Eigentime Identities For A Family Of Treelike Networks

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***Abstract*—In this paper, we study the eigentime identities on a family of treelike networks, which quantifies as the sum of reciprocals of all nonzero normalized Laplacian eigenvalues. Firstly, we calculate the constant term and monomial coefficient of characteristic polynomial. By using the Vieta theorem, we then obtain the sum of reciprocals of all nonzero eigenvalues of normalized Laplacian matrix. Finally, we obtain the scalings of the eigentime identity on a family of weighted treelike networks. The larger the size, the lower the efficiency for a family of treelike networks.**

Keywords—treelike networks； eigentime identity；normalized Laplacian spectrum

I. INTRODUCTION

In the past decade, the study of networks associated with complex systems has received the attentions of researchers from different scientific fields[1]. The eigentime identity is a global characteristic of the network, which reflects the architecture and the efficiency of the whole network. Julaiti et al. used the relation between normalized Laplacian spectra and eigentime identity to derive the explicit solution to the eigentime identity for random walks on the Cayley networks[2].

In this paper, we use normalized normalized Laplacian spectrum to get the eigentime identity on a family of treelike networks. We obtain the normalized Laplacian spectra by writing the normalized Laplacian matrix of the networks. Due to the different structure of the network, the form of normalized Laplacian matrix is also different. So, using this method to research the eigentime identity has great flexibility .Besides, normalized Laplacian spectra is related to a lot of the performance index of the networks[3, 4], such as mixing time, return-to-origin probability, eigentime identity[5].

The organization of this paper is as follows. In next section we introduce the definition of eigentime identity. In Section 3, we give the model of a family of treelike networks and get the scalings of eigentime identity with network size on a family of treelike networks. In the last section we draw the conclusions.

II. EIGENTIME IDENTITY

In this section, we introduce the concept of the eigentime identity.

Let $F_{ij}(n)$ be mean-first passage time from node i to node j in G_n, which is the expect time for a particle starting off from node i to arrive at node j for the first time. Let $d_i(n)$ be the degree of node i and E_n be the number of edges in G_n. The stationary distribution for random walks on G_n is $\pi=(\pi_1,\pi_2,...,\pi_N)^T$, where $\pi_i=\frac{d_i(n)}{2E_n}$, obeying relations $\sum_{i=1}^{N}\pi_i=1$ and $\pi^T M_n=\pi^T$, where M_n be the Markov matrix of G_n. Let H_n represent the eigentime identity for random walks On G_n, which is defined as the expected time for a walker going from a node i to another node j, chosen randomly from all nodes accordingly to the stationary distribution [6]. That is,

$$H_n=\sum_{j=1}^{N_n}\pi_j F_{ij}(n), \qquad (1)$$

where N_n is the number of nodes of G_n. H_n quantifies the expected time taken by a particle starting from node i to get to a node (target) j randomly chosen according the stationary distribution. Since H_n do not rely on the starting node , it can be rewritten as

$$H_n = \sum_{i=1}^{N_n} \pi_i \sum_{j=1}^{N_n} \pi_j F_{ij}(n) = \sum_{j=1}^{N_n} \pi_j \sum_{i=1}^{N_n} \pi_i F_{ij}(n). \quad (2)$$

The rightmost expression in above equation indicates that the eigentime identity H_n is actually the average trapping time of a special trapping problem, which involves a double weighted average: the former is over all the source nodes to a given trapping (target) node j. The later is the average with respect to the first one taken over the stationary distribution. Because trapping is a fundamental mechanism for various other dynamical processes, H_n contains much information about trapping and diverse processes taking place on complex systems [6].

According to previous results [6], let L_n be the normalized Laplacian matrix of G_n . H_n can be expressed in terms of the nonzero eigenvalues of L_n as

$$H_n = \sum_{i=1}^{N} \frac{1}{\lambda_i}, \quad (3)$$

where we assume $\lambda_1 = 0$.

In the following, we introduce a family of treelike networks inspired by the models in [7, 8, 9, 10]. According to their constructions, we will obtain the scalings of eigentime identity for a family of reelike networks with network size.

