Mixed Fractional Differentiation Operators in Hölder Spaces

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***Abstract*— We study mixed fractional derivative in Marchaud form of function of two variables in Hölder spaces of different orders in each variables. We consider Hölder spaces defined both by first order differences in each variable and also by the mixed second order difference, the main interest being in the evaluation of the latter for the mixed fractional derivative in the cases Hölder class defined by usual Hölder condition.**

Keywords—functions of two variables, fractional derivative of Marchaud form, mixed fractional derivative, mixed fractional integral, Hölder space.

I. INTRODUCTION (*Heading 1*)

The classical result of G.Hardi and D.Littlewood (1928, see [1, §3]) is known that the fractional integral $\left(I_{a+}^{\alpha} f\right)(x)=\Gamma^{-1}(\alpha)\left(t_{+}^{\alpha-1} * f\right)(x),\ 0<\alpha<1$ maps isomorphically the space $H_0^{\lambda}([0,1])$ of Hölder order $\lambda \in (0,1)$ functions with a condition $f(0)=0$ on a similar space of a higher order $\lambda+\alpha$ provided that $\lambda+\alpha<1$. Further, this result was generalized in various directions: a space with a power weight, generalized Hölder spaces, spaces of the Nikolsky type, etc. A detailed review of these and some other similar results can be found in [1].

In the multidimensional case, the statement about the properties of a map in Hölder spaces for a mixed fractional Riemann – Liouville integral was studied in [2] - [7].

Mixed fractional derivatives form Marchaud

$$\left(D_{a+,c+}^{\alpha,\beta}\varphi\right)(x,y)=\frac{\varphi(x,y)}{\Gamma(1-\alpha)\Gamma(1-\beta)(x-a)^{\alpha}(y-c)^{\beta}}+$$

$$+\frac{\alpha\beta}{\Gamma(1-\alpha)\Gamma(1-\beta)}\int_a^x\int_c^y \frac{\varphi(x,y)-\varphi(t,\tau)}{(x-t)^{1+\alpha}(y-\tau)^{1+\beta}}\,dt\,d\tau, \quad x>a,\ y>c, \tag{1}$$

were not studied in the Hölder space. This paper is devoted to the study of the properties of a map in Holder spaces, defined by the usual Hölder condition for functions of two variables.

Consider the operator (1) in a rectangle $Q=\{(x,y):\ 0<x<b,\ 0<y<d\}$.

For a continuous function $\varphi(x,y)$ on R^2 we introduce the notation

$$\left(\overset{1,0}{\Delta}_h \varphi\right)(x,y)=\varphi(x+h,y)-\varphi(x,y),$$

$$\left(\overset{0,1}{\Delta}_\eta \varphi\right)(x,y)=\varphi(x,y+\eta)-\varphi(x,y),$$

$$\left(\overset{1,1}{\Delta}_{h,\eta} \varphi\right)(x,y)=\varphi(x+h,y+\eta)-\varphi(x+h,y)-$$

$$-\varphi(x,y+\eta)+\varphi(x,y),$$

so that

$$\varphi(x+h,y+\eta)=\left(\overset{1,1}{\Delta}_{h,\eta} \varphi\right)(x,y)+\left(\overset{1,0}{\Delta}_h \varphi\right)(x,y)+$$

$$+\left(\overset{0,1}{\Delta}_\eta \varphi\right)(x,y)+\varphi(x,y). \tag{2}$$

Everywhere in the sequel by C, C_1, C_2 etc we denote positive constants which may different values in different occurences and even in the same line.

Definition 1.*Let* $\lambda,\gamma \in (0,1]$. *We say that* $\varphi \in H^{\lambda,\gamma}(Q)$, *if*

$$|\varphi(x_1,y_1)-\varphi(x_2,y_2)| \le C_1|x_1-x_2|^{\lambda}+C_2|y_1-y_2|^{\gamma} \tag{3}$$

for all $(x_1,y_1),\ (x_2,y_2) \in Q$. *Condition* (3) *is equivalent to the couple of the separate conditions*

$$\left|\left(\overset{1,0}{\Delta}_h \varphi\right)(x,y)\right| \le C_1|h|^{\lambda}, \qquad \left|\left(\overset{0,1}{\Delta}_\eta \varphi\right)(x,y)\right| \le C_2|\eta|^{\gamma} \tag{4}$$

uniform with respect to another variable.

