Journal of Multidisciplinary Engineering Science and Technology (JMEST)
ISSN: 2458-9403
Vol. 6 Issue 10, October - 2019

Gradient Estimates for Weak Solutions to Nonhomogeneous Elliptic Equations under Natural Growth

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***Abstract*—This paper deals with the gradient estimates for weak solutions to a class of nonhomogeneous elliptic equations under natural growth. Final estimates for such equation under the natural growth are derived by choosing the appropriate test function and other methods.**

Keywords*—*under natural growth; weak solution; gradient estimate

I. INTRODUCTION

Let's consider the weak solution of the following general nonhomogeneous elliptic equation

$$\mathrm{div}A(x,\nabla u)=B(x,\nabla u)+\mathrm{div}(|\nabla u|^{p-2}\nabla u), \quad (1.1)$$

where $1<p<+\infty$, Ω is a bounded domain in $\mathbb{R}^n (n\geq 2)$ and $A=A(x,\xi):\Omega\times\mathbb{R}^n\to\mathbb{R}^n$ is assumed to be a Carathéodory vectorial-valued function which is measurable in x for each ξ and continuous in ξ for almost everywhere x. Moreover, for given $p\in(1,\infty)$ the structural conditions on the function A are given as follows:

$$\langle A(x,\xi),\xi\rangle\geq C_1|\xi|^p; \quad (1.2)$$

$$|A(x,\xi)|\leq C_2|\xi|^{p-1}, \quad (1.3)$$

for all $\xi\in\mathbb{R}^n, x\in\Omega$, and some positive constants $C_i>0, i=1,2$. The nonhomogeneous term $B=B(x,\xi):\ \Omega\times\mathbb{R}^n\to\mathbb{R}^n$ satisfies the natural growth condition:

$$|B(x,\xi)|\leq C_3|\xi|^p. \quad (1.4)$$

As usual, the solution of (1.1) is taken in a weak sense. The definition of weak solution is given as follows.

Definition1.1. A function $u\in W^{1,p}_{loc}(\Omega)\cap L^\infty(\Omega)$ is a local weak solution of (1.1) if A and B satisfy the conditions (1.2)-(1.4), and for any $\varphi\in W^{1,p}_0(\Omega)\cap L^\infty(\Omega)$ with compact support, one has

$$\int_\Omega\langle A(x,\nabla u),\nabla\varphi\rangle dx=\int_\Omega B(x,\nabla u)\varphi dx +\int_\Omega\left\langle|\nabla u|^{p-2}\nabla u,\nabla\varphi\right\rangle dx. \quad (1.5)$$

E. Acerbi and G. Mingione[1] obtained $L^{p(x)}$-type gradient estimates for weak solutions of quasilinear elliptic equation of *p*-Laplacian type

$$\mathrm{div}(|Du|^{p-2}Du)=\mathrm{div}(|F|^{p-2}F);$$

S. S. Byun and L. Wang[2] obtained $W^{1,p}$, $2\leq p<\infty$, regularity for weak solutions of the general nonlinear elliptic problem

$$\mathrm{div}a(\nabla u,x)=\mathrm{div}f;$$

F. Yao[3] obtained gradient estimates in Orlicz spaces for weak solutions of A-harmonic equations under the assumptions that A satisfies some proper conditions and the given function satisfies some moderate growth conditions. G. Li[4] considered the regularity for weak solutions of the A-harmonic equations

$$-\mathrm{div}A(x,\nabla u)+B(x,\nabla u)=0;$$

Recently, Q. Zhao[5] obtained the regularity for very weak solutions of the nonhomogeneous A-harmonic equations

$$-\mathrm{div}A(x,u,Du)=B(x,Du).$$

The purpose of this paper is to study a class of A-harmonic equations which have both *p*-laplacian type and nonhomogeneous terms and to consider the L^p-type estimates for such equation under natural growth. Here, our approach is based on the paper[6] in which the L^p-type estimates were derived under natural growth.

Now the main result of this paper will be stated as follows.

Theorem 1.1. Suppose $B_{2R}\subset\Omega$ and $u\in W^{1,p}_{loc}(\Omega)\bigcap L^{\infty}(\Omega)$ is the weak solution of equation (1.1) under assumptions (1.2)-(1.4), then the following estimation holds

$$\int_{B_R}|\nabla u|^p dx\le C\int_{B_{2R}}|u-u_R|^p dx. \tag{1.6}$$

The rest of the paper is organized as follows. Section 2 is devoted to introduce some useful lemmas. Section 3 focus on proving our main theorem.

II. Technical Tools

In this section some basic inequalities and lemmas needed in proving the main conclusion will be introduced.

