Exact Analytical Solution for MHD Flow and Heat Transfer of Jeffrey Fluid over a Stretching Sheet with Viscous Dissipation

[1]Wahid, N. S., [2*]Hafidzuddin, M. E. H. [1,3]Arifin, N. M., [4]Turkyilmazoglu, M., [1]Rahmin, N. A. A.
[1]Department of Mathematics, Universiti Putra Malaysia, Serdang Selangor 43400 UPM, Malaysia
[2]Centre of Foundation Studies For Agricultural Science, Universiti Putra Malaysia, Serdang Selangor 43400 UPM, Malaysia
[3]Institute for Mathematical Research, Universiti Putra Malaysia, Serdang Selangor 43400 UPM, Malaysia
[4]Mathematics Department, University of Hacettepe, Beytepe, Ankara, Turkey
*ezadhafidz@upm.edu.my

***Abstract*—The magnetohydrodynamic (MHD) flow and heat transfer of Jeffrey fluid over a stretching sheet with the presence of viscous dissipation is studied analytically. The governing partial differential equations are transformed to non-linear ordinary differential equations with the facilitation of sophisticated similarity variables, which are then being solved analytically by using exact analytical method. The effects of the physical parameters on the velocity and temperature distributions are presented through graphs, discussed and compared with the previous work of the same problem that has been solved numerically to verify the present analytical method being used. The skin friction coefficient and the local Nusselt number are also computed and analyzed.**

Keywords—Magnetohydrodynamic; Jeffrey fluid; Viscous dissipation; Stretching sheet; Analytical solution.

I. INTRODUCTION

Nowadays, the study of boundary layer flows of non-Newtonian fluids has been the topic of great interest to investigators and researchers, as it is known to be so significant in real life application. Jeffrey model is one kind of the non-Newtonian fluids, which is a relatively simpler linear model that used time derivatives instead of convected derivatives, which are used by most fluid models. Until now, the investigation towards this fluid flow is still being studied by the researchers in various aspects either by the implementation of numerical method or analytical method.

Nadeem et al. [1] have analytically analyzed the boundary layer flow of a Jeffrey fluid over an exponentially stretching surface by taking concern of thermal radiation towards two cases of heat transfer. Later, by implementing the same method, three-dimensional shrinking flow of Jeffrey fluid in a rotating system is presented by Hayat et al. [2] with the consideration of magnetic field. Also, the exact solution of the combined effect of heat and mass transfer in Jeffrey fluid over a stretching sheet in the presence of heat source or sink has been investigated by Qasim [3]. Moreover, by employing the exact method also, Turkyilmazoglu and Pop [4] has investigated unique and multiple solutions of the flow and heat transfer of a Jeffrey fluid near the stagnation point on a both stretching and shrinking sheet. Then, Hussain et al. [5] performed an analysis on the flow of Jeffrey fluid over exponentially stretching sheet with the consideration of various effects such as thermophoresis, Brownian motion, thermal radiation, and viscous dissipation effects. Some of the recent studies about Jeffrey fluid can be found in [6]-[8].

The main interest of this work is to solve the mathematical model for magnetohydrodynamic (MHD) boundary layer flow and heat transfer of Jeffrey fluid past a linear stretching sheet with the presence of viscous dissipation by using exact analytical method as the extension to [9], that has solved the problem numerically by using a finite-difference method, namely the Keller-box method. The profiles are plotted and discussed for the variations of different involved parameters such as magnetic parameter, Prandtl number, Eckert number, and Deborah number. The skin friction coefficient and local Nusselt number have been computed and analyzed through the tabulation of data and then being compared to the previous results to verify the validity of the technique and the method that we used in this paper.

