{3}{2}}} | \nabla u |^{\varepsilon} dx + \beta C \int_{Q_{\frac{3}{2}}} \left| \frac{u}{R} \right|^{\varepsilon} dx. \tag{3.13}
\]

For the second integral formula at the right end of the upper formula, take \( t \) such that \( \max\{1, \frac{nr}{n+r}\} \leq t < r \), then by Lemma 2.1,

\[
\beta C \int_{Q_{\frac{3}{2}}} \left| \frac{u}{R} \right|^{\varepsilon} dx \leq \beta \varepsilon \left( \int_{Q_{\frac{3}{2}}} | \nabla u |^{\varepsilon} dx \right)^{\frac{r}{\varepsilon}}. \tag{3.14}
\]

it doesn’t effect \( \nabla u \) and \( A \)-harmonic equation when \( u \) plus a constant, so assuming the integral average of \( u \) is zero in \( Q_{\frac{3}{2}} \), then we have

\[
|I_2| \leq \beta C \theta \int_{Q_{\frac{3}{2}}} | \nabla u |^{\varepsilon} dx + \beta \varepsilon \left( \int_{Q_{\frac{3}{2}}} | \nabla u |^{\varepsilon} dx \right)^{\frac{r}{\varepsilon}}. \tag{3.15}
\]

Combining the inequalities (3.7), (3.8), (3.11), (3.15), we obtain

\[
\alpha \int_{Q_{\frac{3}{2}}} | \nabla u |^{\varepsilon} dx \leq C \beta \varepsilon \left( \int_{Q_{\frac{3}{2}}} | \nabla u |^{\varepsilon} dx \right)^{\frac{r}{\varepsilon}} \\
+ C \beta \theta \int_{Q_{\frac{3}{2}}} | \nabla u |^{\varepsilon} dx \tag{3.16}
\]

Divide the two sides of the formula above by \( |Q_{\frac{3}{2}}| = \omega_n R^n \) ( here \( \omega_n \) is the volume of unit cube in \( \mathbb{R}^n \) ), then

\[
\alpha \int_{Q_{\frac{3}{2}}} | \nabla u |^{\varepsilon} dx \leq C \beta \varepsilon \left( \int_{Q_{\frac{3}{2}}} | \nabla u |^{\varepsilon} dx \right)^{\frac{r}{\varepsilon}} + C \beta \theta \int_{Q_{\frac{3}{2}}} | \nabla u |^{\varepsilon} dx \tag{3.17}
\]

Since \( \Omega \) is bounded, \( \Omega \subset Q_{\frac{3}{2}}, R < R_0 \), the formula above becomes
where $C = C(n, p, r, \alpha, \beta, R_i, \Omega)$. Noticing that the case we considered is that $r$ is close enough to $P$, then $r$ can be removed from the parameter of $C$. For $1 < r < R$, then (3.19) is a weak reverse Hölder inequality about $\nabla u$.

Choosing $g = \|\nabla u\|$, $G = 0$ in $Q_{2r}$ and $g = G = 0$ in $Q_{2r} \setminus Q_{2r}$ with $q = r$. Then we arrive at the following inequality in $Q_{2r} \subset \Omega$, that is

$$\int_{Q_{2r}} |\nabla u|^r \, dx \leq C\left[ \int_{Q_{2r}} g^r \, dx + C\left[ \int_{Q_{2r}} g^r \, dx \right] \right].$$

In the case (2), let $w = -\eta^r(u - u_0) \in W_0^{1, r}(Q_{2r})$, where $\eta \in C^1(\Omega) \subset C_c(\Omega)$ is a cutoff function, $0 \leq \eta \leq 1$, $|\nabla \eta| \leq C/R$, and $\eta = 1$ in $Q_1$.

Extending the function $u - u_0$ with zero to $R^* \setminus \Omega$ continuously. Then by Lemma 2.2, there exist $\phi \in W_0^{1, r}(Q_{2r})$ and $H(x) \in L^{1, r}(Q_{2r})$, such that

$$|\nabla u|^r \nabla w = \nabla \phi + H$$

and

$$\|H\|_{1, \frac{r}{r-1}} \leq C\|\nabla \phi\|_{1, \frac{r}{r-1}}^{\frac{r}{r-1}} \leq C\|\nabla \phi\|_{1, \frac{r}{r-1}}^{\frac{r}{r-1}},$$

(3.23)

where $C$ is a constant only depending on $n$, $r$ and $\Omega$. Note that for Hodge decomposition (3.21), (3.22) and (3.23), we have $u - u_0 = 0$, $H$ and $\nabla \phi$ are equal to zero when $u - u_0 \in R^* \setminus \Omega$. By the Minkowski inequality and the selection of $\eta$, we have

$$\|\nabla \eta^r(u - u_0)\|_{1, \frac{r}{r-1}}^{\frac{r}{r-1}} \leq C\|\nabla [((u - u_0)\nabla \eta)]\|_{1, \frac{r}{r-1}}^{\frac{r}{r-1}},$$

