# Generalizations of Pythagoras Theorem to Polygons*
# Geometry / Algebra

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***Abstract*** **– The celebrated Greek mathematician (more precisely a geometer) Pythagoras was born on the Island of Samos (Greece) during the period about 569 B.C. He left Samos for Egypt around 535 B.C. to study with the priests in temples. He gave a legendary result known as Pythagoras theorem:**

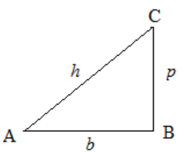
***Sum of squares of two mutually perpendicular sides in a right triangle equals the square of the hypotenuse.***

***Keywords – Geometry; Algebra; Theory of Equations; polygons*.**

## 1. Introduction

The celebrated Greek mathematician (more precisely a geometer) Pythagoras was born on the Island of Samos (Greece) during the period about 569 B.C. He left Samos for Egypt around 535 B.C. to study with the priests in temples. He gave a legendary result known as Pythagoras theorem:

*Sum of squares of two mutually perpendicular sides in a right triangle equals the square of the hypotenuse*, i.e.




$$b^2 + p^2 = h^2,$$

**Fig. 1**

where $b$, $p$, $h$ are the lengths of base, perpendicular and hypotenuse of the triangle. In the present paper an attempt has been made to extend above result by considering the left hand expression as the sum of squares of 3 integers becoming the square of the fourth integer. The first paper in this series deals with quadrilaterals. In terms of symbols we explore the solutions of the quadratic equation

$$a^2 + b^2 + c^2 = d^2. \qquad (1.1)$$

Interpreting the result geometrically the integers $a$, $b$, $c$, $d$ satisfying above relation shall represent the lengths of consecutive sides AB, BC, CD and DA respectively of a quadrilateral ABCD composed of 2 right triangles ABC and ACD with right angles at their vertices B and C.

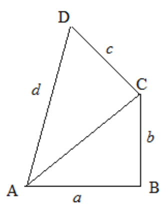



**Fig. 2**

The first fifteen Sections deal with the direct sum of squares of some positive integers making the square of a fourth integer. In the latter part, some special identities involving complicated terms are also discussed.

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* Many results of the paper were worked out during stay of the first author at Divine Word Univ., Madang (PNG).

## 2. Identities of the type $a^2 + n^2 + b^2 = (b + 1)^2$

Expanding the right hand member, and dropping the common terms one easily derives the value of $b$ in terms of $a$ and $n$ in order to satisfy above identity:

$$b = (a^2 + n^2 - 1) / 2, \qquad (2.1)$$

where $a$ and $n$ assume some suitable integral values making $b$ integer. For different integral values $a$ = 1, 2, 3, 4, 5, etc. above relation yields the following values of $b$:

| $a$ | $b$ | Remark |
|---|---|---|
| 1 | $n^2 / 2$ | $b$ is integer for even $n$. |
| 2 | $(n^2 + 3)/2 = (n^2 + 1)/2 + 1$ | $b$ is integer for odd $n$. |
| 3 | $(n^2 + 8)/2 = n^2/2 + 4$ | As for $a$ = 1. |
| 4 | $(n^2 + 15)/2 = (n^2 + 1)/2 + 7$ | As for $a$ = 2. |
| 5 | $(n^2 + 24)/2 = n^2/2 + 12$ | As for $a$ = 1. |
| 6 | $(n^2 + 35)/2 = (n^2 + 1)/2 + 17$ | As for $a$ = 2. |
| 7 | $(n^2 + 48)/2 = n^2/2 + 24$ | As for $a$ = 1. |
| 8 | $(n^2 + 63)/2 = (n^2 + 1)/2 + 31$ | As for $a$ = 2. |
| 9 | $(n^2 + 80)/2 = n^2/2 + 40$ | As for $a$ = 1. |
| 10 | $(n^2 + 99)/2 = (n^2 + 1)/2 + 49$ | As for $a$ = 2. |

etc. As such, there exist identities of above type for every integer $a$. Thus, we have the following theorems.

**Theorem 2.1.** For $a$ = 1, there hold the identities:

| $n$ | $b$ | Identity |
|---|---|---|
| 2 | 2 | $1^2 + 2^2 + 2^2 = 3^2$ |

| | | |
|---|---|---|
| 4 | 8 | $1^2 + 4^2 + 8^2 = 9^2$ |
| 6 | 18 | $1^2 + 6^2 + 18^2 = 19^2$ |
| 8 | 32 | $1^2 + 8^2 + 32^2 = 33^2$ |
| 10 | 50 | $1^2 + 10^2 + 50^2 = 51^2$ |
| 12 | 72 | $1^2 + 12^2 + 72^2 = 73^2$ |

etc. //

**Theorem 2.2.** For $a$ = 2, here hold the identities:

| *n* | *b* | Identity | Reference |
|---|---|---|---|
| 1 | 2 | $2^2 + 1^2 + 2^2 = 3^2$ | Theo. 2.1 |
| 3 | 6 | $2^2 + 3^2 + 6^2 = 7^2$ | |
| 5 | 14 | $2^2 + 5^2 + 14^2 = 15^2$ | |
| 7 | 26 | $2^2 + 7^2 + 26^2 = 27^2$ | |
| 9 | 42 | $2^2 + 9^2 + 42^2 = 43^2$ | |
| 11 | 62 | $2^2 + 11^2 + 62^2 = 63^2$ | |
| 13 | 86 | $2^2 + 13^2 + 86^2 = 87^2$ | |

etc. //

**Theorem 2.3.** For $a$ = 3, there hold the identities:

| *n* | *b* | Identity | Reference |
|---|---|---|---|
| 2 | 6 | $3^2 + 2^2 + 6^2 = 7^2$ | Theo. 2.2 |
| 4 | 12 | $3^2 + 4^2 + 12^2 = 13^2$ | |
| 6 | 22 | $3^2 + 6^2 + 22^2 = 23^2$ | |
| 8 | 36 | $3^2 + 8^2 + 36^2 = 37^2$ | |
| 10 | 54 | $3^2 + 10^2 + 54^2 = 55^2$ | |
| 12 | 76 | $3^2 + 12^2 + 76^2 = 77^2$ | |

etc. //

**Theorem 2.4.** For $a$ = 4, there hold the identities:

| *n* | *b* | Identity | Reference |
|---|---|---|---|
| 1 | 8 | $4^2 + 1^2 + 8^2 = 9^2$ | Theo. 2.1 |
| 3 | 12 | $4^2 + 3^2 + 12^2 = 13^2$ | Theo. 2.3 |
| 5 | 20 | $4^2 + 5^2 + 20^2 = 21^2$ | |
| 7 | 32 | $4^2 + 7^2 + 32^2 = 33^2$ | |
| 9 | 48 | $4^2 + 9^2 + 48^2 = 49^2$ | |
| 11 | 68 | $4^2 + 11^2 + 68^2 = 69^2$ | |

etc. //

**Theorem 2.5.** For $a$ = 5, here hold the identities:

| *n* | *b* | Identity | Reference |
|---|---|---|---|
| 2 | 14 | $5^2 + 2^2 + 14^2 = 15^2$ | Theo. 2.2 |
| 4 | 20 | $5^2 + 4^2 + 20^2 = 21^2$ | Theo. 2.4 |
| 6 | 30 | $5^2 + 6^2 + 30^2 = 31^2$ | |
| 8 | 44 | $5^2 + 8^2 + 44^2 = 45^2$ | |
| 10 | 62 | $5^2 + 10^2 + 62^2 = 63^2$ | |
| 12 | 84 | $5^2 + 12^2 + 84^2 = 85^2$ | |