Note that

$\varphi(x,y) \in H^{\lambda,\gamma} \Rightarrow$

$\left|\left(\overset{1,1}{\Delta}_{h,\eta} \varphi\right)(x,y)\right| \le C_\theta |h|^{\theta\lambda} |\eta|^{(1-\theta)\gamma} \le C \min\left\{|h|^{\lambda},\ |\eta|^{\gamma}\right\}$, where

$\theta \in [0,1]$. (5)

By $H_0^{\lambda,\gamma}(Q)$ we define a subspace of functions $f \in H_0^{\lambda,\gamma}(Q)$, vanishing at the boundaries $x=0$ and $y=0$ of Q.

A one –dimensional statements

The following statements are known [1]. We use the schemes of the proofs to make the presentation easier for two-dimensional case.

Lemma 1. *If* $f(x)\in H^{\lambda+\alpha}([0,b])$ *and* $0<\lambda,\ 0<\alpha+\lambda<1$, *then*

$$g(x)=\frac{f(x)-f(0)}{|x|^{\alpha}}\in H^{\lambda}([0,b]) \text{ and } \|g\|_{H^{\lambda}}\le C\|f\|_{H^{\lambda+\alpha}},$$

where C doesn't depend from $f(x)$.

Proof. Let $h>0;\ x,x+h\in[0,b]$. We consider the difference

$$|g(x+h)-g(x)|\le\frac{|f(x+h)-f(x)|}{(x+h)^{\alpha}}+|f(x)-f(0)|\frac{(x+h)^{\alpha}-x^{\alpha}}{x^{\alpha}(x+h)^{\alpha}}$$

Since $f\in H^{\lambda+\alpha}$, we have

$$|f(x+h)-f(x)|\le C_1h^{\lambda+\alpha},\qquad |f(x)-f(0)|\le C_2x^{\lambda+\alpha}.$$

Using these inqualities we obtain

$$|g(x+h)-g(x)|\le C_1\frac{h^{\lambda+\alpha}}{(x+h)^{\alpha}}+C_2x^{\lambda}\frac{(x+h)^{\alpha}-x^{\alpha}}{(x+h)^{\alpha}}=G_1+G_2$$

.

For G_1 we have

$$G_1=C_1h^{\lambda}\left(\frac{h}{x+h}\right)^{\alpha}\le Ch^{\lambda}.$$

Let's estimate G_2, here we shall consider two cases: $x\le h$ and $x>h$. In the first case, we use inequality $\left|\sigma_1^{\mu}-\sigma_2^{\mu}\right|\le|\sigma_1-\sigma_2|^{\mu},\ (\sigma_1\ne\sigma_2)$ and obtain

$$G_2\le C_2\frac{x^{\lambda}h^{\alpha}}{(x+h)^{\alpha}}\le Ch^{\lambda}.$$

In second case, using $(1+t)^{\alpha}-1\le\alpha t,\ t>0$ we have

$$G_2=C_2\frac{x^{\lambda+\alpha}}{(x+h)^{\alpha}}\left|\left(1+\frac{h}{x}\right)^{\alpha}-1\right|\le Chx^{\lambda-1}\le Ch^{\lambda},$$

which completes the proof.

The Marchaud fractional differentiation operator has a form:

$$\left(\mathrm{D}_{0+}^{\alpha}f\right)(x)=\frac{f(x)}{x^{\alpha}\Gamma(1-\alpha)}+\frac{\alpha}{\Gamma(1-\alpha)}\int_0^x\frac{f(x)-f(t)}{(x-t)^{1+\alpha}}dt\ ,(6)$$

where $0<\alpha<1$.

Theorem 1. *If* $f(x)\in H^{\lambda+\alpha}([0,b]),\ 0<\alpha+\lambda<1$, *that*

$$\left(\mathrm{D}_{0+}^{\alpha}f\right)(x)=\frac{f(0)}{x^{\alpha}\Gamma(1-\alpha)}+\psi(x)\quad,\qquad(7)$$

where $\psi(x)\in H^{\lambda}([0,b])$ *and* $\psi(0)=0$, *thus* $\|\psi\|_{H^{\lambda}}\le C\|f\|_{H^{\lambda+\alpha}}$.

