Journal of Multidisciplinary Engineering Science and Technology (JMEST)
ISSN: 2458-9403
Vol. 6 Issue 4, April - 2019

Remark On The Global Integrability For Very Weak Solutions To Elliptic Boundary Value Problems

YANXIA ZHOU
College of Science
North China University of Science and Technology
Hebei Tangshan, China

YAQI YANG
College of Science
North China University of Science and Technology
Hebei Tangshan, China

XIUJUAN XU*
College of Science
North China University of Science and Technology
Hebei Tangshan, China
*Corresponding Author
xxjluck@126.com

***Abstract*—The very weak solution to elliptic boundary value problems is considered. By making use of the Hodge decomposition and other tools, the global integrability to elliptic boundary value problems is established.**

Keywords—sintegrability; very weak solution; boundary value problem; p-harmonic equation

I. INTRODUCTION AND PRELIMINARY LEMMAS

Throughout this paper Ω will stand for a bounded regular domain in $\mathbb{R}^n (n \geq 2)$. By a regular domain we understand any domain of finite measure for which the estimates (2.4) and (2.5) below for the Hodge decomposition are satisfied, see [10], [11]. A Lipschitz domain, for example, is regular.

Let 1 < p < n. We shall examine the boundary value problem of the p- harmonic equation

$$\begin{cases} -div(|\nabla u(x)|^{p-2}\nabla u(x)) = -divf(x), & x \in \Omega, \\ u(x) = \theta(x), & x \in \partial\Omega. \end{cases} \tag{1.1}$$

Where $\theta(x) \in W^{1,q}(\Omega), q > r.$

This paper deals with very weak solutions to (1.1).

Definition1.1. A function $u \in \theta + W_0^{1,r}(\Omega)$, $\max\{1, p-1\} < r < p-1$, is called a very weak solution to the boundary value problem (1.1) if

$$\int_\Omega \langle |\nabla u|^{p-2}\nabla u, \nabla\varphi\rangle dx = \int_\Omega \langle f(x), \nabla\varphi\rangle dx. \tag{1.2}$$

holds true for all $\varphi \in W_0^{1,r/(r-p+1)}(\Omega)$.

A function $u \in \theta + W_0^{1,p}(\Omega)$ is called a weak solution to the boundary value problem (1.1) if (1.2) holds true for all $\varphi \in W_0^{1,p}(\Omega)$. This is natural setting of problem (1.1). The words very weak in Definition 1.1 mean that the integrable exponent r of u can be smaller than the natural one p. We refer the readers to [1]. Theorem 1, page602, and [3], Theorems 1 and 2, page 251.

In this paper we will need the definition of weak $L^t - spaces$ or Marcinkiewicz space (see [4] , Chapter 1, Section 2): for $t > 0$, the weak $L^t - space$, $L^t_{weak}(\Omega)$, consists of all Measurable functions f such that

$$\left|\{x \in \Omega : |f(x)| > s\}\right| \leq \frac{k}{s^t}$$

for some positive constant $k = k(f)$ and every $s > 0$, where $|E|$ is the $n-$dimensional Lebesgue measure of E. Note that if $f \in L^t_{weak}(\Omega)$ for some $t > 1$, then $f \in L^\tau(\Omega)$ for every $1 \leq \tau < t$.

Integrability is a very important property in the regularity of nonlinear elliptic PDEs and systems, see [5]. In [6], [7], the authors considered regularity properties of the $p-$harmonic type equations

$$-div\left(|\nabla u|^{p-2}\nabla u(x)\right) = -divf,$$

with r sufficiently close to p, and obtained an estimate for the operator H which carries a given vector function f into the gradient field ∇u. In the present paper, we consider very weak solutions to boundary value problems of (1.1). The main result of this paper is the following theorem.