etc. //

**Theorem 2.6.** For $a$ = 6, there hold the identities:

| *n* | *b* | Identity | Reference |
|---|---|---|---|
| 1 | 18 | $6^2 + 1^2 + 18^2 = 19^2$ | Theo. 2.1 |
| 3 | 22 | $6^2 + 3^2 + 22^2 = 23^2$ | Theo. 2.3 |
| 5 | 30 | $6^2 + 5^2 + 30^2 = 31^2$ | Theo. 2.5 |
| 7 | 42 | $6^2 + 7^2 + 42^2 = 43^2$ | |
| 9 | 58 | $6^2 + 9^2 + 58^2 = 59^2$ | |
| 11 | 78 | $6^2 + 11^2 + 78^2 = 79^2$ | |

etc. //

**Theorem 2.7.** For $a$ = 7, there hold the identities:

| *n* | *b* | Identity | Reference |
|---|---|---|---|
| 2 | 26 | $7^2 + 2^2 + 26^2 = 27^2$ | Theo. 2.2 |
| 4 | 32 | $7^2 + 4^2 + 32^2 = 33^2$ | Theo. 2.4 |
| 6 | 42 | $7^2 + 6^2 + 42^2 = 43^2$ | Theo. 2.6 |
| 8 | 56 | $7^2 + 8^2 + 56^2 = 57^2$ | |
| 10 | 74 | $7^2 + 10^2 + 74^2 = 75^2$ | |
| 12 | 96 | $7^2 + 12^2 + 96^2 = 97^2$ | |

etc. //

**Theorem 2.8.** For $a$ = 8, there hold the identities:

| $n$ | $b$ | Identity | Reference |
|---|---|---|---|
| 1 | 32 | $8^2 + 1^2 + 32^2 = 33^2$ | Theo. 2.1 |
| 3 | 36 | $8^2 + 3^2 + 36^2 = 37^2$ | Theo. 2.3 |
| 5 | 44 | $8^2 + 5^2 + 44^2 = 45^2$ | Theo. 2.5 |
| 7 | 56 | $8^2 + 7^2 + 56^2 = 57^2$ | Theo. 2.7 |
| 9 | 72 | $8^2 + 9^2 + 72^2 = 73^2$ | |
| 11 | 92 | $8^2 + 11^2 + 92^2 = 93^2$ | |

etc. //

**Theorem 2.9.** For $a$ = 9, there hold the identities:

| $n$ | $b$ | Identity | Reference |
|---|---|---|---|
| 2 | 42 | $9^2 + 2^2 + 42^2 = 43^2$ | Theo. 2.2 |
| 4 | 48 | $9^2 + 4^2 + 48^2 = 49^2$ | Theo. 2.4 |
| 6 | 58 | $9^2 + 6^2 + 58^2 = 59^2$ | Theo. 2.6 |
| 8 | 72 | $9^2 + 8^2 + 72^2 = 73^2$ | Theo. 2.8 |
| 10 | 90 | $9^2 + 10^2 + 90^2 = 91^2$ | |
| 12 | 112 | $9^2 + 12^2 + 112^2 = 113^2$ | |

etc. //

**Theorem 2.10.** For $a$ = 10, there hold the identities:

| $n$ | $b$ | Identity | Reference |
|---|---|---|---|
| 1 | 50 | $10^2 + 1^2 + 50^2 = 51^2$ | Theo. 2.1 |
| 3 | 54 | $10^2 + 3^2 + 54^2 = 55^2$ | Theo. 2.3 |
| 5 | 62 | $10^2 + 5^2 + 62^2 = 63^2$ | Theo. 2.5 |
| 7 | 74 | $10^2 + 7^2 + 74^2 = 75^2$ | Theo. 2.7 |
| 9 | 90 | $10^2 + 9^2 + 90^2 = 91^2$ | Theo. 2.9 |
| 11 | 110 | $10^2 + 11^2 + 110^2 = 111^2$ | |

etc. //

## 3. Identities of the type $a^2 + n^2 + b^2 = (b + 2)^2$

Above type of identities require:

$$b = (a^2 + n^2 - 4) / 4 = (a^2 + n^2) / 4 - 1, \qquad (3.1)$$

where $a$ and $n$ assume some suitable integral values making $b$ integer. For $a$ = 1, Eq. (3.1) yields

$$b = (n^2 + 1) / 4 - 1,$$

which cannot assume integral values for any integer $n$. For instance, when $n$ is even (say $2p$),

$$(n^2 + 1) / 4 = (4p^2 + 1) / 4 = p^2 + 1/4,$$

which is never an integer for any integer $p$. Similarly, for *odd* values of $n$ (say $2p + 1$),

$$(n^2 + 1) / 4 = (4p^2 + 4p + 2) / 4 = p^2 + p + 1/2,$$

is also not integer. Hence, there exist no such identities for $a$ = 1. But, $a = 2 \Rightarrow b = n^2/4$ assuming integral values for any even $n$. Hence, there exist identities for $a$ = 2. In the following, we check for other values of $a$. Different integral values of $a$ = 3, 4, 5, etc. yield the following values of $b$:

| $a$ | $b$ | Remark |
|---|---|---|
| 3 | $(n^2 + 5)/4 = (n^2 + 1)/4 + 1$ | As for $a$ = 1. |
| 4 | $(n^2 + 12)/4 = n^2/4 + 3$ | As for $a$ = 2. |
| 5 | $(n^2 + 21)/4 = (n^2 + 1)/4 + 5$ | As for $a$ = 1. |
| 6 | $(n^2 + 32)/4 = n^2/4 + 8$ | As for $a$ = 2. |
| 7 | $(n^2 + 45)/4 = (n^2 + 1)/4 + 11$ | As for $a$ = 1. |
| 8 | $(n^2 + 60)/4 = n^2/4 + 15$ | As for $a$ = 2. |
| 9 | $(n^2 + 77)/4 = (n^2 + 1)/4 + 19$ | As for $a$ = 1. |
| 10 | $(n^2 + 96)/4 = n^2/4 + 24$ | As for $a$ = 2. |

etc. Conclusively, there exist identities for even values of $a$, but no identities for odd values of $a$.

**Theorem 3.1.** For $a$ = 2, there hold the identities:

| $n$ | $b$ | Identity | Equivalently | Ref. |
|---|---|---|---|---|
| 2 | 1 | $2^2 + 2^2 + 1^2 = 3^2$ | | Th. 2.1 |
| 4 | 4 | $2^2 + 4^2 + 4^2 = 6^2$ | $1^2 + 2^2 + 2^2 = 3^2$ | ” |
| 6 | 9 | $2^2 + 6^2 + 9^2 = 11^2$ | | Th. 3.1 |
| 8 | 16 | $2^2 + 8^2 + 16^2 = 18^2$ | $1^2 + 4^2 + 8^2 = 9^2$ | Th. 2.1 |
| 10 | 25 | $2^2 + 10^2 + 25^2 = 27^2$ | $1^2 + 10^2 + 50^2 = 51^2$ | ” |
| 12 | 36 | $2^2 + 12^2 + 36^2 = 38^2$ | $1^2 + 12^2 + 72^2 = 73^2$ | ” |

