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Characteristic Of Tiling Generated By Root Of Polynomial P(X) = X^4 -3x^3 + X^2-2x-1

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***Abstract*—There are different authors that have extensive studies on the tiling generated by Pisot numbers related to Pisot of degree 3. The connected of Pisot dual tailings play an important role on beta– expansion, it is often referred to Rényi as the first occurrence of beta– expansions [7]. The theory of beta – expansion create a link between symbolic dynamics and a part of number theory. There exists a Pisot whose dual tiles is disconnected [1, 2]. Let beta a Pisot which is the root of equation of fourth degree. At least one of the tiles is not connected [9]. We try to give same property for the tiling generated by Pisot root of this polynomial.**

***Keywords*—component; (beta – expansions , Pisot number, Disconnect, Fractal)**

I. INTRODUCTION

From history we have representation of real numbers in different bases. Sumerian and Babylonian mathematics was based on a sexegesimal, or base 60, numeric system. The Mayan and other Mesoamerican cultures used a vigesimal number system based on base 20. Al-Khwarizmi had a important contribution to mathematics usin the Hindu numerical system (1 - 9 and 0). From integer base expansions R´enyi on 1957 was first that extend representation of real numbers to noninteger bases . He study β – expansions from dynamical viewpoint. Many authors working on representation to noninteger bases for real numbers get a connection with different fields in mathematics from combinatorics ,dynamics systems, topology , number theory and ergodic theory.

We give for a real number $x \geq 0$ by their decimal expansions

$$x = a_k 10^k + a_{k-1} 10^{k-1} + \cdots + a_1 10 + a_0 + a_{-1} 10^{-1} + a_{-2} 10^{-2} + \cdots$$

Where $a_k \in \{0,1,\cdots,9\}$ and $k \geq 0$

On this paper we are interested in numbers in $x \in [0\,,1]$ their decimal expansions :

$x = \sum_{k\geq 1} \frac{a_k}{10^k}$ with $a_k \in \{0,1,\cdots,9\}$

Some property : [8]

1. All but countably many numbers have a unique representation

2. Numbers with no unique representation have two different rapresentations

$$\frac{9}{16} = \frac{5}{10} + \frac{6}{10^2} + \frac{2}{10^3} + \frac{5}{10^4}$$

$$= \frac{5}{10} + \frac{6}{10^2} + \frac{2}{10^3} + \frac{4}{10^4} + \frac{9}{10^5} + \frac{9}{10^6} + \cdots$$

3. The set with two expansions :

$$\left\{\frac{k}{10^n} \mid 1 \leq k \leq 10^n - 1\,, n \geq 1\right\}$$

The number 10 is base of this expansions and the set $\{0,1,\cdots,9\}$ digit set.

Let $\beta > 1$ be a real number that is not an integer. A representation in base β of a positive real number x is in form :

$$x = a_k\beta^k + a_{k-1}\beta^{k-1} + \cdots + a_1\beta + a_0 + a_{-1}\beta^{-1} + a_{-2}\beta^{-2} + \cdots$$

It is denoted by :

$$x = a_k a_{k-1} \cdots a_1 a_0 . a_{-1} a_{-2} \cdots$$

A greedy expansions in base β of a positive real number x is in form

$$x = \sum_{i=-k}^{\infty} a_{-i}\beta^{-i}$$

With $a_{-i} \in \mathcal{A}_\beta = [0,\beta) \cap \mathbb{Z}$ and greedy condition

$$\left| x - \sum_{N_0 \leq k \leq N} a_k \beta^{-k} \right| < \beta^{-N}$$

For all $N \geq N_0$.

$.a_{-1}a_{-2}\cdots$ is the fractional part of x denote by $\{x\}$. And $a_k a_{k-1} \cdots a_1 a_0$ is the integer part of x denote by $[x]$. The digit a_k obtained by greedy algorithm are integer from the set $\mathcal{A}_\beta = \{0,\cdots,\beta-1\}$ if β is an integer or the set $\mathcal{A}_\beta = \{0,\cdots,[\beta]\}$ if β is not an integer. This expansion for $x \in [0,1)$ is produced by iterating the beta transform :

$T_\beta : x \longrightarrow \beta x - [\beta x]$ where $[\beta x] \in \mathcal{A}_\beta$

