Solving Higher Index Differential Algebraic Equations By Partitioning Technique Multistep Method

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***Abstract*—In this paper we describe a new algorithm for solving higher index differential algebraic equations (DAEs) system. The stated system is transformed to first order system and that reduced to index one or zero. We developed method where the system is partitioned into stiff and non-stiff subsystem. The code is constructed and the resulting method is shown to be efficient in term computational effort and accuracy.**

Keywords—DAE; Mixed multistep; partitioning; index

I. Introduction

The general form of DAEs[1] is given by

$$F(t, y, y') = 0 \qquad (1)$$

with consistent initial values

$$y(x_0) = y_0, \qquad y'(x_0) = y_1$$

When $\partial F/\partial y'$ is non-singular, (1) is an ODEs. The existence of algebraic constraints on the variable is expressed by the singularity of the Jacobian matrix $\partial F/\partial y'$. They are more difficult to handle than ODEs due to the existence of algebraic equations[9]. The algebraic constraints may appear explicitly as in the system

$$y' = f(t, y, z) \qquad (2)$$

$$0 = l(t, y, z)$$

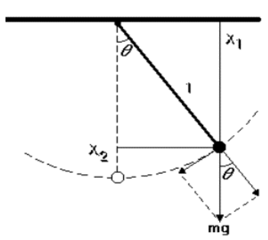
The system (2) is called a semi-explicit system of DAEs. A system of DAEs is characterized by its index, which is the number of differentiations required to convert it into a system of ODEs. Here the Jacobian matrix l_y is assumed to be non-singular for all t, therefore the system has index one. Since most DAEs arising in the applications are in semi-explicit form, then the technique to be developed is for solving DAEs in this form.

In solution of ODEs, partitioning of the equations into stiff and non-stiff subsystems has been very successful in terms of computational effort [7][8]. Some of the earlier works on partitioning are given in Enright and Kamel [10]. Other research on partitioning are discussed by Hall and Suleiman [4] and Hairer et al [3]. DAEs consist of algebraic equations which are treated as stiff [2][5]. However, the same cannot be said of the ODEs counterpart which consist of non-stiff and stiff subsystems that can be treated by non-stiff and stiff methods respectively. Therefore, in this paper we look into partitioned system that allows us switching from stiff to non-stiff when necessary so that substantial saving can be gained if this part is done efficiently.

For many dynamical modelling problems, the stated system has been transformed using the mechanics to a first order system. In addition to substituting for higher order derivatives to obtain a first order system, they may be changes made to the components of F that reduce the index to the value one or value zero. A further change is often made that replaces constraints for a system by their total derivatives. This change is done because it reduces the index of the problem. After a sufficient number of replacements of the constraints by their derivatives, a system is obtained such that it has index of value one or value zero.

II. Problem Formulation

PLANAR PENDULUM



In rectangular coordinate:

$$x_1'' = -\lambda x_1$$
$$x_2'' = g - \lambda x_2$$

Where g is the force of gravity and λ is a Lagrange multiplier. The terms represent the force which holds the solution onto the constraint:

$$x_1^2 + x_2^2 = 1$$

which expresses the condition that the rod has fixed length 1. After rewriting the two second order equations as four first order ODEs, a DAE system with four differential and one algebraic equations result:

$$x_1' = x_3$$
$$x_2' = x_4$$
$$x_3' = -\lambda x_1$$
$$x_4' = g - \lambda x_2$$
$$x_1^2 + x_2^2 = 1$$

The system above is an index three semi-explicit nonlinear DAE. By differentiating the algebraic constraint in the system, will lead to a new algebraic constraint.

$$0 = x_1 x_3 + x_2 x_4$$

The index of the system reduces from 3 to 2 by replacing the algebraic constraint by its derivative:

$$x_1' = x_3$$
$$x_2' = x_4$$
$$x_3' = -\lambda x_1$$
$$x_4' = g - \lambda x_2$$
$$0 = x_1 x_3 + x_2 x_4$$

Differentiating one more time resulting an algebraic constraint

$$0 = -\lambda + x_3^2 + x_4^2 - x_2 g$$

The ODEs with this new algebraic equation will form the DAE system of index 1.

$$x_1' = x_3$$
$$x_2' = x_4$$
$$x_3' = -\lambda x_1$$
$$x_4' = g - \lambda x_2$$
$$0 = -\lambda + x_3^2 + x_4^2 - x_2 g$$

The index of the system is reduced from index 1 to index 0 by differentiating again the algebraic constraint:

$$x_1' = x_3$$
$$x_2' = x_4$$
$$x_3' = -\lambda x_1$$
$$x_4' = -\lambda x_2 - g$$
$$\lambda' = 3x_4$$

The system is solved using the Adams method with Predict Evaluate Correct Evaluate (PECE) mode and Backward Difference Formula (BDF) on non-stiff and stiff equations respectively[6]. Initially, the ODEs are treated as non-stiff and solved with a variable stepsize variable order componentwise (VSVOC) Adam method. The equations have a transient phase that can be solved using simple iteration. Once the transient phase dies off, we switch to stiff method which is BDF method. The technique of placing the nonstiff equations into the stiff subsystem is called dynamic partitioning[8].