etc. //

**Theorem 3.2.** For $a$ = 4, there hold the following identities:

| $n$ | $b$ | Identity | Equivalently | Ref. |
|---|---|---|---|---|
| 2 | 4 | $4^2 + 2^2 + 4^2 = 6^2$ | $2^2 + 1^2 + 2^2 = 3^2$ | Th. 2.1 |
| 4 | 7 | $4^2 + 4^2 + 7^2 = 9^2$ | | Th. 3.2 |
| 6 | 12 | $4^2 + 6^2 + 12^2 = 14^2$ | $2^2 + 3^2 + 6^2 = 7^2$ | Th. 2.2 |
| 8 | 19 | $4^2 + 8^2 + 19^2 = 21^2$ | | Th. 3.2 |
| 10 | 28 | $4^2 + 10^2 + 28^2 = 30^2$ | $2^2 + 5^2 + 14^2 = 15^2$ | Th. 2.2 |
| 12 | 39 | $4^2 + 12^2 + 39^2 = 41^2$ | | Th. 3.2 |

etc. //

**Theorem 3.3.** For $a$ = 6, there hold the identities:

| $n$ | $b$ | Identity | Equivalently | Ref. |
|---|---|---|---|---|
| 2 | 9 | $6^2 + 2^2 + 9^2 = 11^2$ | | Th. 3.1 |
| 4 | 12 | $6^2 + 4^2 + 12^2 = 14^2$ | $3^2 + 2^2 + 6^2 = 7^2$ | Th. 2.2 |
| 6 | 17 | $6^2 + 6^2 + 17^2 = 19^2$ | | Th. 3.3 |
| 8 | 24 | $6^2 + 8^2 + 24^2 = 26^2$ | $3^2 + 4^2 + 12^2 = 13^2$ | Th. 2.3 |
| 10 | 33 | $6^2 + 10^2 + 33^2 = 35^2$ | | Th. 3.3 |
| 12 | 44 | $6^2 + 12^2 + 44^2 = 46^2$ | $3^2 + 6^2 + 22^2 = 23^2$ | Th. 2.3 |

etc. //

**Theorem 3.4.** For $a$ = 8, there hold the identities:

| $n$ | $b$ | Identity | Equivalently | Ref. |
|---|---|---|---|---|
| 2 | 16 | $8^2 + 2^2 + 16^2 = 18^2$ | $4^2 + 1^2 + 8^2 = 9^2$ | Th. 2.1 |
| 4 | 19 | $8^2 + 4^2 + 19^2 = 21^2$ | | Th. 3.2 |
| 6 | 24 | $8^2 + 6^2 + 24^2 = 26^2$ | $4^2 + 3^2 + 12^2 = 13^2$ | Th. 2.3 |
| 8 | 31 | $8^2 + 8^2 + 31^2 = 33^2$ | | Th. 3.4 |
| 10 | 40 | $8^2 + 10^2 + 40^2 = 42^2$ | $4^2 + 5^2 + 20^2 = 21^2$ | Th. 2.4 |
| 12 | 51 | $8^2 + 12^2 + 51^2 = 53^2$ | | Th. 3.4 |

etc. //

**Theorem 3.5.** For $a$ = 10, there hold the identities:

| $n$ | $b$ | Identity | Equivalently | Ref. |
|---|---|---|---|---|
| 2 | 25 | $10^2 + 2^2 + 25^2 = 27^2$ | | Th. 3. 1 |
| 4 | 28 | $10^2 + 4^2 + 28^2 = 30^2$ | $5^2 + 2^2 + 14^2 = 15^2$ | Th. 2.2 |
| 6 | 33 | $10^2 + 6^2 + 33^2 = 35^2$ | | Th. 3.3 |
| 8 | 40 | $10^2 + 8^2 + 40^2 = 42^2$ | $5^2 + 4^2 + 20^2 = 21^2$ | Th. 2.4 |
| 10 | 49 | $10^2 + 10^2 + 49^2 = 51^2$ | | Th. 3.5 |
| 12 | 60 | $10^2 + 12^2 + 60^2 = 62^2$ | $5^2 + 6^2 + 30^2 = 31^2$ | Th. 2.5 |

etc. //

## 4. Identities of the type $a^2 + n^2 + b^2 = (b + 3)^2$

Above type of identities require:

$$b = (a^2 + n^2 - 9) / 6 = (a^2 + n^2 - 3) / 6 - 1, \qquad (4.1)$$

where $a$ and $n$ assume some suitable integral values making $b$ integer. For different integral values $a$ = 1, 2, 3, 4, 5, etc. above relation yields the following values of $b$:

| $a$ | $b$ | Remark |
|---|---|---|
| 1 | $(n^2 - 8)/6 = (n^2 + 4)/6 - 2$ | $b$ is not integer for any integer $n$. |
| 2 | $(n^2 - 5)/6 = (n^2 + 1)/6 - 1$ | ” |
| 3 | $n^2 / 6$ | $b$ is integer for $n$ = integral multiple of 6. |
| 4 | $(n^2 + 7)/6 = (n^2 + 1)/6 + 1$ | As for $a$ = 2 above. |
| 5 | $(n^2 + 16)/6 = (n^2 + 4)/6 + 2$ | As for $a$ = 1 above. |
| 6 | $(n^2 + 27)/6 = (n^2 + 3)/6 + 4$ | $b$ is integer for $n$ = 3, 9, 15, 21, 27, etc. |
| 7 | $(n^2 + 40)/6 = (n^2 + 4)/6 + 6$ | As for $a$ = 1 above. |
| 8 | $(n^2 + 55)/6 = (n^2 + 1)/6 + 9$ | As for $a$ = 2 above. |
| 9 | $(n^2 + 72)/6 = n^2/6 + 12$ | As for $a$ = 3 above. |
| 10 | $(n^2 + 91)/6 = (n^2 + 1)/6 + 15$ | As for $a$ = 2 above. |
| 11 | $(n^2 + 112)/6 = (n^2 + 4)/6 + 18$ | As for $a$ = 1 above. |
| 12 | $(n^2 + 135)/6 = (n^2 + 3)/6 + 22$ | As for $a$ = 6 above. |

etc. Thus, there exist identities whenever $a$ is an integral multiple of 3. Consequently, we have the following theorems.

**Theorem 4.1.** For $a$ = 3, there hold the identities:

| $n$ | $b$ | Identity | Equivalently | Ref. |
|---|---|---|---|---|
| 6 | 6 | $3^2 + 6^2 + 6^2 = 9^2$ | $1^2 + 2^2 + 2^2 = 3^2$ | Th. 2.1 |
| 12 | 24 | $3^2 + 12^2 + 24^2 = 27^2$ | $1^2 + 4^2 + 8^2 = 9^2$ | ” |

| | | | | |
|---|---|---|---|---|
| 18 | 54 | $3^2 + 18^2 + 54^2 = 57^2$ | $1^2 + 6^2 + 18^2 = 19^2$ | ” |
| 24 | 96 | $3^2 + 24^2 + 96^2 = 99^2$ | $1^2 + 8^2 + 32^2 = 33^2$ | ” |
| 30 | 150 | $3^2 + 30^2 + 150^2 = 153^2$ | $1^2 + 10^2 + 50^2 = 51^2$ | ” |
| 36 | 216 | $3^2 + 36^2 + 216^2 = 219^2$ | $1^2 + 12^2 + 72^2 = 73^2$ | ” |