Proof. We present (6) as

$$\left(\mathrm{D}_{0+}^{\alpha}f\right)(x)=\frac{f(0)}{x^{\alpha}\Gamma(1-\alpha)}+\frac{f(x)-f(0)}{x^{\alpha}\Gamma(1-\alpha)}+\frac{\alpha}{\Gamma(1-\alpha)}\int\frac{f(x)-f(t)}{(x-t)^{1+\alpha}}dt$$

,

receive equality (7), where

$$\psi(x)=\psi_1(x)+\psi_2(x)=\frac{f(x)-f(0)}{x^{\alpha}\Gamma(1-\alpha)}+\frac{\alpha}{\Gamma(1-\alpha)}\int_0^x\frac{f(x)-f(t)}{(x-t)^{1+\alpha}}dt$$

.

Here $\psi_1(x)\in H^{\lambda}([0,\ b])$ by Lemme 1. It is enough to show $\psi_2(x)\in H^{\lambda}([0,\ b])$.

Let $h>0;\ x,x+h\in[0,b]$. Let's consider the difference

$$\psi_2(x+h)-\psi_2(x)=\int_0^x\frac{f(x+h)-f(x)}{(x+h-t)^{1+\alpha}}dt+\int_x^{x+h}\frac{f(x+h)-f(t)}{(x+h-t)^{1+\alpha}}dt+$$

$$+\int_0^x[f(x)-f(t)]\left[\frac{1}{(x+h-t)^{1+\alpha}}-\frac{1}{(x-t)^{1+\alpha}}\right]dt=I_1+I_2+I_3.$$

Since $f\in H^{\lambda+\alpha}([0,b])$, then we have for I_1

$$|I_1|\le Ch^{\lambda+\alpha}\int_0^x(t+h)^{-1-\alpha}dt\le C_1h^{\lambda}.$$

Let's estimate I_2. We have

$$|I_2|\le C\int_x^{x+h}(x+h-t)^{\lambda-1}dt=C_2h^{\lambda}$$

For I_3,

$$|I_3|\le Ch^{\lambda}\int_0^{\frac{x}{h}}t^{\lambda}\left|\frac{1}{(1+t)^{1+\alpha}}-\frac{1}{t^{1+\alpha}}\right|dt\le C_3h^{\lambda}.$$

Finally, it remains to note that $\psi_2(0)=0$, since

$$|\psi_2(x)|\le C\int_0^x t^{\lambda-1}dt.$$

Main result.

Lemma 2. *Let* $f(x,y)\in H^{\lambda,\gamma}(Q)$, $\alpha<\lambda\le1,\beta<\gamma\le1$. *Then for the mixed fractional differential operator (1) the representation*

$$\left(\mathrm{D}_{0+,0+}^{\alpha,\beta}f\right)(x,y)=\frac{f(0,0)x^{-\alpha}y^{-\beta}+y^{-\beta}\psi_1(x)+x^{-\alpha}\psi_2(y)+\psi(x,y)}{\Gamma(1-\alpha)\Gamma(1-\beta)}$$

, (8)

and

$$|\psi_1(x)|\le C_1x^{\lambda-\alpha},\quad |\psi_2(y)|\le C_2y^{\gamma-\beta},\qquad(9)$$

$$|\psi(x,y)|\le Cx^{\theta\lambda-\alpha}y^{(1-\theta)\gamma-\beta}\qquad(10)$$

where

$$\psi_1(x)=x^{-\alpha}[f(x,0)-f(0,0)]+\alpha\int_0^x[f(x,0)-f(t,0)](x-t)^{-\alpha-1}dt,$$

$$\psi_2(y)=y^{-\beta}[f(0,y)-f(0,0)]+\beta\int_0^y\frac{f(0,y)-f(0,\tau)}{(y-\tau)^{1+\beta}}d\tau,$$

$$\psi(x,y)=\frac{1}{x^{\alpha}y^{\beta}}\left(\overset{1,1}{\Delta}_{x,y}f\right)(0,0)+$$