II. METHODOLOGY

A. Modeling

We consider a steady two-dimensional (2D) laminar boundary layer flow of incompressible, electrically conducting Jeffrey fluid with the presence of magnetohydrodynamic (MHD) over a stretching sheet with viscous dissipation. Here *x*-axis is chosen in the direction of the sheet motion, while *y*-axis normal to *x*-axis. Let $u_w(x) = cx$ describes the stretching velocity along the *x*-direction. A uniform transverse magnetic field of strength B_0 is applied parallel to the *y*-axis. The boundary layer equations governing the flow and heat transfer of Jeffrey fluid can be written as follows [9]:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{1}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{\nu}{1+\lambda_1}\times\left[\frac{\partial^2 u}{\partial y^2}+\lambda_2\left(u\frac{\partial^3 u}{\partial x\partial y^2}+v\frac{\partial^3 u}{\partial y^3}+\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x\partial y}-\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2}\right)\right]-\frac{\sigma B_0^2 u}{\rho}, \tag{2}$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\left(\frac{\partial^2 T}{\partial y^2}\right)+\frac{\mu}{\rho C_p}\left(\frac{\partial u}{\partial y}\right)^2+\frac{\sigma B_0^2 u^2}{\rho C_p}, \tag{3}$$

with a set of initial and boundary conditions,

$$u=u_w,\quad v=0,\quad T=T_w=T_\infty+A\left(\frac{x}{L}\right)^2 \text{ at } y=0,$$

$$u\to 0,\quad \frac{\partial u}{\partial y}\to 0,\quad T\to T_\infty \text{ as } y\to\infty, \tag{4}$$

where u and v are the velocity components in x- and y- directions respectively, while $\nu=\frac{\mu}{\rho}$ is the kinematic viscosity, where μ is the coefficient of fluid viscosity and ρ is the fluid density. Moreover, σ is for fluid electrical conductivity, B_0 for uniform magnetic field, λ_1 is the ratio of relaxation and retardation times which held fixed at zero, λ_2 is the relaxation time, T for temperature, $\alpha=\frac{\kappa}{\rho C_p}$ stands for thermal diffusivity where C_p is specific heat at constant pressure and κ is the thermal conductivity. Further, T_w, T_∞, A and L stand for constant surface temperature, ambient fluid temperature, a constant and characteristic length.

By using similarity transformation,

$$u=cxf'(\eta),\ v=-(c\nu)^{\frac{1}{2}}f(\eta),$$

$$\eta=\left(\frac{c}{\nu}\right)^{\frac{1}{2}}y,\qquad \theta(\eta)=\frac{T-T_\infty}{T_w-T_\infty}, \tag{5}$$

the continuity equation (1) is automatically satisfied and equation (2), (3) become

$$f'''-(1+\lambda_1)[f'^2-ff'']+\beta(f''^2-ff^{iv})-(1+\lambda_1)[Mf']=0, \tag{6}$$

$$\theta''+Pr(f\theta'-2f'\theta)+PrEcf''^2+MPrEcf'^2=0, \tag{7}$$

subject to the new transformed initial and boundary conditions,

$$f=0,\ f'=1,\ \theta=1,\text{ at } \eta=0,$$
$$f'\to 0,\ f''\to 0,\ \theta\to 0,\text{ as } \eta\to\infty, \tag{8}$$

where f and θ are the dimensionless stream function and the dimensionless temperature, accordingly. The prime denotes differentiation with respect to η. Noted that $\beta=c\lambda_2$ is the Deborah number, $\mathrm{Ec}=\frac{c^2l^2}{AC_p}$ is the Eckert number, $M=\frac{\sigma B_0^2}{\rho c}$, stands for the magnetic parameter and $\mathrm{Pr}=\frac{\mu c_p}{k}$ is for the Prandtl number.

According to [9], the skin friction coefficient and local Nusselt number through dimensionless scale are as the following, respectively.

$$\frac{1}{2}Re_x^{\frac{1}{2}}C_f=f''(0), \tag{9}$$

$$Re_x^{-\frac{1}{2}}Nu_x=-\theta'(0), \tag{10}$$

in which $Re_x=u_w x/\nu$ depicts the local Reynolds number.