etc. //

**Theorem 4.2.** For $a = 6$, there hold the identities:

| ***n*** | ***b*** | **Identity** | **Equivalently** | **Ref.** |
|---|---|---|---|---|
| 3 | 6 | $6^2 + 3^2 + 6^2 = 9^2$ | $2^2 + 1^2 + 2^2 = 3^2$ | Th. 2.1 |
| 9 | 18 | $6^2 + 9^2 + 18^2 = 21^2$ | $2^2 + 3^2 + 6^2 = 7^2$ | Th. 2.2 |
| 15 | 42 | $6^2 + 15^2 + 42^2 = 45^2$ | $2^2 + 5^2 + 14^2 = 15^2$ | ” |
| 21 | 78 | $6^2 + 21^2 + 78^2 = 81^2$ | $2^2 + 7^2 + 26^2 = 27^2$ | ” |
| 27 | 126 | $6^2 + 27^2 + 126^2 = 129^2$ | $2^2 + 9^2 + 42^2 = 43^2$ | ” |
| 33 | 186 | $6^2 + 33^2 + 186^2 = 189^2$ | $2^2 + 11^2 + 62^2 = 63^2$ | ” |

etc. //

**Theorem 4.3.** For $a = 9$, there hold the identities:

| ***n*** | ***b*** | **Identity** | **Equivalently** | **Ref.** |
|---|---|---|---|---|
| 6 | 18 | $9^2 + 6^2 + 18^2 = 21^2$ | $3^2 + 2^2 + 6^2 = 7^2$ | Th. 2.2 |
| 12 | 36 | $9^2 + 12^2 + 36^2 = 39^2$ | $3^2 + 4^2 + 12^2 = 13^2$ | Th. 2.3 |
| 18 | 66 | $9^2 + 18^2 + 66^2 = 69^2$ | $3^2 + 6^2 + 22^2 = 23^2$ | ” |
| 24 | 108 | $9^2 + 24^2 + 108^2 = 111^2$ | $3^2 + 8^2 + 36^2 = 37^2$ | ” |
| 30 | 162 | $9^2 + 30^2 + 162^2 = 165^2$ | $3^2 + 10^2 + 54^2 = 55^2$ | ” |
| 36 | 228 | $9^2 + 36^2 + 228^2 = 231^2$ | $3^2 + 12^2 + 76^2 = 77^2$ | ” |

etc. //

**Theorem 4.4.** For $a = 12$, there hold the identities:

| ***n*** | ***b*** | **Identity** | **Equivalently** | **Ref.** |
|---|---|---|---|---|
| 3 | 24 | $12^2 + 3^2 + 24^2 = 27^2$ | $4^2 + 1^2 + 8^2 = 9^2$ | Th. 2.1 |
| 9 | 36 | $12^2 + 9^2 + 36^2 = 39^2$ | $4^2 + 3^2 + 12^2 = 13^2$ | Th. 2.3 |
| 15 | 60 | $12^2 + 15^2 + 60^2 = 63^2$ | $4^2 + 5^2 + 20^2 = 21^2$ | Th. 2.4 |
| 21 | 96 | $12^2 + 21^2 + 96^2 = 99^2$ | $4^2 + 7^2 + 32^2 = 33^2$ | ” |
| 27 | 144 | $12^2 + 27^2 + 144^2 = 147^2$ | $4^2 + 9^2 + 48^2 = 49^2$ | ” |
| 33 | 204 | $12^2 + 33^2 + 204^2 = 207^2$ | $4^2 + 11^2 + 68^2 = 69^2$ | ” |

etc. //

**Theorem 4.5.** For $a = 15$, there hold the identities

| ***n*** | ***b*** | **Identity** | **Equivalently** | **Ref.** |
|---|---|---|---|---|
| 6 | 42 | $15^2 + 6^2 + 42^2 = 45^2$ | $5^2 + 2^2 + 14^2 = 15^2$ | Th. 2.2 |
| 12 | 60 | $15^2 + 12^2 + 60^2 = 63^2$ | $5^2 + 4^2 + 20^2 = 21^2$ | Th. 2.4 |
| 18 | 90 | $15^2 + 18^2 + 90^2 = 93^2$ | $5^2 + 6^2 + 30^2 = 31^2$ | Th. 2.5 |
| 24 | 132 | $15^2 + 24^2 + 132^2 = 135^2$ | $5^2 + 8^2 + 44^2 = 45^2$ | ” |
| 30 | 186 | $15^2 + 30^2 + 186^2 = 189^2$ | $5^2 + 10^2 + 62^2 = 63^2$ | ” |
| 36 | 252 | $15^2 + 36^2 + 252^2 = 255^2$ | $5^2 + 12^2 + 84^2 = 85^2$ | ” |

etc. //

**Theorem 4.6.** For $a = 18$, there hold the identities:

| ***n*** | ***b*** | **Identity** | **Equivalently** | **Ref.** |
|---|---|---|---|---|
| 3 | 54 | $18^2 + 3^2 + 54^2 = 57^2$ | $6^2 + 1^2 + 18^2 = 19^2$ | Th. 2.1 |
| 9 | 66 | $18^2 + 9^2 + 66^2 = 69^2$ | $6^2 + 3^2 + 22^2 = 23^2$ | Th. 2.3 |
| 15 | 90 | $18^2 + 15^2 + 90^2 = 93^2$ | $6^2 + 5^2 + 30^2 = 31^2$ | Th. 2.5 |
| 21 | 126 | $18^2 + 21^2 + 126^2 = 129^2$ | $6^2 + 7^2 + 42^2 = 43^2$ | Th. 2.6 |
| 27 | 174 | $18^2 + 27^2 + 174^2 = 177^2$ | $6^2 + 9^2 + 58^2 = 49^2$ | ” |
| 33 | 234 | $18^2 + 33^2 + 234^2 = 237^2$ | $6^2 + 11^2 + 78^2 = 79^2$ | ” |

etc. //

**Theorem 4.7.** For $a = 21$, there hold the identities:

| ***n*** | ***b*** | **Identity** | **Equivalently** | **Ref.** |
|---|---|---|---|---|
| 6 | 78 | $21^2 + 6^2 + 78^2 = 81^2$ | $7^2 + 2^2 + 26^2 = 27^2$ | Th. 2.2 |
| 12 | 96 | $21^2 + 12^2 + 96^2 = 99^2$ | $7^2 + 4^2 + 32^2 = 33^2$ | Th. 2.4 |
| 18 | 126 | $21^2 + 18^2 + 126^2 = 129^2$ | $7^2 + 6^2 + 42^2 = 43^2$ | Th. 2.6 |
| 24 | 168 | $21^2 + 24^2 + 168^2 = 171^2$ | $7^2 + 8^2 + 56^2 = 57^2$ | Th. 2.7 |
| 30 | 222 | $21^2 + 30^2 + 222^2 = 225^2$ | $7^2 + 10^2 + 74^2 = 75^2$ | ” |
| 36 | 288 | $21^2 + 36^2 + 288^2 = 291^2$ | $7^2 + 12^2 + 96^2 = 97^2$ | ” |

etc. //

**Theorem 4.8.** For $a = 24$, there hold the identities:

| ***n*** | ***b*** | **Identity** | **Equivalently** | **Ref.** |
|---|---|---|---|---|