III. Eigentime identity of a family of treelike networks

A. Construction of a family of treelike networks

Initially $(n = 0)$, G_0 *consists of only a central node. To form* G_1*, we create* 4 *nodes and attach them to the central node. For any* $n \ge 1$, G_n *is obtained from* G_{n-1} *by performing the following operations. For each outermost node of* G_{n-1}*,* 4 *nodes are generated and attached them to the outermost node. Let* $G_n = G(V_n, E_n)$, $G_n = G(V_n, E_n)$ *be its associated network, with node set* $V_n(|V_n| = N_n)$ *and edge set* $E_n(|E_n| = N_n - 1)$*. In Fig.1, we schematically illustrate the process of the first three iterations. From the construction of a family treelike networks, one can see that* G_n*, is characterized by the parameter* n*. Let* $N_i(n)$ *denote the number of nodes in* G_n*, which are given birth to at iteration* i*. It is easy to check that* $N_i = 4^i$*. The total number of nodes in* G_n*,* N_n*, satisfies the following relationship,*

$$N_n = \sum_{i=0}^{n} N_i = \frac{4^{n+1} - 1}{3}. \quad (4)$$

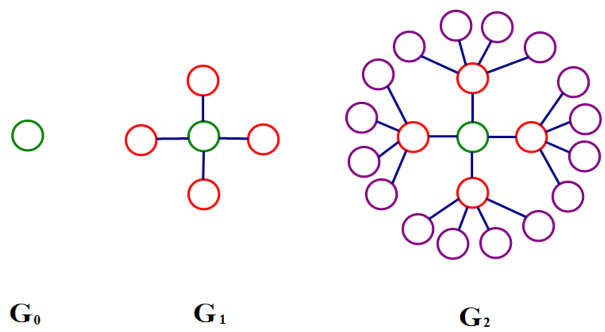

Figure 1: Take a family of treelike G_0, G_1 and G_2 for example.

B. The characteristic polynomial of the normalized Laplacian matrix

In this subsection, we use the elementary matrix operations to reduce the related matrix to lower

triangle matrix.

Let A_n and D_n be the adjacency matrix and diagonal degree matrix of G_n . Then, its normalized

Laplacian matrix is $L_n = I_n - D_n^{-1} A_n$. The N_n nodes can be divided into $n+1$ levels: the 0th level contains only one node (i.e., the central node) labeled by 1; the ith $(1 \le i \le n)$ level has $N_i(n) = N_i - N_{i-1}$ nodes, which are labeled sequentially by $N_{i-1}+1, N_{i-1}+2, \cdots, N_i$.

We now address the eigenvalue problem of L_n . By definition, all eigenvalues of L_n are actually the roots of characteristic equation $\det(L_n - \lambda I_n) = 0$. Let $X_n = L_n - \lambda I_n$.. Then $\det(X_n)$ is a determinant of order $\frac{4^{n+1}-1}{3}$. Next we apply the row operations of determinant to transform $\det(X_n)$ into a lower triangle determinant.

Now, we calculate the $\det(X_n)$ of G_n, let R_k represent the kth row of X_n and its variants after

being performed row operations. In order to have a lower triangle matrix, the row operations are performed as follows.

- First, we keep rows $R_k(N_{n-1}+1 \le k \le N_n)$ unchanged and define their diagonal entries as $f_1(\lambda)=1-\lambda$ as in the original matrix.

- For i from 1 to $k\,(N_{n-i-1}+1 \le k \le N_{n-i})$ n, we repeat the following two operations: (i) For each, we multiply R_k by $f_i(\lambda)$; m(ii) For each $k\,(N_{n-i-1}+1 \le k \le N_{n-i})$, we add the sum of $R_{4k+1}, R_{4k+}2, R_{4k+3}, \cdots, R_{4k+4}$ times $\frac{1}{5}$ to R_k. Assuming that $N_{-1}=0$, after performing the two operations, for $i \le n$, the diagonal entry of $R_k\,(N_{n-i-1}+1 \le k \le N_{n-i})$ becomes $f_{i+1}(\lambda)$, while the other entries of R_k on the right-hand side of the diagonal entry $f_{i+1}(\lambda)$ are zeros. Finally, we add R_2 times $\frac{1}{5}$ to R_1.