Let $1 = d_{-1}\,\beta^{-1} + d_{-2}\,\beta^{-2} + \cdots$ be an expansion of 1 defined by the algorithm

$$c_{-i} = \beta c_{-i+1} - [\beta c_{-i+1}],\ d_{-i} = [\beta c_{-i+1}],\ \text{with } c_0 = 1$$

where $[x]$ denoted the maximal integer not exceeding x. This expansion is achived as a trajectory of e $T_\beta^n(1)$ $(n = 1,2,\ldots\ldots)$. $d_\beta(1) = d_{-1}, d_{-2}, \cdots$ is called $\beta-$ expansion of 1. Parry has shown that a sequence $x = a_1, a_2, \cdots$ of non negative integers give a $\beta-$ expansion of pozitiv real number iff satisfies lexicographical condition:[6]

$$\forall p \geq 0\ ,\qquad \sigma^p(d) < lex\ d^*(1)$$

with

$$d^*(1) = \begin{cases} \mathrm{d_\beta(1)} & \text{if } \mathrm{d_\beta(1)} \text{ is infinite} \\ (d_{-1}, d_{-2}\ldots, d_{-n+1,}\ (\mathrm{d_{-n}} - 1))^{\omega} & \text{if } \mathrm{d_\beta(1)} \\ & = \mathrm{d_{-1}}, \cdots \mathrm{d_{-n}} \end{cases}$$

where the string of simbols w , w^ω is a periodic expansions $w, w, \cdots$ and σ is the shift defined by

$$\sigma((a_i)_{i\leq M}) = (a_{i-1})_{i\leq M}$$

so the sequence $x = a_1, a_2, a_3 \ldots$ is called admissible.

II. Pisot numbers

Definition 1. A number α is an algebraic integer if it is the root of a monic integer polynomial. There is a unique monic integer polynomial $p(x)$, called the minimal polynomial, for which α is a root and the degree of $p(x)$ is minimal.

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

$$a_0, a_1, \ldots, a_{n-1}, a_n \in \mathbb{Z}\ ,\ a_n = 1$$

Exampel 1. $\alpha = \frac{5+\sqrt{5}}{2}$, root of polynomial,

$$p(x) = x^2 - 5x + 5$$

Definition 2. If α is an algebraic integer, and $p(x)$ is its minimal polynomial, then we say that all of the other roots of $p(x)$ are the conjugates of α.

Definition 3. A Pisot number β satisfies the following conditions:

- β is an algebraic integer
- $\beta > 1$
- all of β's conjugates $\beta_2, \beta_3, \cdots, \beta_d$ (where d is the degree of the minimal polynomial of β) are strictly less than 1 in modulus, $|\beta_i| < 1\ , i = 2, 3, \cdots, d$

Definition 4. A Salem number τ is a real algebraic integer $\tau > 1$ such that all of τ's conjugates are less than or equal to 1 in modulus, and at least one conjugate is equal to 1 in modulus.

$\mathbb{Q}(\beta)$ minmum field containing the rational numbers $\mathbb{Q}$ and Pisot β. The properties of $\beta-$ expansions are strongly releted to symbolic dynamic. The closure of a set of infinite sequences, appearing as $\beta-$ expansions is called a $\beta-$ shift. It is a symbolic dynamical system that is closed shift-invariant subset of $\mathcal{A}_\beta{}^N$. $\beta-$ shift is finite iff $T_\beta^n(1) = 0$ for some n, and it is sofic iff the orbit $\{T_\beta^n(1)\}$ is finite.[3]

In Exampels 2 we take to polynomials with the root a Pisot numbers at firs Fig .1 and polynomials with the root a Salem numbers at Fig. 2

Examples 2

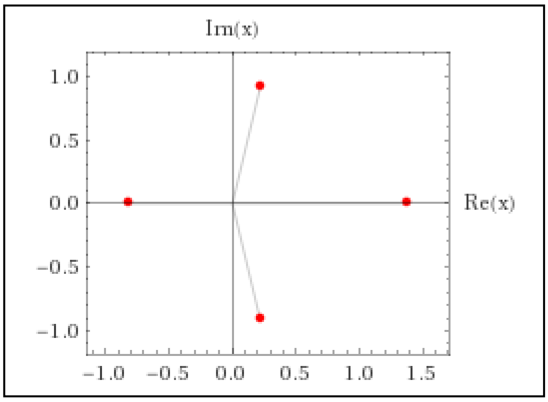

Fig .1 *Pisot number 1. 3803… and the other roots of* $x^4 - x^3 - 1$

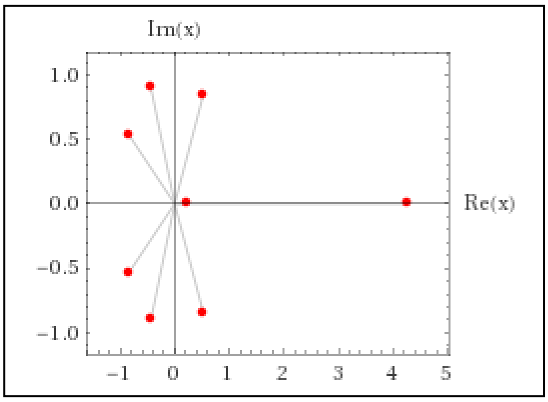

Fig . 2. *Salem number 4. 26845… and the other roots of* $x^8 - 3x^7 - 4x^6 - 5x^5 - 3x^4 - 5x^3 - 4x^2 - 3x + 1$