Theorem 1.1. let $\theta \in W^{1,q}(\Omega), q > r$. There exists $\varepsilon_0 = \varepsilon_0(n, p) > 0$ such that for every very weak solution $u \in \theta + W_0^{1,r}(\Omega)$, $\max\{1, p-1\} < r < p < n$, to the boundary value problem (1.1), we have

$$u \in \begin{cases} \theta + L^{q^*}_{weak}(\Omega), & for\ q < n, \\ \theta + L^{\tau}_{weak}(\Omega), & for\ q = n\ and\ any\ \tau < \infty, \\ \theta + L^{\infty}(\Omega), & for\ q > n, \end{cases} \tag{1.3}$$

provided that $|p - r| < \varepsilon_0$.

Note that very weak solutions u to the boundary value problem (1.1) are taken from the Sobolev space $W^{1,r}(\Omega)$. The embedding theorem guarantees that the integrability of u reaches r^*. Our result (1.3) improves such integrability. We remark that the key point in the proof of Theorem 1.1 is the choice of appropriate test functions.

In order to prove Theorem 1.1, we need the following two lemmas.

Lemma1.1. For $1<p<2$ and any $X,Y\in \mathbb{R}$, one has

$$\langle |X|^{p-2}X-|Y|^{p-2}Y, X-Y\geq |X-Y|\left((|X-Y|+|Y|)^{p-1}-|Y|^{p-1}\right).$$

The proof of can be found in [8], Lemma 1.1.

Lemma 1.2. Let $s_0>0$ and let $\phi:(s_0,\infty)\to[0,\infty)$ be a decreasing function such that for every r,s with $r>s>s_0$,

$$\phi(r)\leq \frac{c}{(r-s)^{\alpha}}(\phi(s))^{\beta},$$

where c,α,β are positive constants. Then

(1) If $\beta>1$ we have that $\phi(s_0+d)=0$, where

$$d^{\alpha}=c2^{\alpha\beta/(\beta-1)}(\phi(s_0))^{\beta-1};$$

(2) If $\beta<1$ we have that

$$\phi(s)\leq 2^{u/(1-\beta)}\left(c^{1/(1-\beta)}+(2s_0)^{u}\phi(s_0)\right)s^{-u},$$

where $u=\alpha/1-\beta$.

II. PROOF OF THEOREM 1.1

For any $L>0$ we take

$$v=\begin{cases} u-\theta+L & for \quad u-\theta<-L,\\ 0 & for \quad -L\leq u-\theta\leq L,\\ u-\theta-L & for \quad u-\theta>L,\end{cases} \tag{2.1}$$

so that, by our assumptions, we have $v\in W_0^{1,r}(\Omega)$ and

$$\nabla v=(\nabla u-\nabla\theta)\cdot 1_{\{|u-\theta|>L\}}, \tag{2.2}$$

where 1_E is the characteristic function for the set E, that is, $1_E=1$ if $x\in E$ and $1_E=0$ otherwise. We introduce the Hodge decomposition of the vector field $|\nabla v|^{p-2}\nabla v\in L^{1,r/(r-p+1)}(\Omega)$. Accordingly,

$$|\nabla v|^{r-p}\nabla v=\nabla\phi+h, \tag{2.3}$$

where ϕ is in $W_0^{1,r/(r-p+1)}(\Omega)$ and h is a divergence free vector field of class $L^{r/(r-p+1)}(\Omega,\mathbb{R}^n)$. The reader is referred to [3], [1] for estimates concerning such decomposition. We have

$$\|\nabla\phi\|_{r/r-p+1}\leq C(n,p)\|\nabla v\|_r^{r-p+1} \tag{2.4}$$

and

$$\|h\|_{r/(r-p+1)}\leq C(n,p)|p-r|\|\nabla v\|_r^{r-p+1} \tag{2.5}$$

Let's take ϕ as the test function in the formula, on the left-hand side, we have

$$\int_{\{|u-\theta|>L\}}\langle|\nabla u|^{p-2}\nabla u-|\nabla\theta|^{p-2}\nabla\theta,h>dx+\int_{\{|u-\theta|>L\}}\langle|\nabla\theta|^{p-2}\nabla\theta,h\rangle dx$$
$$-\int_{\{|u-\theta|>L\}}\langle|\nabla\theta|^{p-2}\nabla\theta,|\nabla u-\nabla\theta|^{r-p}(\nabla u-\nabla\theta)\rangle dx \tag{2.6}$$