B. *Exact Analytical Method*

A numerical solution of (6) and (7) has already been given in [9]. In this section, we instead present the exact analytical solution.

a. *Velocity field*

As according to Crane [10], nice exact formula representing the boundary layer flow in different configurations can be obtained where we can assume that (6) possesses solution of exponential type

$$f(\eta)=\frac{1}{\lambda}\left(1-e^{-\lambda\eta}\right). \tag{11}$$

By using the boundary conditions and some steps of substitution, as a result, we obtain the value of λ and the exact velocity field function $f'(\eta)$ as shown below

$$\lambda=\sqrt{\frac{-(1+\lambda_1)(-1-M)}{1+\beta}},\qquad \lambda>0, \tag{12}$$

$$f'(\eta)=e^{-\sqrt{\frac{-(1+\lambda_1)(-1-M)}{1+\beta}}\eta}. \tag{13}$$

b. *Temperature field*

As referring to [11] and [12], the exact solution for temperature field can be obtained. We suppose the following transformation, where $t=e^{-\lambda\eta}$ and we then propose the following relations between the derivatives with respect to η and t,

$$\frac{d}{d\eta}\theta=-\lambda t\frac{d}{dt}\theta,\qquad \frac{d^2}{d\eta^2}\theta=\lambda^2\left[t^2\frac{d^2}{dt^2}\theta+t\frac{d}{dt}\theta\right] \tag{14}$$

By applying the transformation, we obtain

$$t\theta''(t)+(n-mt)\theta'(t)+2m\theta(t)+\mathrm{Pr}\,\mathrm{Ec}\,t-m\,M\,Ec\,t=0, \tag{15}$$

associated to the boundary conditions

$$\theta(0)=0,\qquad \theta(1)=1. \tag{16}$$

Therefore, solving the ordinary differential equation (15), we obtain the exact solution for temperature field

$$\theta(\eta) = \frac{1}{2}\frac{\left(m^2(e^{-\lambda\eta})^2 - 2m\,e^{-\lambda\eta}(n+1) + n^2 + n\right)\mathrm{Ec}(M\,m - \mathrm{Pr})}{m^2(n+1)} + \frac{1}{2}\frac{\mathrm{Ec}(M\,m - \mathrm{Pr})\left(2m\,e^{-\lambda\eta} - n\right)}{m^2} - \frac{1}{2}H\frac{\left(e^{-\lambda\eta}\right)^{-n+1}(\mathrm{Ec}\,M\,m - \mathrm{Ec}\,\mathrm{Pr} - 2n - 2)}{(n+1)}, \quad (17)$$

where

$$H = \frac{hypergeometric\left([-1-n],[-n+2],m\,e^{-\lambda\eta}\right)}{hypergeometric([-1-n],[-n+2],m)},$$
$$n = 1 - \frac{\mathrm{Pr}}{\lambda^2}, \text{ and } m = -\frac{\mathrm{Pr}}{\lambda^2}. \quad (18)$$

III. RESULT AND DISCUSSION

To facilitate the analysis process, we have used Maple software to calculate and solve the equation, as well as plotting the graph. So, in this section, we will discuss and analyze the results that we obtained, which we will explore the effects of various influential parameters on the velocity $f'(\eta)$ and temperature $\theta(\eta)$ distributions for Jeffrey fluid. We found out that it gives good agreement and well coincide with the numerical solution provided by [9].

TABLE 1. COMPARISON OF SKIN FRICTION COEFFICIENT

M	β	$f''(0)$	
		Ahmad & Ishak [9]	**Present**
0.2	0	-1.0955	-1.0954
	0.5	-0.8945	-0.8944
	1.5	-0.6930	-0.6928

Impacts of Deborah number β and magnetic parameter M on the non-dimensional velocity field $f'(\eta)$ are plotted in Figs. 1 and 2 respectively. Fig. 1 illustrates the impact of Deborah number β on velocity field $f'(\eta)$ as well as the magnetic parameter M, meanwhile Fig. 2 depicts the relation of velocity field with Deborah number but with constant value of M. It is noted that the velocity field increases as the value of Deborah number β increases, but opposite behavior occurs for magnetic parameter M.