| | | | | |
|---|---|---|---|---|
| 3 | 96 | $24^2 + 3^2 + 96^2 = 99^2$ | $8^2 + 1^2 + 32^2 = 33^2$ | Th. 2.1 |
| 9 | 108 | $24^2 + 9^2 + 108^2 = 111^2$ | $8^2 + 3^2 + 36^2 = 37^2$ | Th. 2.3 |
| 15 | 132 | $24^2 + 15^2 + 132^2 = 135^2$ | $8^2 + 5^2 + 44^2 = 45^2$ | Th. 2.5 |
| 21 | 168 | $24^2 + 21^2 + 168^2 = 171^2$ | $8^2 + 7^2 + 56^2 = 57^2$ | Th. 2.7 |
| 27 | 216 | $24^2 + 27^2 + 216^2 = 219^2$ | $8^2 + 9^2 + 72^2 = 73^2$ | Th. 2.8 |
| 33 | 276 | $24^2 + 33^2 + 276^2 = 279^2$ | $8^2 + 11^2 + 92^2 = 93^2$ | ” |

etc. //

**Theorem 4.9.** For $a = 27$, there hold the identities:

| *n* | *b* | Identity | Equivalently | Ref. |
|---|---|---|---|---|
| 6 | 126 | $27^2 + 6^2 + 126^2 = 129^2$ | $9^2 + 2^2 + 42^2 = 43^2$ | Th. 2.2 |
| 12 | 144 | $27^2 + 12^2 + 144^2 = 147^2$ | $9^2 + 4^2 + 48^2 = 49^2$ | Th. 2.4 |
| 18 | 174 | $27^2 + 18^2 + 174^2 = 177^2$ | $9^2 + 6^2 + 58^2 = 59^2$ | Th. 2.6 |
| 24 | 216 | $27^2 + 24^2 + 216^2 = 219^2$ | $9^2 + 8^2 + 72^2 = 73^2$ | Th. 2.8 |
| 30 | 270 | $27^2 + 30^2 + 270^2 = 273^2$ | $9^2 + 10^2 + 90^2 = 91^2$ | Th. 2.9 |
| 36 | 336 | $27^2 + 36^2 + 336^2 = 339^2$ | $9^2 + 12^2 + 112^2 = 113^2$ | ” |

etc. //

**Theorem 4.10.** For $a = 30$, there hold the identities:

| *n* | *b* | Identity | Equivalently | Ref. |
|---|---|---|---|---|
| 3 | 150 | $30^2 + 3^2 + 150^2 = 153^2$ | $10^2 + 1^2 + 50^2 = 51^2$ | Th. 2.1 |
| 9 | 162 | $30^2 + 9^2 + 162^2 = 165^2$ | $10^2 + 3^2 + 54^2 = 55^2$ | Th. 2.3 |
| 15 | 186 | $30^2 + 15^2 + 186^2 = 189^2$ | $10^2 + 5^2 + 62^2 = 63^2$ | Th. 2.5 |
| 21 | 222 | $30^2 + 21^2 + 222^2 = 225^2$ | $10^2 + 7^2 + 74^2 = 75^2$ | Th. 2.7 |
| 27 | 270 | $30^2 + 27^2 + 270^2 = 273^2$ | $10^2 + 9^2 + 90^2 = 91^2$ | Th. 2.9 |
| 33 | 330 | $30^2 + 33^2 + 330^2 = 333^2$ | $10^2 + 11^2 + 110^2 = 111^2$ | Th. 2.10 |

etc. //

## 5. Identities of the type $a^2 + n^2 + b^2 = (b + 4)^2$

Above type of identities require:

$$b = (a^2 + n^2 - 16) / 8, \qquad (5.1)$$

where $a$ and $n$ assume some suitable integral values making $b$ integer. For different integral values $a = 1, 2, 3, 4, 5$, etc. above relation yields the following values of $b$:

| *a* | *b* | Remark |
|---|---|---|
| 1 | $(n^2 - 15)/8 = (n^2 + 1)/8 - 2$ | $b$ is not integer for any integer $n$. |
| 2 | $(n^2 - 12)/8 = (n^2 - 4)/8 - 1$ | $b$ is integer for $n = 2$, 6, 10, 14, 18, 22, etc. |
| 3 | $(n^2 - 7)/8 = (n^2 + 1)/8 - 1$ | As for $a = 1$ above. |
| 4 | $n^2 / 8$ | $b$ is integer for $n$ = integral multiple of 4. |
| 5 | $(n^2 + 9)/8 = (n^2 + 1)/8 + 1$ | As for $a = 1$ above. |
| 6 | $(n^2 + 20)/8 = (n^2 - 4)/8 + 3$ | As for $a = 2$ above. |
| 7 | $(n^2 + 33)/8 = (n^2 + 1)/8 + 4$ | As for $a = 1$ above. |
| 8 | $(n^2 + 48)/8 = n^2/8 + 6$ | As for $a = 4$ above. |
| 9 | $(n^2 + 65)/8 = (n^2 + 1)/8 + 8$ | As for $a = 1$ above. |
| 10 | $(n^2 + 84)/8 = (n^2 - 4)/8 + 11$ | As for $a = 2$ above. |
| 11 | $(n^2 + 105)/8 = (n^2 + 1)/8 + 13$ | As for $a = 1$ above. |
| 12 | $(n^2 + 128)/8 = n^2/8 + 16$ | As for $a = 4$ above. |

etc. Thus, there exist identities for every even $a$. Hence, we have the following theorems.

**Theorem 5.1.** For $a = 2$, there hold the following identities for integral values of $n$, $b$.

| *n* | *b* | Identity | Ref. |
|---|---|---|---|
| 6 | 3 | $2^2 + 6^2 + 3^2 = 7^2$ | Th. 2.2 |
| 10 | 11 | $2^2 + 10^2 + 11^2 = 15^2$ | |
| 14 | 23 | $2^2 + 14^2 + 23^2 = 27^2$ | |
| 18 | 39 | $2^2 + 18^2 + 39^2 = 43^2$ | |
| 22 | 59 | $2^2 + 22^2 + 59^2 = 63^2$ | |
| 26 | 83 | $2^2 + 26^2 + 83^2 = 87^2$ | |
| 30 | 111 | $2^2 + 30^2 + 111^2 = 115^2$ | |

etc. //

**Theorem 5.2.** For $a = 4$, there exist identities for $n$ = 4, 8, 12, 16, 20, etc.

| *n* | *b* | Identity | Equivalently | Ref. |
|---|---|---|---|---|
| 4 | 2 | $4^2 + 4^2 + 2^2 = 6^2$ | $2^2 + 2^2 + 1^2 = 3^2$ | Th. 2.1 |
| 8 | 8 | $4^2 + 8^2 + 8^2 = 12^2$ | $1^2 + 2^2 + 2^2 = 3^2$ | ” |

| | | | | |
|---|---|---|---|---|
| 12 | 18 | $4^2 + 12^2 + 18^2 = 22^2$ | $2^2 + 6^2 + 9^2 = 11^2$ | Th. 3.1 |
| 16 | 32 | $4^2 + 16^2 + 32^2 = 36^2$ | $1^2 + 4^2 + 8^2 = 9^2$ | Th. 2.1 |
| 20 | 50 | $4^2 + 20^2 + 50^2 = 54^2$ | $2^2 + 10^2 + 25^2 = 27^2$ | Th. 3.1 |
| 24 | 72 | $4^2 + 24^2 + 72^2 = 76^2$ | $1^2 + 6^2 + 18^2 = 19^2$ | Th. 2.1 |
| 28 | 98 | $4^2 + 28^2 + 98^2 = 102^2$ | $2^2 + 14^2 + 49^2 = 51^2$ | Th. 3.1 |
| 32 | 128 | $4^2 + 32^2 + 128^2 = 132^2$ | $1^2 + 8^2 + 32^2 = 33^2$ | Th. 2.1 |