A. *Characteristic of tiling generated by a root of polynomial*

$$P(x) = x^4 - 3x^3 + x^2 - 2x - 1$$

Let $Fin(\beta)$ be a set consisting of all finite beta expansions. Consider the condition (F)

$$Fin(\beta) = \mathbb{Z}[1\backslash\beta]_{\geq 0}.$$

It is easily seen that if $\beta > 1$ is an integer, then (F) holds [9]. Conversely the condition (F) implies that $\beta > 1$ is a Pisot number. $\mathbb{Z}[1\backslash\beta]_{\geq 0}$ is a set of all nonnegative elements in $\mathbb{Z}[1\backslash\beta]$. $Per\,(\beta)$ is a set of periodic points for T_β the set of a points whose orbits under T_β are finite.
Schmid had proved [5]

- If $\mathbb{Q} \cap [0,1) \subset Per\,(\beta)$ then β is either a Pisot or a Salem number
- If β is a Pisot number then $Per\,(\beta) = \mathbb{Q} \cap [0,1)$

Let take a dual tiling.Take

$$\beta = \beta^{(1)}, \beta^{(2)}, \dots, \beta^{(r_1)}$$

And

$$\beta^{(r_1+1)}, \overline{\beta^{(r_1+1)}}, \cdots, \beta^{(r_1+r_2)}, \overline{\beta^{(r_1+r_2)}},$$

be a conjugates of β. $x^{(j)}$ $(j = 1,2,\dots\dots r_1 + 2r_2)$ the corresponding conjugates of $x \in \mathbb{Q}(\beta)$.
Let be ϕ a map define as :

$$\phi: \mathbb{Q}(\beta) \longrightarrow \mathbb{R}^{r_1+2r_2-1)}$$

by

$$\phi(x) = \begin{pmatrix} x^{(2)}, \dots, x^{(r_1)}, \Re\left(x^{(r_1+1)}\right), \Im\left(x^{(r_1+1)}\right), \\ \cdots, \Re\left(x^{(r_1+r_2)}\right), \Im\left(x^{(r_1+r_2)}\right) \end{pmatrix}$$

Let $A = a_{-1}, a_{-2}\dots\dots$be greedy in base β of an element $\mathbb{Z}[\beta] \cap [0,1)$.
The set elements of $\mathbb{Z}[1\backslash\beta]_{\geq 0}$. whose greedy expansion has suffix A , denote by S_A.
A tile $T_A = \overline{\phi(S_A)}$. T_A is a collections of left infinite admissibile sequences[2,4]

$$\cdots a_3 a_2 a_1 a_0 \oplus A = \cdots a_3 a_2 a_1 a_0 . A$$

Realized by the map Φ into $\mathbb{R}^{n-1}$.

Theorem 1. [1, 9]. Let $\beta > 1$ be a pisot unit. Set $\eta = max_{k\geq 1} T_\beta^k(1)$ which gives the smallest tile T_η. If

$$T_\eta \cap \left(T_\eta - \Phi(\beta^{-1})\right) \neq \phi$$

Then each Pisot dual tile is arcwise connected.

Theorem 2. [1, 9] Let β be a Pisot unit of degree 3 or 4 defined by the monic polynomial $p(x) \in \mathbb{Z}[x]$. If $\deg\beta = 3$ or $p(0) = 1$ then each tile is connected. If $\deg\beta = 4$ and
$p(0) = -1$ then each tile is connected iff

$a + c - 2\lfloor\beta\rfloor \neq 1$ for

$$p(x) = x^4 - ax^3 - bx^2 - cx - 1.$$

In fact if $\deg\beta = 4, p(0) = -1$ and $a + c - 2\lfloor\beta\rfloor = 1$ there exists a disconnected tile.

Lemma 1.[9] If β is a Pisot unit of degree 4 with minimal polynomial $p(x) = x^4 - ax^3 - bx^2 - cx - 1$ and if the negativ root γ of the polynomial $q(x) = x^2 - \lfloor\beta\rfloor x - 1$ has the property $p(\gamma) > 0$ then at least on of the tiles is not connected.