(3.24)

By the conditions (H1), (H2), Lemma 2.4, Hodge decomposition (3.21), and the Definition1.1, we obtain

$$\alpha \int_{\Omega} |\nabla u| \, dx \leq \int_{\Omega} \left\{ A(x, \nabla u)|\nabla \eta|^r \eta^r \nabla u \right\} \, dx$$

$$= \int_{\Omega} \left\{ A(x, \nabla u)|\nabla u|^r \eta^r \nabla u - |\nabla \eta|^r \eta^r \nabla (u - u_0) \right\} \, dx$$

(3.25)
\[ + \int_{\Omega} \left\{ A(x, \nabla u), \eta' \nabla (u-u_0) \right\} dx \\
- \int_{\Omega} \left\{ A(x, \nabla u), \eta \nabla (u-u_0) \right\} dx \\
\leq C \int_{\Omega} \eta'' |x|^{-\alpha} |u-u_0| \eta'' dx \\
+ \int_{\Omega} \left\{ A(x, \nabla u), \eta \nabla (u-u_0) \right\} dx \\
\leq 2 \left[ \int_{\Omega} |\nabla u|^m dx \right]^{\frac{\alpha-m}{\alpha-m}} + CR' \int_{\Omega} |u'_0| dx. \] 

Then
\[ I_4 = \theta_1 \int_{\Omega} |\nabla u|^{m} dx + C \int_{\Omega} |H|^{\alpha-m} dx. \]

The estimate of \( I_4 \) is given below. By Young’s inequality, (3.27) and the Hölder inequality, for any \( \theta_i > 0 \), we have
\[ I_4 = \theta_1 \int_{\Omega} |\nabla u|^{m} \left( \int_{\Omega} |\nabla u|^{m} dx \right)^{-1} dx. \]

For the second integral formula at the right end of the upper formula, noticing that \( \mathcal{Q} \) is \( r \)-Poincaré thick. By (3.25) we get
\[ C \int_{\Omega} \eta'' |x|^{-\alpha} |u-u_0| \eta'' dx \\
\leq CR' \int_{\Omega} |\nabla (u-u_0)|^{\alpha-m} dx \] 

Then by Minkowski inequality and the Hölder inequality, we have
\[ \left( \int_{\Omega} |\nabla (u-u_0)|^{\alpha-m} dx \right)^{\frac{\alpha}{m}} \\
\leq \left( \int_{\Omega} |\nabla u|^{\alpha-m} dx \right)^{\frac{\alpha}{m}} + C \int_{\Omega} |u'_0|^{\alpha-m} dx \]

Combining the inequalities (3.29),(3.30), (3.34), (3.35), we obtain
\[ \int_{\Omega} \eta'' |x|^{-\alpha} |u-u_0| \eta'' dx \\
\leq C(\theta_1 + \theta_2 + \epsilon) \int_{\Omega} |\nabla u|^{m} dx \\
+ CR' \int_{\Omega} |\nabla u|^{\alpha-m} dx \]

where \( C = C(n, p, \alpha, \beta, K, \Omega) \).

Choosing \( \theta_1 \), \( \theta_2 \), \( \theta_3 \), and \( \epsilon_0 > 0 \) small enough, there exist \( r_i = p - \epsilon_0 < p \), such that \( \theta = C(\theta_1 + \theta_2 + \theta_3 + \epsilon) < 1 \) when \( \epsilon < \epsilon_0 \). By (3.36),
\[ \int_{\Omega} |\nabla u|^{m} dx \leq \int_{\Omega} |\nabla u|^{m} dx + C \int_{\Omega} |u'_0|^{\alpha-m} dx \]

(3.37)
where \( t = \frac{nr}{n+r} < r \). Let \( g = |\nabla u|^r, G = 0 \). Then we arrive at the following inequality when \( \varepsilon < \varepsilon_0 \), that is
\[
\int_{\Omega_k} g^r \, dx \leq \Theta \int_{\Omega_k} g^r \, dx + C \int_{\Omega_k} g \, dx \quad \text{(3.38)}
\]
\[
+ C \int_{\Omega_k} |G|^{\frac{r}{r-1}} \, dx,
\]
where \( C = C(n, p, \alpha, \beta, K, \Omega) \). Then by (3.20), (3.38) and Lemma 2.5, there exists \( r' \), and \( r' > r \), such that \( u \in W^{1,r'}(\Omega) \). For \( r' \), repeating the above process, the integrability of \( \nabla u \) is improved over and over again. In this way, there must be an integrable exponent \( r_1 \) and \( r_2 \), satisfying \( r_1 < p < r_2 \), such that \( u \in W^{1,r}(\Omega) \), \( \forall \tau \in (r_1, r_2) \). The proof is complete.

REFERENCES