Lemma 2.1[7] (Young's inequality) Suppose that $a>0, b>0, p>1, q>1$, and $\frac{1}{p}+\frac{1}{q}=1$, then

$$ab\le\frac{a^p}{p}+\frac{b^q}{q}.$$

In particularly, the above inequality is the Cauchy inequality if $p=q=2$.

Suppose that $\varepsilon>0$, a and b are replaced by $\varepsilon^{1/p}a$ and $\varepsilon^{-1/p}b$ respectively in the above inequality, then one has the following lemma.

Lemma 2.2[7] (Young's inequality with ε) Suppose that $a>0, b>0, p>1, q>1$, and $\frac{1}{p}+\frac{1}{q}=1$, then

$$ab\le\frac{\varepsilon a^p}{p}+\frac{\varepsilon^{-q/p}b^q}{q}\le\varepsilon a^p+\varepsilon^{-q/p}b^q.$$

In particularly, the above inequality is the Cauchy inequality with ε if $p=q=2$.

Moreover, we give the following lemma.

Lemma 2.3[8] Suppose $f(\tau)$ is a nonnegative bounded function defined on $0\le R_0\le t\le R_1$ if one has

$$f(\tau)\le A(t-\tau)^{-\alpha}+B+\theta f(t)$$

for $R_0\le\tau<t\le R_1$, where A,B,α,θ are nonnegative constants, and $\theta<1$, then there exists a constant C that only depends on α and θ, such that for all $\rho,R,R_0\le\rho<R\le R_1$, one has

$$f(\rho)\le C\left[A(R-\rho)^{-\alpha}+B\right].$$

III. Proof of Main Result

Taking $\varphi=(u-u_{2R})e^{\beta|u-u_{2R}|}\eta^p$ (β is determined later) as a test function in Definition 1.1, where $u_{2R}=\frac{1}{|B_{2R}|}\int_{B_{2R}}u\,dx$, $\eta\in C_0^{\infty}(\mathbb{R}^n)$ is a cutoff function satisfying

$$0\le\eta\le1,\quad \eta\equiv1\, in\, B_R,\quad \eta\equiv0\ in\,\mathbb{R}^n\setminus B_{2R},\quad |\nabla\eta|\le\frac{C}{R}. \tag{3.1}$$

Then one has

$$\begin{aligned}\int_\Omega\langle A(x,\nabla u),\nabla\varphi\rangle dx&=\int_\Omega B(x,\nabla u)\varphi dx\\&+\int_\Omega\left\langle|\nabla u|^{p-2}\nabla u,\nabla\varphi\right\rangle dx.\end{aligned} \tag{3.2}$$

And thus

$$\begin{aligned}&\int_\Omega\left\langle A(x,\nabla u)-|\nabla u|^{p-2}\nabla u, e^{\beta|u-u_{2R}|}\eta^p\nabla u\right\rangle dx\\&+\int_\Omega\left\langle A(x,\nabla u),\beta|u-u_{2R}|e^{\beta|u-u_{2R}|}\eta^p\nabla u\right\rangle dx\\&=-\int_\Omega\left\langle A(x,\nabla u), p(u-u_{2R})e^{\beta|u-u_{2R}|}\eta^{p-1}\nabla\eta\right\rangle dx\\&+\int_\Omega\left\langle|\nabla u|^{p-2}\nabla u,\beta|u-u_{2R}|e^{\beta|u-u_{2R}|}\eta^p\nabla u\right\rangle dx\\&+\int_\Omega\left\langle|\nabla u|^{p-2}\nabla u, p(u-u_{2R})e^{\beta|u-u_{2R}|}\eta^{p-1}\nabla\eta\right\rangle dx\\&+\int_\Omega B(x,\nabla u)\cdot(u-u_{2R})e^{\beta|u-u_{2R}|}\eta^p dx.\end{aligned} \tag{3.3}$$

Using (1.2), it yields

$$\begin{aligned}&\int_\Omega\left\langle A(x,\nabla u)-|\nabla u|^{p-2}\nabla u, e^{\beta|u-u_{2R}|}\eta^p\nabla u\right\rangle dx\\&+\int_\Omega\left\langle A(x,\nabla u),\beta|u-u_{2R}|e^{\beta|u-u_{2R}|}\eta^p\nabla u\right\rangle dx\\&\ge C_1\int_\Omega e^{\beta|u-u_{2R}|}|\eta\nabla u|^p dx\\&\quad+C_1\beta\int_\Omega|u-u_{2R}|e^{\beta|u-u_{2R}|}|\eta\nabla u|^p dx.\end{aligned} \tag{3.4}$$