The proof of above lemma and theorems that follow is made by Sh. Akiyama and N. Gjini

Theorem 3[9]. Let β is a Pisot unit of degree 4 with minimal polynomial
$p(x) = x^4 - ax^3 - bx^2 - cx - 1$ each tile is arcwise connected except for the following cases:

$$\begin{cases} a \geq 5 \\ c = a - 3 \\ \frac{5-3a}{2} \leq b \leq -a \end{cases}, \quad \begin{cases} a \geq 3 \\ c = a - 1 \\ \frac{1-a}{2} \leq b \leq -1 \end{cases}$$

$$\begin{cases} a \geq 3 \\ c = a + 1 \\ \frac{1+a}{2} \leq b \leq a - 1 \end{cases}, \quad \begin{cases} a \geq 1 \\ c = a + 3 \\ \frac{5+3a}{2} \leq b \leq 2a + 2 \end{cases}$$

Let si this polynomial .

$$P(x) = x^4 - 3x^3 + x^2 - 2x - 1$$

The roots are Fig. 3

$$x = \frac{3}{4} - \frac{1}{4\sqrt{\frac{3}{19 - 116\sqrt[3]{\frac{2}{3\sqrt{12309} - 115}} + 2\times 2^{2/3}\sqrt[3]{3\sqrt{12309} - 115}}}} + \frac{1}{2}\sqrt{\left(\frac{19}{6} + \frac{29}{3}\sqrt[3]{\frac{2}{3\sqrt{12309} - 115}} - \frac{1}{3}\sqrt[3]{\frac{1}{2}\left(3\sqrt{12309} - 115\right)} - \frac{31}{2}\sqrt{\frac{3}{19 - 116\sqrt[3]{\frac{2}{3\sqrt{12309} - 115}} + 2\times 2^{2/3}\sqrt[3]{3\sqrt{12309} - 115}}}\right)}$$

roots	$x^4 - 3x^3 + x^2 - 2x - 1 = 0$

$x \approx -0.358434$

$x \approx 2.93132$

$x \approx 0.213559 - 0.951921\, i$

$x \approx 0.213559 + 0.951921\, i$

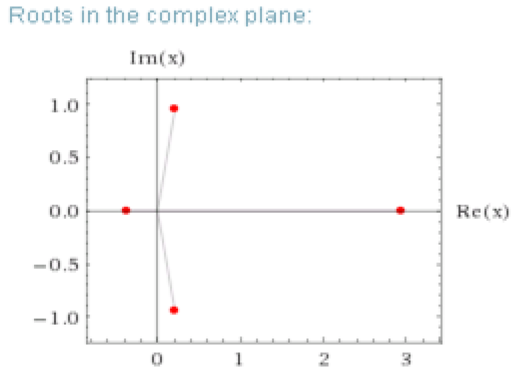

Fig. 3 The roots $P(x) = x^4 - 3x^3 + x^2 - 2x - 1$

Let see what happen for $\gamma = 1 - \sqrt{2}$ this is a negativ root of $x^2 - \lfloor\beta\rfloor x - 1$

$$p(\gamma) = \gamma^4 - 3\gamma x^3 + \gamma^2 - 2\gamma - 1$$

$$p(\gamma) = \left(1-\sqrt{2}\right)^4 - 3\left(1-\sqrt{2}\right)^3 + \left(1-\sqrt{2}\right)^2 - 2\left(1-\sqrt{2}\right) - 1 = 3\sqrt{2} - 4 > 0$$

From above results we see that this polynomial had the same conditions of Lemma 1. so at least one of the tiles is not connected. By our calculation we can demonstrate $d_\beta(1)$:

And $\beta -$ expansions of 1 is given by greedy algorithm as,

$$1 = \frac{3}{\beta} - \frac{1}{\beta^2} + \frac{2}{\beta^3} + \frac{1}{\beta^4}$$

$$= \frac{2}{\beta} + \frac{2}{\beta^2} + \frac{1}{\beta^3} + \frac{3}{\beta^4} + \frac{1}{\beta^5}$$

$$= \frac{2}{\beta} + \frac{2}{\beta^2} + \frac{1}{\beta^3} + \frac{2}{\beta^4} + \frac{4}{\beta^5} - \frac{1}{\beta^6} + \frac{2}{\beta^7} + \frac{1}{\beta^8}$$

$$= \frac{2}{\beta} + \frac{2}{\beta^2} + \frac{1}{\beta^3} + \frac{2}{\beta^4} + \frac{4}{\beta^5} + \frac{2}{\beta^6} + \frac{1}{\beta^7} + \frac{3}{\beta^8} + \frac{1}{\beta^8}$$

$d_\beta(1) = 2, 2, 1, 3, 1 \; = 2, 2, 1, 2, 4, 2, 1\; 3, 1 \; \cdots$

$\beta -$ expansions of 1 have $d_{-3} < \lfloor\beta\rfloor$
We need to study the orbit for the tiles generated by this Pisot numbers for further studies.

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