Moreover, impacts of Deborah number β, magnetic parameter M, Prandtl number Pr and Eckert number Ec on the non-dimensional temperature field $\theta(\eta)$ are plotted in Figs. 3-5 respectively. Fig. 3 displays the effects of Deborah number β and magnetic parameter M to temperature field $\theta(\eta)$, and it can be concluded that the increment of Deborah number β results in decrement of temperature field $\theta(\eta)$. Also, the magnetic parameter M enhances the temperature field $\theta(\eta)$. The impact of Prandtl number Pr on temperature is shown in Fig. 4, where the high value of Prandtl number Pr decays the temperature field $\theta(\eta)$. Meanwhile, the high value of Eckert number Ec which is the parameter for viscous dissipation enhances the temperature field $\theta(\eta)$ as plotted in Fig. 5.

Variations of skin friction coefficient $f''(0)$ with Deborah number β for various value of magnetic parameter M are shown in Table 1 and Table 2 in the tabulation of numerical data. As have been mentioned earlier, it can be seen in Table 1, that the values of the skin friction coefficient $f''(0)$ of present study are well coincide with the previous result obtained numerically by [9]. Also, from Table 2, it can be seen that the higher the value of magnetic parameter M, the lower the value of skin friction coefficient $f''(0)$ and the higher the value of Deborah number β, the higher the value of skin friction coefficient $f''(0)$.

The variations of Nusselt number $-\theta'(0)$ also being analyze in the form of tabulation data in Table 3, 4 and 5. It is noticed in Table 3 and Table 4 that the increment of Deborah number β increases the value of Nusselt number $-\theta'(0)$, meanwhile, the increase in Eckert number Ec and magnetic parameter M lessen the local Nusselt number $-\theta'(0)$. It is also depicted in Table 5 that the increment of Prandtl number Pr increases the value of Nusselt number $-\theta'(0)$.

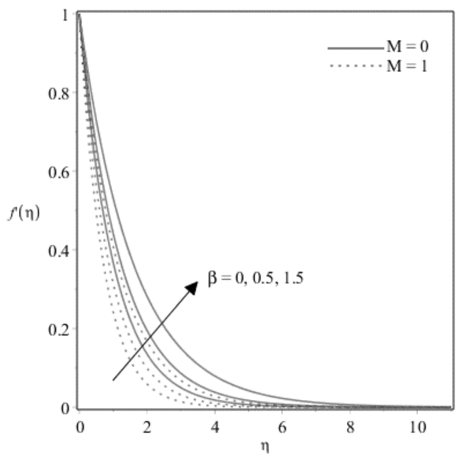

Fig. 1. *Plots of velocity field for Deborah number and magnetic field parameter.*

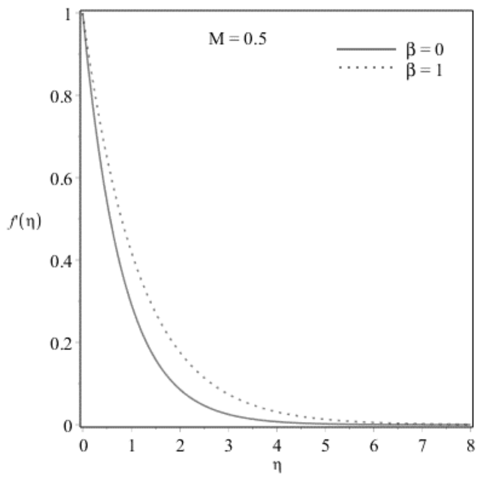

Fig. 2. *Plots of velocity field for Deborah number and constant magnetic field parameter M.*