Moreover, by (1.3), (1.4) and Young's inequality with ε, one has

$$\begin{aligned}&-\int_\Omega\left\langle A(x,\nabla u), p\eta^{p-1}(u-u_{2R})e^{\beta|u-u_{2R}|}\nabla\eta\right\rangle dx\\&+\int_\Omega\left\langle|\nabla u|^{p-2}\nabla u,\beta|u-u_{2R}|e^{\beta|u-u_{2R}|}\eta^p\nabla u\right\rangle dx\\&+\int_\Omega\left\langle|\nabla u|^{p-2}\nabla u, p(u-u_{2R})e^{\beta|u-u_{2R}|}\eta^{p-1}\nabla\eta\right\rangle dx\\&+\int_\Omega B(x,\nabla u)\cdot(u-u_{2R})e^{\beta|u-u_{2R}|}\eta^p dx\\&\le pC_2\int_\Omega|\eta\nabla u|^{p-1}|u-u_{2R}|e^{\beta|u-u_{2R}|}|\nabla\eta|dx\\&\quad+\beta\int_\Omega|\nabla u|^p|u-u_{2R}|e^{\beta|u-u_{2R}|}\eta^p dx\\&\quad+p\int_\Omega|\nabla u|^{p-1}\eta^{p-1}|u-u_{2R}|e^{\beta|u-u_{2R}|}|\nabla\eta|dx\\&\quad+C_3\int_\Omega|\nabla u|^p|u-u_{2R}|e^{\beta|u-u_{2R}|}\eta^p dx\\&=C\int_\Omega\left|e^{\frac{\beta|u-u_{2R}|}{p}}\eta\nabla u\right|^{p-1}|u-u_{2R}|e^{\frac{\beta|u-u_{2R}|}{p}}|\nabla\eta|dx\\&\quad+(\beta+C_3)\int_\Omega|u-u_{2R}|e^{\beta|u-u_{2R}|}|\eta\nabla u|^p dx\\&\le C\left\{\varepsilon\int_\Omega e^{\beta|u-u_{2R}|}|\eta\nabla u|^p dx\right.\\&\qquad\left.+C(\varepsilon)\int_\Omega e^{\beta|u-u_{2R}|}|(u-u_{2R})\nabla\eta|^p dx\right\}\\&\quad+(\beta+C_3)\int_\Omega|u-u_{2R}|e^{\beta|u-u_{2R}|}|\eta\nabla u|^p dx.\end{aligned} \tag{3.5}$$

where $C = C(C_2, p)$. Putting (3.1), (3.3) and (3.5) together yields

$$\begin{aligned} & C_1\int_\Omega e^{\beta|u-u_{2R}|}|\eta\nabla u|^p\,dx \\ & +C_1\beta\int_\Omega |u-u_{2R}|e^{\beta|u-u_{2R}|}|\eta\nabla u|^p\,dx \\ \le & (pC_2+p)\Big\{\varepsilon\int_\Omega e^{\beta|u-u_{2R}|}|\eta\nabla u|^p\,dx \\ & +C(\varepsilon)\int_\Omega e^{\beta|u-u_{2R}|}|(u-u_{2R})\nabla\eta|^p\,dx\Big\} \\ & +(\beta+C_3)\int_\Omega |u-u_{2R}|e^{\beta|u-u_{2R}|}|\eta\nabla u|^p\,dx. \end{aligned} \tag{3.6}$$

Let β satisfy $C_1\beta > \beta + C_3$, it yields

$$\begin{aligned} & \int_\Omega e^{\beta|u-u_{2R}|}|\eta\nabla u|^p\,dx \\ \le & C\varepsilon\int_\Omega e^{\beta|u-u_{2R}|}|\eta\nabla u|^p\,dx \\ & +C(\varepsilon)\int_\Omega e^{\beta|u-u_{2R}|}|(u-u_{2R})\nabla\eta|^p\,dx \end{aligned} \tag{3.7}$$

Due to the natural growth condition and $u \in L^\infty(\Omega)$, there exists a large enough positive constant M , such that $|u| \le M$, it follows that

$$\begin{aligned} \int_{B_R}|\nabla u|^p\,dx \le & C\varepsilon\int_{B_{2R}}|\nabla u|^p\,dx \\ & +\frac{C(\varepsilon)}{R^P}\int_{B_{2R}}|u-u_{2R}|^p\,dx. \end{aligned}$$

Selecting a small enough constant $0<\varepsilon<1$, and recalling the Lemma 2.3 , it yields

$$\int_{B_R}|\nabla u|^p\,dx \le C\int_{B_{2R}}|u-u_{2R}|^p\,dx\,,$$

where $C = C(C_1, C_2, C_3, \beta, p, M)$. This completes the proof of Theorem 1.1 .

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