Therefore, equation (1.2) can be transformed into

$$\begin{aligned}&\int_{\{|u-\theta|>L\}}\langle|\nabla u|^{p-2}\nabla u-|\nabla\theta|^{p-2}\nabla\theta,|\nabla u-\nabla\theta|^{r-p}(\nabla u-\nabla\theta)\rangle dx\\ &=\int_{\{|u-\theta|>L\}}\langle|\nabla u|^{p-2}\nabla u-|\nabla\theta|^{p-2}\nabla\theta,h\rangle dx\\ &+\int_{\{|u-\theta|>L\}}\langle|\nabla\theta|^{p-2}\nabla\theta,h\rangle dx\\ &-\int_{\{|u-\theta|>L\}}\langle|\nabla\theta|^{p-2}\nabla\theta,|\nabla u-\nabla\theta|^{r-p}(\nabla u-\nabla\theta)\rangle dx\\ &+\int_{\{|u-\theta|>L\}}\langle f(x),\nabla\phi\rangle dx\\ &=I_1+I_2+I_3+I_4\end{aligned} \tag{2.7}$$

Now let's talk about two cases.

Case1 : $p\geq 2$. Since for any $X,Y\in\mathbb{R}^n$ (see [9], page 72)

$$2^{2-p}|X-Y|^{p}\leq\langle|X|^{p-2}X-|Y|^{p-2}Y,X-Y\rangle,$$

the left-hand side of (2.6) can be estimated as

$$\begin{aligned}&\int_{\{|u-\theta|>L\}}\langle|\nabla u|^{p-2}\nabla u-|\nabla\theta|^{p-2}\nabla\theta,|\nabla u-\nabla\theta|^{r-p}(\nabla u-\nabla\theta)\rangle dx\\ &\geq 2^{2-p}\int_{\{|u-\theta|>L\}}|\nabla u-\nabla\theta|^{r}dx\end{aligned} \tag{2.8}$$

We now estimate $|I_1|$, $|I_2|$ and $|I_3|$. By an elementary inequality (see [10]): for any $X,Y\in\mathbb{R}^n$ and $\varepsilon>0$,

$$\left||X|^{\varepsilon}X-|Y|^{\varepsilon}Y\right|\leq\begin{cases}(1+\varepsilon)\left(|Y|+|X-Y|^{\varepsilon}\right)|X-Y| & for\ \varepsilon>0,\\ \dfrac{1-\varepsilon}{2^{\varepsilon}(1+\varepsilon)}|X-Y|^{1+\varepsilon} & for -1<\varepsilon\leq 0,\end{cases} \tag{2.9}$$

and using Hölder inequality, (2.5) and Young inequality, we obtain

$$\begin{aligned}|I_1|&\leq(p-1)\int_{\{|u-\theta|>L\}}\langle|\nabla\theta|+|\nabla u-\nabla\theta|^{p-2}|\nabla u-\nabla\theta||h|dx\\ &\leq 2^{p-2}(p-1)(\int_{\{|u-\theta|>L\}}|\nabla\theta|^{p-2}|\nabla u-\nabla\theta||h|dx\\ &\quad+\int_{\{|u-\theta|>L\}}|\nabla u-\nabla\theta|^{p-1}|h|dx)\\ &\leq 2^{p-2}(p-1)(\|\nabla\theta\|_r^{p-2}\|\nabla u-\nabla\theta\|_r\|h\|_{r/(r-p+1)}\\ &\quad+\|\nabla u-\nabla\theta\|_r^{p-1}\|h\|_{r/(r-p+1)})\\ &\leq 2^{p-2}(p-1)C(n,p)|p-r|\left(\|\nabla\theta\|_r^{p-2}\|\nabla u-\nabla\theta\|_r^{r-p+2}+\|\nabla u-\nabla\theta\|_r^{r}\right)\\ &\leq 2^{p-2}(p-1)C(n,p)|p-r|\left(C(\varepsilon)\|\nabla\theta\|_r^{r}+(1+\varepsilon)\|\nabla u-\nabla\theta\|_r^{r}\right);\end{aligned} \tag{2.10}$$

here and in the sequel, $\|\cdot\|_r=\|\cdot\|_{r,\{|u-\theta|>L\}}$, we omit the subscript for the sake of simplicity.