etc. //

**Theorem 5.3.** For $a = 6$, there exist identities for $n$ = 2, 6, 10, 14, 18, 22, 26, 30, etc.

| *n* | *b* | Identity | Ref. |
|---|---|---|---|
| 2 | 3 | $6^2 + 2^2 + 3^2 = 7^2$ | Th. 2.2 |
| 6 | 7 | $6^2 + 6^2 + 7^2 = 11^2$ | |
| 10 | 15 | $6^2 + 10^2 + 15^2 = 19^2$ | |
| 14 | 27 | $6^2 + 14^2 + 27^2 = 31^2$ | |
| 18 | 43 | $6^2 + 18^2 + 43^2 = 47^2$ | |
| 22 | 63 | $6^2 + 22^2 + 63^2 = 67^2$ | |
| 26 | 87 | $6^2 + 26^2 + 87^2 = 91^2$ | |
| 30 | 115 | $6^2 + 30^2 + 115^2 = 119^2$ | |

etc. //

**Theorem 5.4.** For $a = 8$, there exist identities for $n$ = 4, 8, 12, 16, 20, etc.

| *n* | *b* | Identity | Equivalently | Ref. |
|---|---|---|---|---|
| 4 | 8 | $8^2 + 4^2 + 8^2 = 12^2$ | $2^2 + 1^2 + 2^2 = 3^2$ | Th. 2.1 |
| 8 | 14 | $8^2 + 8^2 + 14^2 = 18^2$ | $4^2 + 4^2 + 7^2 = 9^2$ | Th. 3.2 |
| 12 | 24 | $8^2 + 12^2 + 24^2 = 28^2$ | $2^2 + 3^2 + 6^2 = 7^2$ | Th. 2.2 |
| 16 | 38 | $8^2 + 16^2 + 38^2 = 42^2$ | $4^2 + 8^2 + 19^2 = 21^2$ | Th. 3.2 |
| 20 | 56 | $8^2 + 20^2 + 56^2 = 60^2$ | $2^2 + 5^2 + 14^2 = 15^2$ | Th. 2.2 |
| 24 | 78 | $8^2 + 24^2 + 78^2 = 82^2$ | $4^2 + 12^2 + 39^2 = 41^2$ | Th. 3.2 |

etc. //

**Theorem 5.5.** For $a = 10$, there exist identities for $n$ = 2, 6, 10, 14, 18, 22, 26, 30, etc.

| *n* | *b* | Identity | Reference |
|---|---|---|---|
| 2 | 11 | $10^2 + 2^2 + 11^2 = 15^2$ | Theo. 5.1 |
| 6 | 15 | $10^2 + 6^2 + 15^2 = 19^2$ | Theo. 5.3 |
| 10 | 23 | $10^2 + 10^2 + 23^2 = 27^2$ | |
| 14 | 35 | $10^2 + 14^2 + 35^2 = 39^2$ | |
| 18 | 51 | $10^2 + 18^2 + 51^2 = 55^2$ | |
| 22 | 71 | $10^2 + 22^2 + 71^2 = 75^2$ | |
| 26 | 95 | $10^2 + 26^2 + 95^2 = 99^2$ | |
| 30 | 123 | $10^2 + 30^2 + 123^2 = 127^2$ | |

etc. //

**Theorem 5.6.** For $a = 12$, there exist identities for $n$ = 4, 8, 12, 16, 20, etc.

| *n* | *b* | Identity | Equivalently | Ref. |
|---|---|---|---|---|
| 4 | 18 | $12^2 + 4^2 + 18^2 = 22^2$ | $6^2 + 2^2 + 9^2 = 11^2$ | Th. 3.1 |
| 8 | 24 | $12^2 + 8^2 + 24^2 = 28^2$ | $3^2 + 2^2 + 6^2 = 7^2$ | Th. 2.2 |
| 12 | 34 | $12^2 + 12^2 + 34^2 = 38^2$ | $6^2 + 6^2 + 17^2 = 19^2$ | Th. 3.3 |
| 16 | 48 | $12^2 + 16^2 + 48^2 = 52^2$ | $3^2 + 4^2 + 12^2 = 13^2$ | Th. 2.3 |
| 20 | 66 | $12^2 + 20^2 + 66^2 = 70^2$ | $6^2 + 10^2 + 33^2 = 35^2$ | Th. 3.3 |
| 24 | 88 | $12^2 + 24^2 + 88^2 = 92^2$ | $3^2 + 6^2 + 22^2 = 23^2$ | Th. 2.3 |

etc. //

## 6. Identities of the type $a^2 + n^2 + b^2 = (b + 5)^2$

Above type of identities require:

$$b = (a^2 + n^2 - 25) / 10, \qquad (6.1)$$

where $a$ and $n$ assume some suitable integral values making $b$ integer. For different integral values $a$ = 1, 2, 3, 4, 5, etc. above relation yields the following values of $b$:

| *a* | *b* | Remark |
|---|---|---|
| 1 | $(n^2 - 24)/10 = (n^2 - 4)/10 - 2$ | $b$ is +ve integer for $n$ = 8, 12, 18, 22, 28, 32, etc. |
| 2 | $(n^2 - 21)/10 = (n^2 - 1)/10 - 2$ | $b$ is +ve integer for $n$ = 9, 11, 19, 21, 29, 31 etc. |
| 3 | $(n^2 - 16)/10 = (n^2 + 4)/10 - 2$ | $b$ is +ve integer for $n$ = 6, 14, 16, 24, 26, 34, etc. |
| 4 | $(n^2 - 9)/10 = (n^2 + 1)/10 - 1$ | $b$ is +ve integer for $n$ = 7, 13, 17, 23, 27, 33, etc. |

| | | |
|---|---|---|
| 5 | $n^2 / 10$ | $b$ is +ve integer for $n$ = 10, 20, 30, 40, 50, etc. |
| 6 | $(n^2 + 11)/10 = (n^2 + 1)/10 + 1$ | $b$ is +ve integer for $n$ = 3, 7, 13, 17, 23, 27, etc. |
| 7 | $(n^2 + 24)/10 = (n^2 + 4)/10 + 2$ | $b$ is +ve integer for $n$ = 4, 6, 14, 16, 24, 26, etc. |
| 8 | $(n^2 + 39)/10 = (n^2 - 1)/10 + 4$ | $b$ is +ve integer for $n$ = 1, 9, 11, 19, 21, 29, etc. |
| 9 | $(n^2 + 56)/10 = (n^2 - 4)/10 + 6$ | $b$ is +ve integer for $n$ = 2, 8, 12, 18, 22, 28, etc. |
| 10 | $(n^2 + 75)/10 = (n^2 + 5)/10 + 7$ | $b$ is +ve integer for $n$ = 5, 15, 25, 35, 45, etc. |
| 11 | $(n^2 + 96)/10 = (n^2 - 4)/10 + 10$ | As for $a$ = 9 above. |
| 12 | $(n^2 + 119)/10 = (n^2 - 1)/10 + 12$ | As for $a$ = 8 above. |

etc. Hence, we have the following theorems:

**Theorem 6.1.** For $a$ = 1, there hold the identities for integral values of $n$, $b$.

| *n* | *b* | Identity | Ref. |
|---|---|---|---|
| 8 | 4 | $1^2 + 8^2 + 4^2 = 9^2$ | Th. 2.1 |
| 12 | 12 | $1^2 + 12^2 + 12^2 = 17^2$ | |
| 18 | 30 | $1^2 + 18^2 + 30^2 = 35^2$ | |
| 22 | 46 | $1^2 + 22^2 + 46^2 = 51^2$ | |
| 28 | 76 | $1^2 + 28^2 + 76^2 = 81^2$ | |
| 32 | 100 | $1^2 + 32^2 + 100^2 = 105^2$ | |

etc. //

**Theorem 6.2.** For $a$ = 2, there hold the identities for integral values of $n$, $b$.

| *n* | *b* | Identity | Ref. |
|---|---|---|---|
| 9 | 6 | $2^2 + 9^2 + 6^2 = 11^2$ | Th. 3.1 |
| 11 | 10 | $2^2 + 11^2 + 10^2 = 15^2$ | Th. 5.1 |
| 19 | 34 | $2^2 + 19^2 + 34^2 = 39^2$ | |
| 21 | 42 | $2^2 + 21^2 + 42^2 = 47^2$ | |
| 29 | 82 | $2^2 + 29^2 + 82^2 = 87^2$ | |
| 31 | 94 | $2^2 + 31^2 + 94^2 = 99^2$ | |

etc. //

**Theorem 6.3.** For $a$ = 3, there hold the identities for integral values of $n$, $b$.

| *n* | *b* | Identity | Ref. |
|---|---|---|---|
| 6 | 2 | $3^2 + 6^2 + 2^2 = 7^2$ | Th. 2.2 |
| 14 | 18 | $3^2 + 14^2 + 18^2 = 23^2$ | |
| 16 | 24 | $3^2 + 16^2 + 24^2 = 29^2$ | |
| 24 | 56 | $3^2 + 24^2 + 56^2 = 61^2$ | |
| 26 | 66 | $3^2 + 26^2 + 66^2 = 71^2$ | |
| 34 | 114 | $3^2 + 34^2 + 114^2 = 119^2$ | |

etc. //

**Theorem 6.4.** For $a$ = 4, there hold the following identities:

| *n* | *b* | Identity | Ref. |
|---|---|---|---|
| 7 | 4 | $4^2 + 7^2 + 4^2 = 9^2$ | Th. 3.2 |
| 13 | 16 | $4^2 + 13^2 + 16^2 = 21^2$ | |
| 17 | 28 | $4^2 + 17^2 + 28^2 = 33^2$ | |
| 23 | 52 | $4^2 + 23^2 + 52^2 = 57^2$ | |
| 27 | 72 | $4^2 + 27^2 + 72^2 = 77^2$ | |
| 33 | 108 | $4^2 + 33^2 + 108^2 = 113^2$ | |

etc. //

**Theorem 6.5.** When $a$ = 5, there hold the identities for positive integral multiples of $n$ = 10.

| *n* | *b* | Identity | Equivalently | Ref. |
|---|---|---|---|---|
| 10 | 10 | $5^2 + 10^2 + 10^2 = 15^2$ | $1^2 + 2^2 + 2^2 = 3^2$ | Th. 2.1 |
| 20 | 40 | $5^2 + 20^2 + 40^2 = 45^2$ | $1^2 + 4^2 + 8^2 = 9^2$ | ” |
| 30 | 90 | $5^2 + 30^2 + 90^2 = 95^2$ | $1^2 + 6^2 + 18^2 = 19^2$ | ” |
| 40 | 160 | $5^2 + 40^2 + 160^2 = 165^2$ | $1^2 + 8^2 + 32^2 = 33^2$ | ” |
| 50 | 250 | $5^2 + 50^2 + 250^2 = 255^2$ | $1^2 + 10^2 + 50^2 = 51^2$ | ” |
| 60 | 360 | $5^2 + 60^2 + 360^2 = 365^2$ | $1^2 + 12^2 + 72^2 = 73^2$ | ” |

etc. //

**Theorem 6.6.** For $a$ = 6, there hold the following identities:

| *n* | *b* | Identity | Ref. |
|---|---|---|---|
| 3 | 2 | $6^2 + 3^2 + 2^2 = 7^2$ | Th. 2.2 |

| | | | |
|---|---|---|---|
| 7 | 6 | $6^2 + 7^2 + 6^2 = 11^2$ | Th. 5.3 |
| 13 | 18 | $6^2 + 13^2 + 18^2 = 23^2$ | |
| 17 | 30 | $6^2 + 17^2 + 30^2 = 35^2$ | |
| 23 | 54 | $6^2 + 23^2 + 54^2 = 59^2$ | |
| 27 | 74 | $6^2 + 27^2 + 74^2 = 79^2$ | |

etc. //

**Theorem 6.7.** For $a = 7$, there hold the identities:

| *n* | *b* | Identity | Ref. |
|---|---|---|---|
| 4 | 4 | $7^2 + 4^2 + 4^2 = 9^2$ | Th. 3.2 |
| 6 | 6 | $7^2 + 6^2 + 6^2 = 11^2$ | Th. 5.3 |
| 14 | 22 | $7^2 + 14^2 + 22^2 = 27^2$ | |
| 16 | 28 | $7^2 + 16^2 + 28^2 = 33^2$ | |
| 24 | 60 | $7^2 + 24^2 + 60^2 = 65^2$ | |
| 26 | 70 | $7^2 + 26^2 + 70^2 = 75^2$ | |

etc. //

**Theorem 6.8.** For $a = 8$, there hold the identities:

| *n* | *b* | Identity | Ref. |
|---|---|---|---|
| 1 | 4 | $8^2 + 1^2 + 4^2 = 9^2$ | Th. 2.1 |
| 9 | 12 | $8^2 + 9^2 + 12^2 = 17^2$ | |
| 11 | 16 | $8^2 + 11^2 + 16^2 = 21^2$ | |
| 19 | 40 | $8^2 + 19^2 + 40^2 = 45^2$ | |
| 21 | 48 | $8^2 + 21^2 + 48^2 = 53^2$ | |
| 29 | 88 | $8^2 + 29^2 + 88^2 = 93^2$ | |

etc. //

**Theorem 6.9.** For $a = 9$, there hold the identities for integral values of *n*, *b*.

| *n* | *b* | Identity | Ref. |
|---|---|---|---|
| 2 | 6 | $9^2 + 2^2 + 6^2 = 11^2$ | Th. 3.1 |
| 8 | 12 | $9^2 + 8^2 + 12^2 = 17^2$ | Th. 6.8 |
| 12 | 20 | $9^2 + 12^2 + 20^2 = 25^2$ | |
| 18 | 38 | $9^2 + 18^2 + 38^2 = 43^2$ | |
| 22 | 54 | $9^2 + 22^2 + 54^2 = 59^2$ | |
| 28 | 84 | $9^2 + 28^2 + 84^2 = 89^2$ | |

etc. //

**Theorem 6.10.** For $a = 10$, there hold the identities for integral values of *n*, *b*.

| *n* | *b* | Identity | Equivalently | Ref. |
|---|---|---|---|---|
| 5 | 10 | $10^2 + 5^2 + 10^2 = 15^2$ | $2^2 + 1^2 + 2^2 = 3^2$ | Th. 2.1 |
| 15 | 30 | $10^2 + 15^2 + 30^2 = 35^2$ | $2^2 + 3^2 + 6^2 = 7^2$ | Th. 2.2 |
| 25 | 70 | $10^2 + 25^2 + 70^2 = 75^2$ | $2^2 + 5^2 + 14^2 = 15^2$ | ,, |
| 35 | 130 | $10^2 + 35^2 + 130^2 = 135^2$ | $2^2 + 7^2 + 26^2 = 27^2$ | ,, |
| 45 | 210 | $10^2 + 45^2 + 210^2 = 215^2$ | $2^2 + 9^2 + 42^2 = 43^2$ | ,, |