$$+\frac{\alpha}{y^{\beta}}\int_0^x\left(\overset{1,1}{\Delta}_{x-t,y}f\right)(t,0)\frac{dt}{(x-t)^{1+\alpha}}+$$

$$+\frac{\beta}{x^{\alpha}}\int_0^y\left(\overset{1,1}{\Delta}_{x,y-\tau}f\right)(0,\tau)\frac{d\tau}{(y-\tau)^{1+\beta}}+$$

$$+\alpha\beta\int_0^x\int_0^y\left(\overset{1,1}{\Delta}_{x-t,y-\tau}f\right)(t,\tau)\frac{dtd\tau}{(x-t)^{1+\alpha}(y-\tau)^{1+\beta}}.$$

Proof. Representation (8) itself is easily obtained by means of (2). Since $f \in H^{\lambda,\gamma}(Q)$, inequalities (9) are obvious. Estimate (10) is obtained by means of (5), i.e.

$$\psi(x,y) \le C\left[x^{\lambda\theta} y^{(1-\theta)\gamma} + \alpha y^{(1-\theta)} \int_0^x \frac{dt}{(x-t)^{1-\theta\lambda}} + \beta x^{\theta\lambda} \int_0^y \frac{d\tau}{(y-\tau)^{1-(1-\theta)\gamma}} + \right.$$

$$\left. + \beta\alpha \int_0^x \int_0^y \frac{(x-t)^{\theta\lambda-1} dt d\tau}{(y-\tau)^{1-(1-\theta)\gamma}} \right]$$

It is easy to receive

$$\psi(x,y) \le C x^{\theta\lambda} y^{(1-\theta)\gamma} \left[1 + \int_0^1 \frac{ds}{s^{1-\theta\lambda}} + \int_0^1 \frac{d\xi}{\xi^{1-(1-\theta)\gamma}} + \int_0^1 \int_0^1 \frac{s^{\theta\lambda-1} ds d\xi}{\xi^{1-(1-\theta)\gamma}} \right] \le$$

$$\le C_3 x^{\theta\lambda} y^{(1-\theta)\gamma}$$

Theorem 2. *Let* $f(x,y) \in H_0^{\lambda,\gamma}(Q)$, $\alpha < \lambda \le 1$, $\beta < \gamma \le 1$. *Then the operator* $\mathrm{D}_{0+,0+}^{\alpha,\beta}$ *continuously maps* $H_0^{\lambda,\gamma}(Q)$ *into* $H_0^{\lambda-\alpha,\gamma-\beta}(Q)$.

Proof. Since $f(x,y) \in H_0^{\lambda,\gamma}(Q)$, by (8) we have

$$\varphi(x,y) = \left(\mathrm{D}_{0+,0+}^{\alpha,\beta} f\right)(x,y) = \psi(x,y).$$

Let $h > 0;\ x, x+h \in [0,b]$. We consider the difference

$$\psi(x+h,y) - \psi(x,y) = \sum_{i=1}^{10} \Phi_k := \frac{\left(\overset{1,1}{\Delta}_{h,y} f\right)(0,0)}{y^{\beta}(x+h)^{\alpha}} +$$

$$+ \frac{\left(\overset{1,1}{\Delta}_{x,y} f\right)(0,0)}{y^{\beta}} \left[\frac{1}{(x+h)^{\alpha}} - \frac{1}{x^{\alpha}} \right] +$$

$$+ \frac{\alpha}{y^{\beta}} \int_0^x \frac{\left(\overset{1,1}{\Delta}_{h,y} f\right)(x,0)}{(x+h-t)^{1+\alpha}} dt + \frac{\alpha}{y^{\beta}} \int_x^{x+h} \frac{\left(\overset{1,1}{\Delta}_{x+h-t,y} f\right)(t,0)}{(x+h-t)^{1+\alpha}} dt +$$

$$+ \frac{\beta}{(x+h)^{\alpha}} \int_0^y \frac{\left(\overset{1,1}{\Delta}_{h,y-\tau} f\right)(0,\tau)}{(y-\tau)^{1+\beta}} d\tau +$$