Using the Hölder inequality, (2.5) and Young's inequality again, $|I_2|$ and $|I_3|$ can be estimated as

$$\begin{aligned}|I_2| &= \left|\int_{\{|u-\theta|>L\}} \langle |\nabla\theta|^{p-2}\nabla\theta, h\rangle dx\right| \\ &\le \int_{\{|u-\theta|>L\}} |\nabla\theta|^{p-1}|h|dx \\ &\le \|\nabla\theta\|_r^{p-1}\|h\|_{r/(r-p+1)} \\ &\le C(n,p)|p-r|\|\nabla\theta\|_r^{p-1}\|\nabla u-\nabla\theta\|_r^{r-p+1} \\ &\le C(n,p)|p-r|\left[C(\varepsilon)\|\nabla\theta\|_r^r+\varepsilon\|\nabla u-\nabla\theta\|_r^r\right],\end{aligned} \tag{2.11}$$

$$\begin{aligned}|I_3| &= \left|-\int_{\{|u-\theta|>L\}} \langle |\nabla\theta|^{p-2}\nabla\theta, |\nabla u-\nabla\theta|^{r-p}(\nabla u-\nabla\theta)\rangle dx\right| \\ &\le \int_{\{|u-\theta|>L\}} |\nabla\theta|^{p-1}|\nabla u-\nabla\theta|^{r-p+1}dx \\ &\le \|\nabla\theta\|_r^{p-1}\|\nabla u-\nabla\theta\|_r^{r-p+1} \\ &\le C(\varepsilon)\|\nabla\theta\|_r^r+\varepsilon\|\nabla u-\nabla\theta\|_r^r.\end{aligned} \quad (2.12)$$

Using the Hölder inequality, (2.5), (2.2) and Young's inequality again, $|I_4|$ can be estimated as

$$\begin{aligned}|I_4| &= \left|\int_{\{|u-\theta|>L\}} \langle f(x), \nabla\phi\rangle dx\right| \\ &\le \int_{\{|u-\theta|>L\}} |f(x)||\nabla\phi|dx \\ &= \int_{\{|u-\theta|>L\}} |f(x)|\cdot\left(|\nabla v|^{r-p}\nabla v-h\right)dx \\ &\le \int_{\{|u-\theta|>L\}} |f(x)|\cdot|\nabla v|^{r-p+1}dx+\int_{\{|u-\theta|>L\}} |f(x)|\cdot|h|dx \\ &= \int_{\{|u-\theta|>L\}} |f(x)|\cdot|\nabla u-\nabla\theta|^{r-p+1}dx+\int_{\{|u-\theta|>L\}} |f(x)|\cdot|h|dx \\ &\le C(\varepsilon)\left(\int_{\{|u-\theta|>L\}} |f(x)|^{\frac{r}{p-1}}dx\right)+C\varepsilon\int_{\{|u-\theta|>L\}} |\nabla u-\nabla\theta|^r dx \\ &\quad +C(\varepsilon)\int_{\{|u-\theta|>L\}} |f(x)|^{\frac{r}{p-1}}dx+c\cdot\varepsilon\int_{\{|u-\theta|>L\}} |h|^{\frac{r}{r-p+1}}dx \\ &\le C(\varepsilon)\int_{\{|u-\theta|>L\}} |f(x)|^{\frac{r}{p-1}}dx+c\cdot\varepsilon\int_{\{|u-\theta|>L\}} |h|^{\frac{r}{r-p+1}}dx \\ &\le C(\varepsilon)\left(\int_{\{|u-\theta|>L\}} |f(x)|^{\frac{r}{p-1}}dx\right)+c\varepsilon\|\nabla u-\nabla\theta\|_r^r \\ &\quad +c\varepsilon|p-r|\int_{\{|u-\theta|>L\}} |\nabla u-\nabla\theta|^r dx \\ &\le C(\varepsilon)\|f(x)\|_{\frac{r}{p-1}}^{\frac{r}{p-1}}+c\varepsilon(1+|p-r|)\|\nabla u-\nabla\theta\|_r^r.\end{aligned} \tag{2.13}$$