The above row operations reduce matrix X_n to lower triangle matrix Y_n In the similar way, the diagonal elements of Y_n are as follows: $f_{n+1}(\lambda)-\frac{1}{25}f_{n-1}(\lambda)$ for the first row, $f_i(\lambda)(1 \le i \le n)$ for those rows starting from $N_{n-i}+1$ to N_{n-i+1}. It follows above row operations that the functions $f_i(\lambda)$ obey the following recursive relation:

$$f_i(\lambda)=\begin{cases} 1-\lambda, & i=1, \\ (1-\lambda)^2-\frac{4}{5}, & i=2, \\ (1-\lambda)f_{i-1}(1-\lambda)-\frac{4}{25}f_{i-2}(\lambda), & 3 \le i \le n. \end{cases} \quad (5)$$

According to the properties of determinants, we have $\det(Xn)=\det(\frac{Y_n}{D(\lambda)})$, where $D(\lambda)$ is the overall factor. From the above procedure, we can obtain

$$D(\lambda)=f_n(\lambda)\prod_{i=1}^{n-1}[f_i(\lambda)]^{4^{n-i+1}}. \quad (6)$$

Thus, we have the characteristic polynomial of L_n,

$$\det(X_n)=\frac{[f_n(\lambda)]^4[f_{n+1}(\lambda)-\frac{1}{25}f_{n-1}(\lambda)]\prod_{i=1}^{n-1}[f_i(\lambda)]^{4^{n-i+1}}}{D(\lambda)} \quad (7)$$
$$=\frac{[f_n(\lambda)]^3[f_{n+1}(\lambda)-\frac{1}{25}f_{n-1}(\lambda)]\prod_{i=1}^{n-1}[f_i(\lambda)]^{3\times4^{n-i}}}{D(\lambda)}.$$

Since the eigenvalues of L_n are the roots of $\det(X_n)=0$, the problem of computing eigenvalues of Ln becomes to find the roots of functions $fi(\lambda)(1 \le i \le n)$ and $f_{n+1}(\lambda)-\frac{1}{25}f_{n-1}(\lambda)$. From Eq. (2), it is obvious that $f_i(\lambda)$ is a polynomial of λ with degree i. $f_n(\lambda)$ generates n eigenvalues and $f_{n+1}(\lambda)-\frac{1}{25}f_{n-1}(\lambda)$ generates $n+1$ eigenvalues. Every $f_i(\lambda)=0\,(i=1,2,\ldots,n-1)$ provides i different roots, each of which is an eigenvalue of Ln having a multiplicity $3\times4^{n-i}$. Thus the total number of eigenvalues is

$$3n+(n+1)+\sum_{i=1}^{n-1}3i\times4^{n-i}=\frac{4^{n+1}-1}{3}=N_n, \quad (8)$$

implying that all eigenvalues of L_n are produced by $f_i(\lambda)=0\,(i=1,2,\ldots,n)$ and $f_{n+1}(\lambda)-\frac{1}{25}f_{n-1}(\lambda)=0.$

C. The eigentime identity on a family of treelike networks

It is difficult to obtain eigentime identity on a family of treelike networks straightforwardly from the

eigenvalues of its normalized Laplacian. For this networks with recursive structures, we compute

the analytical eigentime identity without explicitlycomputing these eigenvalues.