ISSN: 2458-9403
Vol. 6 Issue 12, December - 2019, Special Issue

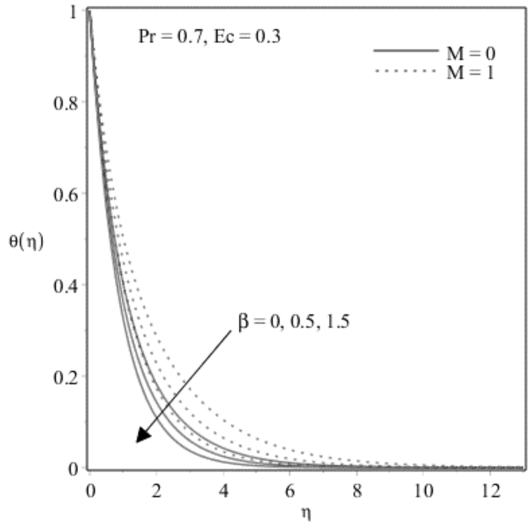

Fig. 3. *Plots of temperature field for Deborah number and magnetic parameter.*

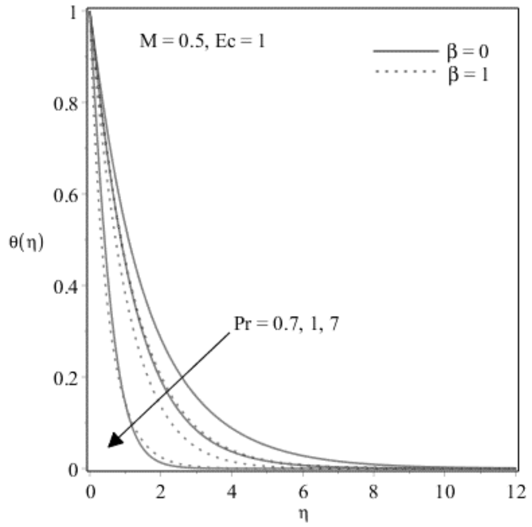

Fig. 4. *Plots of temperature field for Prandtl number and Deborah number.*

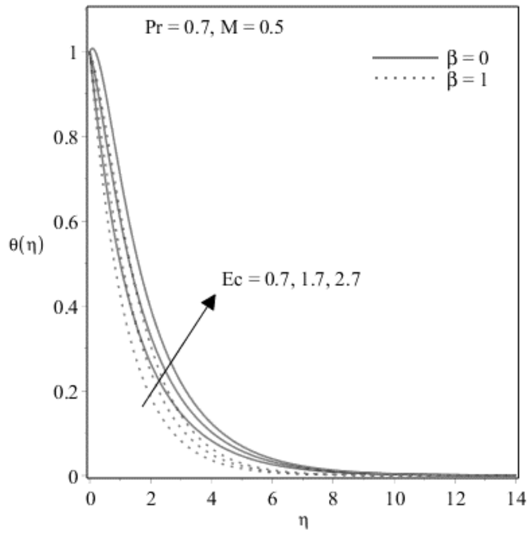

Fig. 5. *Plots of temperature field for Eckert number and Deborah number.*

TABLE 2. SKIN FRICTION COEFFICIENT

M	β	$f''(0)$
0.0	0.5	-0.8164965809
	1	-0.7071067810
	1.5	-0.6324555320
0.5	0.5	-0.9999999996
	1	-0.8660254033
	1.5	-0.7745966690
1	0.5	-1.1547005380
	1	-1.0000000000
	1.5	-0.8944271907

TABLE 1. NUSSELT NUMBER WHEN $\Pr = 0.7,\ M = 0.2$

β	Ec	$-\theta'(0)$
0.5	1.0	0.831095487
	3.0	0.294750507
	5.0	-0.241594475
1	1.0	0.899802780
	3.0	0.433573661
	5.0	-0.032655455
1.5	1.0	0.944983682
	3.0	0.523793489
	5.0	0.102603295

TABLE 2. NUSSELT NUMBER WHEN $\Pr = 0.7,\ \mathrm{Ec} = 0.3$

M	β	$-\theta'(0)$
0.0	0.5	1.064638764
	1.0	1.105689690
	1.5	1.133114735
0.5	0.5	0.956566211
	1.0	1.004782562
	1.5	1.036757584
1.0	0.5	0.863316352
	1.0	0.918981117
	1.5	0.954454832

TABLE 5. NUSSELT NUMBER WHEN $\beta = 1,\ \mathrm{Ec} = 0.3$

M	Pr	$-\theta'(0)$
1.0	0.7	0.918981117
	1.0	1.133333333
	1.5	1.421445057

IV. CONCLUSION

The boundary layer flow and heat transfer of Jeffrey fluid in the presence of magnetohydrodynamic (MHD) over a stretching sheet with viscous dissipation has been discussed. The governing equations were reduced to the ordinary differential equations by using appropriate similarity transformation variables and these ordinary differential equations were then further solved analytically by using exact analytical method.