Combining (2.6)-(2.7), (2.9)-(2.12) we arrive at

$$\begin{aligned}&\int_{\{|u-\theta|>L\}} |\nabla u-\nabla\theta|^r dx \\ &\le C(n,p,\varepsilon)\|\nabla\theta\|_r^r+c(\varepsilon)\|f(x)\|_{\frac{r}{p-1}}^{\frac{r}{p-1}} \\ &\quad +\left(C(n,p)|p-r|+\varepsilon\right)\|\nabla u-\nabla\theta\|_r^r.\end{aligned} \tag{2.14}$$

Case2 : $1<p<2$ .Lemma 1.1 yields

$$\begin{aligned}&\int_{\{|u-\theta|>L\}} \langle |\nabla u|^{p-2}\nabla u-|\nabla\theta|^{p-2}\nabla\theta, |\nabla u-\nabla\theta|^{r-p}(\nabla u-\nabla\theta)\rangle dx \\ &\ge \int_{\{|u-\theta|>L\}} |\nabla u-\nabla\theta|^{r-p+1}\left((|\nabla u-\nabla\theta|+|\nabla\theta|)^{p-1}-|\nabla\theta|^{p-1}\right)dx.\end{aligned}$$

This implies

$$\begin{aligned}&\int_{\{|u-\theta|>L\}} |\nabla u-\nabla\theta|^r dx \\ &\le \int_{\{|u-\theta|>L\}} |\nabla u-\nabla\theta|^{r-p+1}\left((|\nabla u-\nabla\theta|+\nabla\theta)^{p-1}\right)dx \\ &\le \int_{\{|u-\theta|>L\}} \langle |\nabla u|^{p-2}\nabla u-|\nabla\theta|^{p-2}\nabla\theta, |\nabla u-\nabla\theta|^{r-p}(\nabla u-\nabla\theta)\rangle dx \\ &\quad +\int_{\{|u-\theta|>L\}} |\nabla u-\nabla\theta|^{r-p+1}|\nabla\theta|^{p-1}dx \\ &\le \int_{\{|u-\theta|>L\}} \langle |\nabla u|^{p-2}\nabla u-|\nabla\theta|^{p-2}\nabla\theta, |\nabla u-\nabla\theta|^{r-p}(\nabla u-\nabla\theta)\rangle dx \\ &\quad +\varepsilon\int_{\{|u-\theta|>L\}} |\nabla u-\nabla\theta|^r dx+C(\varepsilon)\int_{\{|u-\theta|>L\}} |\nabla\theta|^r dx.\end{aligned} \tag{2.15}$$

By (2.8) and (2.5), $|I_1|$ can be estimated as

$$\begin{aligned}|I_1| &= \left|\int_{\{|u-\theta|>L\}} \langle |\nabla u|^{p-2}\nabla u-|\nabla\theta|^{p-2}\nabla\theta, h\rangle dx\right| \\ &\le \frac{3-p}{2^{p-2}(p-1)}\int_{\{|u-\theta|>L\}} |\nabla u-\nabla\theta|^{p-1}|h|dx \\ &\le \frac{3-p}{2^{p-2}(p-1)}\|\nabla u-\nabla\theta\|_r^{p-1}\|h\|_{r/(r-p+1)} \\ &\le \frac{3-p}{2^{p-2}(p-1)}C(n,p)|p-r|\|\nabla u-\nabla\theta\|_r^r.\end{aligned} \tag{2.16}$$