For each $f_i(\lambda)(1 \le i \le n)$, which is an i degree polynomial, we can rewrite it as

$$f_i(\lambda)=\alpha_i+\beta_i\lambda+\gamma_i\lambda^2+\cdots. \quad (9)$$

Thus, equation $f_i(\lambda)=0$ has i nonzero roots, labeled by $\lambda_1^{(i)}, \lambda_2^{(i)}, \cdots, \lambda_i^{(i)}$. According to Vieta’s formulas, we have

$$\sum_{j=1}^{i}\frac{1}{\lambda_j^{(i)}}=-\frac{\beta_i}{\alpha_i}. \quad (10)$$

Combaring Eqs. (2) and (3), for $i=3,\cdots,n,$ we obtain

$$\alpha_i=\alpha_{i-1}-\frac{4}{25}\alpha_{i-2}, \quad (11)$$

$$\beta_i=\beta_{i-1}-\alpha_{i-1}-\frac{4}{25}\beta_{i-2}, \quad (12)$$

$$\gamma_i=\gamma_{i-1}-\beta_{i-1}-\frac{4}{25}\gamma_{i-2}. \quad (13)$$

Considering the initial conditions $\alpha_1=1,\beta_1=-1,\gamma_1=0,\alpha_2=\frac{1}{5},\ \beta_2=-2,\ \gamma_2=1.$ For $i=1,2,\cdots,n,$, Eqs. (4)-(6) are easily solved to yield

$$\alpha_i=5^{1-i}, \quad (14)$$

$$\beta_i=5^{i-1}[\frac{5(i-2)}{3}+\frac{38}{9}-\frac{2\times 4^{i+1}}{9}], \quad (15)$$

and

$$\gamma_i=\frac{1}{27\times 5^{i-2}}[(2i-3)\times 4^{i+1}+40-\frac{1536\times 4^{i-3}}{3}+13i+\frac{15(i+3)(i-2)}{2}] \quad (16)$$

Therefore, for any $1\le i\le n$

$$\sum_{j=1}^{i}\frac{1}{\lambda_j^{(i)}}=-\frac{\beta_i}{\alpha_i}=\frac{2^{2i+3}}{9}-\frac{5i}{3}-\frac{8}{9}. \quad (17)$$

On the other hand, we an see thatt only zero eigenvalue of L_n is generated by $f_{n+1}(\lambda)-\frac{1}{25}f_{n-1}(\lambda)=0.$

$$f_{n+1}(\lambda)-\frac{1}{25}f_{n-1}(\lambda)=\lambda[(\beta_{n+1}-\frac{1}{25}\beta_{n-1})+(\gamma_{n+1}-\frac{1}{25}\gamma_{n-1})+\cdots]. \quad (18)$$

Notice that the n nonzero roots, $\lambda_1^{(n+1)},\lambda_1^{(n+2)},\ldots,\lambda_n^{(n+1)}$, satisfy

$$\sum_{j=1}^{n}\frac{1}{\lambda_j^{(n+1)}}=-\frac{\gamma_{n+1}-\frac{1}{25}\gamma_{n-1}}{\beta_{n+1}-\frac{1}{25}\beta_{n-1}}. \quad (19)$$

We calculate molecule member element

$$\gamma_{n+1}-\frac{1}{25}\gamma_{n-1}=\frac{1}{54\times 5^{n-1}}[(2n-1)\times 2^{2n+3}-3\times 4^{n+2}-(2n+1)\times 2^{2n+1}+36n+70]. \quad (20)$$

We calculate denominator

$$\beta_{n+1}-\frac{1}{25}\beta_{n-1}=\frac{30-30\times 4^n}{9\times 5^n}. \quad (21)$$

Then

$$\sum_{j=1}^{n}\frac{1}{\lambda_j^{(n+1)}}=-\frac{5[(2n-1)\times 2^{2n-5}-3\times 4^{n+2}]}{6(2^{2n+1}-2^{2n+5}+30)}-\frac{5[36n+70+(2n+1)\times 2^{2n+1}]}{6(2^{2n+1}-2^{2n+5}+30)}. \quad (21)$$