From the results obtained, it can be seen that the present analytical solution show a good agreement as to be compared with the previous numerical solution presented by [9].

ACKNOWLEDGMENT

This work is supported by Putra Grant 9619000 from Universiti Putra Malaysia.

REFERENCES

[1] S. Nadeem, S. Zaheer, and T. Fang, "Effects of thermal radiation on the boundary layer flow of a Jeffrey fluid over an exponentially stretching surface," *Numer. Algorithms*, vol. 57, no. 2, pp. 187–205, 2011.

[2] T. Hayat, M. Awais, S. Asghar, and A. A. Hendi, "Analytic Solution for the Magnetohydrodynamic Rotating Flow of Jeffrey Fluid in a Channel," *J. Fluids Eng.*, vol. 133, no. 6, p. 061201, 2011.

[3] M. Qasim, "Heat and mass transfer in a Jeffrey fluid over a stretching sheet with heat source/sink," *Alexandria Eng. J.*, vol. 52, no. 4, pp. 571–575, 2013.

[4] M. Turkyilmazoglu and I. Pop, "Exact analytical solutions for the flow and heat transfer near the stagnation point on a stretching/shrinking sheet in a Jeffrey fluid," *Int. J. Heat Mass Transf.*, vol. 57, no. 1, pp. 82–88, 2013.

[5] T. Hussain, S. A. Shehzad, T. Hayat, A. Alsaedi, F. Al-Solamy, and M. Ramzan, "Radiative hydromagnetic flow of Jeffrey nanofluid by an exponentially stretching sheet," *PLoS One*, vol. 9, no. 8, pp. 1–9, 2014.

[6] N. A. Mohd Zin, I. Khan, and S. Shafie, "Exact and numerical solutions for unsteady heat and mass transfer problem of Jeffrey fluid with MHD and Newtonian heating effects," *Neural Comput. Appl.*, vol. 30, no. 11, pp. 3491–3507, 2018.

[7] K. Mahmood, M. N. Sadiq, M. Sajid, and N. Ali, "Heat transfer in stagnation-point flow of a Jeffrey fluid past a lubricated surface," *J. Brazilian Soc. Mech. Sci. Eng.*, vol. 41, no. 2, pp. 1–9, 2019.

[8] K. Ramesh and V. Joshi, "Numerical Solutions for Unsteady Flows of a Magnetohydrodynamic Jeffrey Fluid Between Parallel Plates Through a Porous Medium," *Int. J. Comput. Methods Eng. Sci. Mech.*, vol. 0, no. 0, pp. 1–13, 2019.

[9] K. Ahmad and A. Ishak, "MHD flow and heat transfer of a Jeffrey fluid over a stretching sheet with viscous dissipation," *Malaysian J. Math. Sci.*, vol. 10, pp. 311–323, 2016.

[10] L. J. Crane, "Flow past a stretching plate," *Zeitschrift für Angew. Math. und Phys. ZAMP*, vol. 21, no. 4, pp. 645–647, Jul. 1970.

[11] E. H. Aly and A. Ebaid, "Exact analysis for the effect of heat transfer on MHD and radiation Marangoni boundary layer nanofluid flow past a surface embedded in a porous medium," *J. Mol. Liq.*, vol. 215, pp. 625–639, 2016.

[12] A. Ebaid and H. S. Alhawiti, "New Application for the Generalized Incomplete Gamma Function in the Heat Transfer of Nanofluids via Two Transformations," *J. Comput. Eng.*, vol. 2015, pp. 1–6, 2015.