For the case $1<p<2$, $|I_2|$, $|I_3|$ and $|I_4|$ can also be estimated by (2.10) and (2.11). Combining (2.6), (2.13), (2.14), (2.10) and (2.11) and (2.12), we arrive at (2.13).Let $\varepsilon_0 = 1/C(n,p)$. Then for $|p-r|<\varepsilon_0$ we *have* $C(n,p)|p-r|<1$. *Taking* ε small enough, such that $C(n,p)|p-r|+\varepsilon<1$, then the second term on the right-hand side of (2.13) can be absorbed by the left-hand side; thus we obtain

$$\int_{\{|u-\theta|>L\}} |\nabla u-\nabla\theta|^r dx \le C(n,p)\|\nabla\theta\|_r^r+C(\varepsilon)\|f(x)\|_{\frac{r}{p-1}}^{\frac{r}{p-1}}. \tag{2.17}$$

Since $\theta\in W^{1,q}(\Omega), q<r$, by the Hölder equality we obtain

$$\|\nabla\theta\|_r^r=\int_{\{|u-\theta|>L\}}|\nabla\theta|^r dx$$
$$\le\left(\int_{\{|u-\theta|>L\}}|\nabla\theta|^q dx\right)^{r/q}\left|\{|u-\theta|>L\}\right|^{q-r/q} \quad (2.18)$$
$$=\|\nabla\theta\|_q^r\left|\{|u-\theta|>L\}\right|^{q-r/q}.$$

and

$$\|f(x)\|_{\frac{r}{p-1}}^{\frac{r}{p-1}}\le\|f(x)\|_{\frac{q}{p-1}}^{\frac{r}{q}}\cdot\left|\{|u-\theta|>L\}\right|^{\frac{q-r}{q}}. \quad (2.19)$$

by combining (2.18) and (2.19) we get

$$\int_{\{|u-\theta|>L\}}|\nabla u-\nabla\theta|^r dx$$
$$\le C(n,p)\|\nabla\theta\|_q^r\cdot\left|\{|u-\theta|>L\}\right|^{\frac{q-r}{q}}$$
$$+C(\varepsilon)\|f(x)\|_{\frac{q}{p-1}}^{\frac{r}{q}}\cdot\left|\{|u-\theta|>L\}\right|^{\frac{q-r}{q}} \quad (2.20)$$
$$\le C(n,p)\cdot D\cdot\left|\{|u-\theta|>L\}\right|^{\frac{q-r}{q}},$$

where $D=\|\nabla\theta\|_q^r+\|f(x)\|_{\frac{q}{p-1}}^{\frac{r}{q}}$.

We now turn our attention back to the function $v\in W_0(\Omega)$ By the Sobolev embedding theorem, and using (2.2), we have

$$\left(\int_\Omega|v|^{r^*}dx\right)^{1/r^*}$$
$$\le C(n,r)\left(\int_\Omega|\nabla v|^r dx\right)^{1/r} \quad (2.21)$$
$$=C(n,r)\left(\int_{\{|u-\theta|>L\}}|\nabla u-\nabla\theta|^r dx\right)^{1/r}.$$

Since $|\varpi|=(|u-\theta|-L)\cdot 1_{\{|u-\theta|>L\}}$, we have

$$\left(\int_{\{|u-\theta|>L\}}(|u-\theta|-L)^{r^*}dx\right)^{1/r^*}=\left(\int_\Omega|v|^{r^*}dx\right)^{1/r^*}, \quad (2.22)$$

and for $\tilde{L}>L$,

$$(\tilde{L}-L)^{r^*}\left|\{|u-\theta|>\tilde{L}\}\right|=\int_{\{|u-\theta|>\tilde{L}\}}(\tilde{L}-L)^{r^*}dx \quad (2.23)$$
$$\le\int_{\{|u-\theta|>\tilde{L}\}}(|u-\theta|-L)^{r^*}dx\le\int_{\{|u-\theta|>L\}}(|u-\theta|-L)^{r^*}dx.$$

By collecting (2.16)-(2.21), we deduce that

$$\left((\tilde{L}-L)^{r^*}\left|\{|u-\theta|>\tilde{L}\}\right|\right)^{1/r^*}\le C(n,r)\cdot D^{\frac{1}{r}}\left|\{|u-\theta|>L\}\right|^{1/r-1/q}.$$

Thus

$$\left|\{|u-\theta|>\tilde{L}\}\right|\le\frac{1}{(\tilde{L}-L)^{r^*}}\left(C(n,r)\cdot D^{\frac{1}{r}}\right)^{r^*}\left|\{|u-\theta|>L\}\right|^{r^*(1/r-1/q)}. \quad (2.24)$$