$$+ \frac{\alpha}{y^{\beta}} \int_0^x \left(\overset{1,1}{\Delta}_{x-t,y} f\right)(t,0) \left[(x+h-t)^{-1-\alpha} - (x-t)^{-1-\alpha} \right] dt +$$

$$+ \beta\left[(x+h)^{-\alpha} - x^{-\alpha} \right] \int_0^y \frac{\left(\overset{1,1}{\Delta}_{x,y-\tau} f\right)(0,\tau)}{(y-\tau)^{1+\beta}} d\tau +$$

$$+ \alpha\beta \int_0^x \int_0^y \frac{\left(\overset{1,1}{\Delta}_{h,y-\tau} f\right)(x,\tau) dt d\tau}{(x+h-t)^{1+\alpha} (y-\tau)^{1+\beta}} + \alpha\beta \int_x^{x+h} \int_0^y \frac{\left(\overset{1,1}{\Delta}_{x+h-t,y-\tau} f\right)(x,\tau) dt d\tau}{(x+h-t)^{1+\alpha} (y-\tau)^{1+\beta}} +$$

$$+ \alpha\beta \int_0^x \int_0^y \frac{\left(\overset{1,1}{\Delta}_{x-t,y-\tau} f\right)(x,\tau)}{(y-\tau)^{1+\beta}} \left[(x+h-t)^{-1-\alpha} - (x-t)^{-1-\alpha} \right] dt d\tau \quad (11)$$

Since $f(x,y) \in H_0^{\lambda,\gamma}(Q)$ we have

$$|\psi(x+h,y) - \psi(x,y)| \le \sum_{i=1}^{10} |\Phi_k| = C\left[\frac{h^{\lambda}}{y^{\beta}(x+h)^{\alpha}} + \right.$$

$$+ \frac{x^{\lambda}}{y^{\beta}} \left[\frac{1}{(x+h)^{\alpha}} - \frac{1}{x^{\alpha}} \right] + \frac{\alpha h^{\lambda}}{y^{\beta}} \int_0^x \frac{dt}{(x+h-t)^{1+\alpha}} +$$

$$+ \frac{\alpha}{y^{\beta}} \int_x^{x+h} (x+h-t)^{\lambda-1-\alpha} dt + \frac{h^{\theta\lambda}\beta}{(x+h)^{\alpha}} \int_0^y (y-\tau)^{(1-\theta)\gamma-1-\beta} d\tau +$$

$$+ \frac{\alpha}{y^{\beta}} \int_0^x (x-t)^{\lambda} \left[\frac{1}{(x+h-t)^{1+\alpha}} - \frac{1}{(x-t)^{1+\alpha}} \right] dt +$$

$$+ \beta x^{\theta\lambda} \left[(x+h)^{-\alpha} - x^{-\alpha} \right] \int_0^y \frac{d\tau}{(y-\tau)^{1+\beta-(1-\theta)\gamma}} +$$

$$+ \alpha\beta h^{\theta\lambda} \int_0^x \int_0^y \frac{(y-\tau)^{(1-\theta)\gamma-1-\beta} dt d\tau}{(x+h-t)^{1+\alpha}} +$$

$$+ \alpha\beta \int_x^{x+h} \int_0^y \frac{(y-\tau)^{(1-\theta)\gamma-1-\beta} dt d\tau}{(x+h-t)^{1+\alpha-\theta\lambda}} +$$

$$\left. + \alpha\beta \int_0^x \int_0^y \frac{(x-t)^{\theta\lambda}}{(y-\tau)^{1+\beta-(1-\theta)\gamma}} \left[(x+h-t)^{-1-\alpha} - (x-t)^{-1-\alpha} \right] dt d\tau \right],$$

where

$$\int_0^y (y-\tau)^{(1-\theta)\gamma-1-\beta} d\tau < \infty.$$

Using estimations G_1, G_2 of the proof of Lemma 1 and estimations I_1, I_2, I_3 of the proof of the Theorem 1, it is easily possible to receive an estimation

$$|\psi(x+h,y) - \psi(x,y)| \le C h^{\lambda-\alpha}.$$

Rearranging symmetrically representation (11), we can similarly obtain that

$$|\psi(x,y+\eta) - \psi(x,y)| \le C \eta^{\gamma-\beta}.$$

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