III. RESULTS AND DISCUSSION

For this experiment, the consistent initial conditions are as follows:

$$t = 0, \quad L = 1, \quad g = 1$$

$$x_1(0) = 1,\ x_2(0) = 0,\ x_3(0) = 0,\ x_4 = 1,\ \lambda(0) = 1$$

The above system is solved by the code that has been built using mix-multistep method (partitioning technique). The code can also solve ODEs order two directly without reducing to order one system. Table below shows the comparison result between partitioning technique and without using partitioning technique.

Table 1. Comparison of numerical result between without using partitioning technique and using partitioning technique for index 1 system with order 1 and order 2.

	without partitioning		partitioning	
Index	1	1	1	1
order	1	2	1	2
SS	176	545	141	142
FS	6	6	8	9
Total	182	551	149	151

Table 2. Comparison of numerical result between without using partitioning technique and using partitioning technique for index 0 system with order 1 and order 2.

	without partitioning		partitioning	
Index	0	0	0	0
order	1	2	1	2
SS	163	135	46	41
FS	0	1	0	0
Total	163	136	46	41

The true solution is given as below:

variables	solutions
x_1	0.1349949261
x_2	0.9908462897
x_3	-1.710951582
x_4	0.2331035488
λ	3.972538869

To measure the distance of the numerical solution from the constraints, the numerical solution at $t=1$ was substituted into the constraints:

$$G(1) = 1 - x_1^2 - x_2^2$$
$$G(2) = x_1 x_3 + x_2 x_4$$
$$G(3) = x_3^2 + x_4^2 - \lambda - x_3$$

RTOL	G(1)	G(2)	G(3)	STEP
10(-5)	2.61711(-7)	1.18481(-7)	2.2204(-16)	36
10(-6)	8.07846(-9)	-1.8023(-9)	-1.1102(-16)	42
10(-7)	-8.1576(-11)	1.3165(-10)	-2.1132(-16)	51
10(-8)	-4.98990(12)	1.4373(-11)	-1.4502(-16)	66
10(-9)	-3.3606(-13)	4.44283(-13)	6.3421(-16)	81
10(-10)	4.66294(-15)	1.12993(-13)	-1.1485(-16)	113
10(-11)	-8.8817(-16)	2.10942(-15)	-2.2113(-16)	149

IV. Conclusion

By using partitioning technique, the performance of the numerical solution is increase in term of number of step taken for the integration and also increases the accuracy of the solution and achieves acceptable result. The approach (using partitioning) is effective in term of computational effort since the Jacobian has smaller dimension hence requires less number of matrix operation in order to evaluate the Jacobian matrix and also increasing the accuracy. From this experiment we can see that solving index 0 systems gives superior results to those with index 1 system.

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References

[1] C.W.Gear, H.H.Hsu, L.Petzold. Differential Algebraic Equations Revisited. *Proc.ODE Meeting, Oberwolfach, West Germany.*1986

[2] E.Celik, M.Bayram, T.Yeloglu. Solution of Differential Algebraic Equations by Adomian Decompotion Method. *International Journal Pure & Applied Mathematical Sciences.* 3,93-100. 2006

[3] F.Hairer, S.P.Norsett, G.Warner. Solving Ordinary Differential Equations II. Stiff and Differential Equations*. Springer, New York.*1992

[4] G.Hall, M.Suleiman. A Single Code for the Solution of Stiff and Non-stiff ODEs , *SIAM J. Stat. Computing*, 6(3) 684-697. 1985

[5] K.E.Brennan, S.L.Campbell, L.Petzold. Numerical Solution of Initial Value Problems in Differential Algebraic Equations*. Elsevier, New York.*1989

[6] M.Suleiman. Generalized Multistep Adams and Backward Differentiation Methods for the solution of stiff and non-stiff Ordinary Differential Equations. *Phd. Thesis.*1979

[7] M.Suleiman,S.Baok. Using Non convergence of Iteration to Partition ODEs. *Applied Mathematics and Computation,* 49, 111-139. 1992

[8] M.Suleiman, I.Zarina, O.Iskandar. Partitioning Ordinary Differential Equations Using Multistep Formulas. World Academy of Science, Engineering and Technology, 40, 2008

[9] U.M.Ascher, L.Petzold. Projected implicit Runge-Kutta method for Differential Algebraic Equation. *SIAM J. Numer. Anal.,* 28, 1097-1120. 1991

[10] W.H.Enright, M.S.Kamel. Automatic Partitioning of Stiff System and Exploiting the Resulting Structure. ACM Transaction on Mathematical Software. Vol 5 , No 4, 374-385. 1979