Let

$$\phi(s)=\left|\{|u-\theta|>s\}\right|,\alpha=r^*,C=\left[C(n,r)\cdot D^{\frac{1}{r}}\right]^{r^*},$$
$$\beta=r^*(1/r-1/q)$$

and $s_0>0$. then (2.22) becomes

$$\varphi(\tilde{L})\le\frac{C}{(\tilde{L}-L)^\alpha}\varphi(L)^\beta, \quad (2.25)$$

for $\tilde{L}>L>0$.

For the case $q<n$, one has $\beta<1$. In this case, if $s\ge1$, we get from Lemma 1.2 that

$$\left|\{|u-\theta|>s\}\right|\le C(\alpha,\beta,s_0)s^{-t},$$

where $t=\alpha/(1-\beta)=q^*$. For $0<s<1$, one has

$$\left|\{|u-\theta|>s\}\right|\le|\Omega|s^{q^*}s^{-q^*}\le|\Omega|s^{-q^*}.$$

Thus $u\in\theta+L^{q^*}_{weak}(\Omega)$.

For the case $q=n$, one has $\beta>1$. For any $\tau<\infty$, (2.22) implies

$$\phi(\tilde{L})\le\frac{C}{(L-L)^\alpha}\phi(L)=\frac{C}{(L-L)^\alpha}\phi(L)^{1-\alpha/\tau}\phi(L)^{\alpha/\tau}$$
$$\le\frac{C|\Omega|^{\alpha/\tau}}{(L-L)^\alpha}\phi(L)^{1-\alpha/\tau}.$$

As above, we derive

$$u\in\theta+L^\tau_{weak}(\Omega).$$

For the case $q>n$, one has $\beta>1$. Lemma 1.2 implies $\varphi(d)=0$ for some $d=d(\alpha,\beta,s_0,r,D)$. Thus $\left|\{|u-\theta|>d\}\right|=0$, which means $u-\theta\le d$ a.e. in Ω. Therefore

$$u\in\theta+L^\infty(\Omega),$$

completing the proof of Theorem 1.1.

References

[1] E.Giusti, "Metodi diretti nel calcolo delle variazioni," Unione Matematica Italiana, Bologna, 1944. (In Italian.)

[2] T.Iwaniec, G.Martin, "Geometric Function Theory and Nonlinear Analysis," Oxford Mathematical Monographs, Oxford University Press, Oxford, 2001.

[3] L.Greco, T.Iwaniec, C.Sbordone, "Inverting the p-harmonic operator," Manuscr. Math., 92(1977), 249-258.

[4] S.Campanato, "Sistemi ellittici in form a divergenza," Regolarità all'interno. Pubblicazioni della Classe di Scienze: Quaderni, Scuola Normale Superiore Pisa, Pisa, 1980. (In Italian.)

[5] C.B.Morrey, Jr., “Multiple Integrals in the Calculus of Variations,” Die Grundlehren der mathematischen Wissenschaften 130, Springer, New York, 1966.

[6] T.Iwaniec, C.Sbordone, “Weak minima of variational integrals,” J. Reine Angew. Math., 454(1994), 143-161.

[7] T.Iwaniec, C.Scott, B.Stroffolini, “Nonlinear Hodge theory on manifolds with boundary,” Ann. Mat. Pura Appl., (4)177(1999), 37-115.

[8] Hongya Gao, Shuang Liang, Yi Cui, “Integrability for very weak solutions to boundary value problems of p-harmonic equation,” Czechoslovak Mathematical Journal, 66(141), (2016), 101-110.

[9] P.Lindqvist, “Notes on the p-Laplace Equation,” Report. University of Jyväskylä Department of Mathematics and Statistics 102, University of Jyväskylä, Jyväskylä, 2006.

[10] T.Iwaniec, L.Migliaccio, L.Nania, C.Sbordone, “Integrability and removability results for quasiregular mappings in high dimensions,” Math. Scand., 75(1994), 263-279.