From Eqs. (7) and (8), we can obtain that

$$\begin{aligned}H_n&=\sum_{j=1}^{N_n}\frac{1}{\lambda_j^{(n)}}\\&=-[3\sum_{i=1}^{n-1}4^{n-i}\times\frac{\beta_i}{\alpha_i}]-3\frac{\beta_n}{\alpha_n}-\frac{\gamma_{n+1}-\frac{1}{25}\gamma_{n-1}}{\beta_{n+1}-\frac{1}{25}\beta_{n-1}}\\&=5\times 2^{2n+1}+2^{2n-2}-60+\frac{(n-1)\times 2^{2n+!}}{3}\\&+\frac{3\times 2^{2n+4}-2^{2n+2}2^{2n+1}+60n-228}{9}\\&+2^{2n+1}+2^{2n-2}-5n+4\\&-\frac{5[(2n-1)\times 2^{2n-5}-3\times 4^{n+2}]}{6(2^{2n+1}-2^{2n+5}+30)}\\&-\frac{5[36n+70+(2n+1)\times 2^{2n+1}]}{6(2^{2n+1}-2^{2n+5}+30)}\\&\approx\frac{(n-1)\times 2^{2n+3}}{3}\sim n\cdot 4^n.\end{aligned} \quad (22)$$

For very large networks(i.e., $N_n\to\infty$) , the leading term of H_n obeys

$$H_n \sim N_n \ln N_n. \quad (23)$$

IV. conclusions

In this paper, we calculated the eigentime identity a famil of treelike networks and obtained the exact scalings of eigentime identity with network size of the polymer networks. For a family of treelike treelike networks, we showed that all their eigenvalues can be obtained by computing the roots of several small-degree polynomials defined recursively. We got the scalings of eigentime identity for a family of treelike networks with network size N_n is $N_n \ln N_n$. we draw the conclusion that the size of a family of treelike networks affects the efficiency. The larger the size, the lower the efficiency for a family of treelike networks.

ACKNOWLEDGES

Research is supported by the Humanistic and Social Science Foundation from Ministry of Education of China (Grant 14YJAZH012) and National Natural Science Foundation of China (Nos. 11671172, 11501255).

REFERENCES

[1] W. G. Sun , S. Wang and J. Y. Zhang, Counting spanning trees in prism and anti-prism graphs. J. A. A. C. vol. 6, pp.65-75 , 2016.

[2] A. Julaiti, B. Wu, and Z. Z. Zhang, Eigenvalues of normalized Laplacian matrices of fractal trees and dendrimers: Analytical results and applications, J. Chem. Phys. , Vol. 138, pp. 204116, 2013.

[3] P. Xie , Y. Lin, and Z. Z. Zhang, Spectrum of walk matrix for Koch network and its application, J. Chem. Phys., vol. 142, pp. 224106 , 2015.

[4] Z. Z. Zhang, X. Y. Guo, and Y. H. Yi, Spectra of weighted scale-free networks, Sci. Rep., vol. 5, pp. 17469 (2015).

[5] Z. Z. Zhang, X. Y. Guo, and Y. Lin, Eigenvalues for the transition matrix of a small-world scale-free network: Explicit expressions and applications, Phys. Rev. E, vol. 90, pp. 022816, 2014.

[6] D. Aldous, and J. Fill, Reversible Markov chains and random walks on graphs. See http://www.stat.berkeley.edu/aldous/RWG/book.html ,1999.

[7] T. Carletti, and S. Righi, Weighted Fractal Networks, Phys. A, vol. 389, pp. 2134-2142, 2010.

[8] M. F. Dai, , Y. Q. Sun, S. S. Shao, L. F. Xi, and W. Y. Su, Modified box dimension and average weighted receiving time on the weighted fractal networks, Sci. Rep., vol. 74, pp. 47, 2015.

[9] M. F. Dai, D. D.Ye , J. Hou , L. F. Xi, and W. Y. Su, Average weighted trapping time of the node- and edge- weighted fractal networks. Commun. Nonlinear Sci. Numer. Simulat., vol. 39, pp.209-219, 2016.

[10]M. F. Dai, Y. Q.Sun, Y. Sun, L. F. Xi, and S. X. Shao, The entire mean weighted first-passage time on a family of weighted treelike networks, Sci. Rep., vol. 6, pp. 28733, 2016.