etc. //

## 7. Identities of the type $a^2 + n^2 + b^2 = (b + 6)^2$

Above type of identities require:

$$b = (a^2 + n^2 - 36) / 12, \qquad (7.1)$$

where *a* and *n* assume some suitable integral values making *b* integer. For $a = 1, 2, 3, 4, 5$, etc. above relation yields values of *b*:

| *a* | *b* | Remark |
|---|---|---|
| 1 | $(n^2 - 35)/12 = (n^2 + 1)/12 - 3$ | *b* is not an integer for any integer *n*. |
| 2 | $(n^2 - 32) / 12 = (n^2 + 4)/12 - 3$ | ,, |
| 3 | $(n^2 - 27)/12 = (n^2 - 3)/12 - 2$ | ,, |
| 4 | $(n^2 - 20)/12 = (n^2 + 4)/12 - 2$ | ,, |
| 5 | $(n^2 - 11)/12 = (n^2 + 1)/12 - 1$ | ,, |
| 6 | $n^2/ 12$ | *b* is integer when *n* is an integral multiple of 6. |
| 7 | $(n^2 + 13)/12 = (n^2 + 1)/12 + 1$ | *b* is not an integer for any integer *n*. |
| 8 | $(n^2 + 28)/12 = (n^2 + 4)/12 + 2$ | ,, |
| 9 | $(n^2 + 45)/12 = (n^2 - 3)/12 + 4$ | ,, |
| 10 | $(n^2 + 64)/12 = (n^2 + 4)/12 + 5$ | ,, |
| 11 | $(n^2 + 85)/12 = (n^2 + 1)/12 + 7$ | ,, |
| 12 | $(n^2 + 108)/12 = n^2 /12 + 9$ | As for $a = 6$ above. |

etc. Similarly, $a$ = 13 – 17 also do not yield any integral values of $b$ for any integer $n$. But, $a = 18 \Rightarrow b = (n^2 + 288)/12 = n^2/12 + 24$, which in analogy with $a$ = 6, takes integral values for any integral multiple of $n$ = 6. Thus, there hold the following theorems:

**Theorem 7.1.** For $a$ = 6, there hold the identities for integral values of $n$, $b$.

| $n$ | $b$ | Identity | Equivalently | Ref. |
|---|---|---|---|---|
| 6 | 3 | $6^2 + 6^2 + 3^2 = 9^2$ | $2^2 + 2^2 + 1^2 = 3^2$ | Th. 2.1 |
| 12 | 12 | $6^2 + 12^2 + 12^2 = 18^2$ | $1^2 + 2^2 + 2^2 = 3^2$ | ,, |
| 18 | 27 | $6^2 + 18^2 + 27^2 = 33^2$ | $2^2 + 6^2 + 9^2 = 11^2$ | Th. 3.1 |
| 24 | 48 | $6^2 + 24^2 + 48^2 = 54^2$ | $1^2 + 4^2 + 8^2 = 9^2$ | Th. 2.1 |
| 30 | 75 | $6^2 + 30^2 + 75^2 = 81^2$ | $2^2 + 10^2 + 25^2 = 27^2$ | Th. 3.1 |
| 36 | 108 | $6^2 + 36^2 + 108^2 = 114^2$ | $1^2 + 6^2 + 18^2 = 19^2$ | Th. 2.1 |

etc. //

**Theorem 7.2.** For $a$ = 12, there hold the identities for integral values of $n$, $b$.

| $n$ | $b$ | Identity | Equivalently | Ref. |
|---|---|---|---|---|
| 6 | 12 | $12^2 + 6^2 + 12^2 = 18^2$ | $2^2 + 1^2 + 2^2 = 3^2$ | Th. 2.1 |
| 12 | 21 | $12^2 + 12^2 + 21^2 = 27^2$ | $4^2 + 4^2 + 7^2 = 9^2$ | Th. 3.2 |
| 18 | 36 | $12^2 + 18^2 + 36^2 = 42^2$ | $2^2 + 3^2 + 6^2 = 7^2$ | Th. 2.2 |
| 24 | 57 | $12^2 + 24^2 + 57^2 = 63^2$ | $4^2 + 8^2 + 19^2 = 21^2$ | Th. 3.2 |
| 30 | 84 | $12^2 + 30^2 + 84^2 = 90^2$ | $2^2 + 5^2 + 14^2 = 15^2$ | Th. 2.2 |
| 36 | 117 | $12^2 + 36^2 + 117^2 = 123^2$ | $4^2 + 12^2 + 39^2 = 41^2$ | Th. 3.2 |

etc. //

**Theorem 7.3.** For $a$ = 18, there hold the identities for integral values of $n$, $b$.

| $n$ | $b$ | Identity | Equivalently | Ref. |
|---|---|---|---|---|
| 6 | 27 | $18^2 + 6^2 + 27^2 = 33^2$ | $6^2 + 2^2 + 9^2 = 11^2$ | Th. 3.1 |
| 12 | 36 | $18^2 + 12^2 + 36^2 = 42^2$ | $3^2 + 2^2 + 6^2 = 7^2$ | Th. 2.2 |
| 18 | 51 | $18^2 + 18^2 + 51^2 = 57^2$ | $6^2 + 6^2 + 17^2 = 19^2$ | Th. 3.3 |
| 24 | 72 | $18^2 + 24^2 + 72^2 = 78^2$ | $3^2 + 4^2 + 12^2 = 13^2$ | Th. 2.3 |
| 30 | 99 | $18^2 + 30^2 + 99^2 = 105^2$ | $6^2 + 10^2 + 33^2 = 35^2$ | Th. 3.3 |
| 36 | 132 | $18^2 + 36^2 + 132^2 = 138^2$ | $3^2 + 6^2 + 22^2 = 23^2$ | Th. 2.3 |

etc. //

**Note 7.1.** Conclusively, there exist such identities for every integral multiple of $a$ = 6.

## 8. Identities of the type $a^2 + n^2 + b^2 = (b + 7)^2$

Above type of identities require:

$$b = (a^2 + n^2 - 49) / 14, \qquad (8.1)$$

where $a$ and $n$ assume some suitable integral values making $b$ integer. For $a$ = 1, 2, 3, 4, 5, etc. above relation yields values of $b$:

| $a$ | $b$ | Remark |
|---|---|---|
| 1 | $(n^2 - 48)/14 = (n^2 - 6)/14 - 3$ | $b$ is not an integer for any integer $n$. |
| 2 | $(n^2 - 45)/14 = (n^2 - 3)/14 - 3$ | ,, |
| 3 | $(n^2 - 40)/14 = (n^2 + 2)/14 - 3$ | ,, |
| 4 | $(n^2 - 33)/14 = (n^2 - 5)/14 - 2$ | ,, |
| 5 | $(n^2 - 24)/14 = (n^2 + 4)/14 - 2$ | ,, |
| 6 | $(n^2 - 13)/14 = (n^2 + 1)/14 - 1$ | ,, |
| 7 | $n^2/14$ | $b$ is integer for $n$ = an integral multiple of 14. |
| 8 | $(n^2 + 15)/14 = (n^2 + 1)/14 + 1$ | As for $a$ = 6 above. |
| 9 | $(n^2 + 32)/14 = (n^2 + 4)/14 + 2$ | As for $a$ = 5 above. |
| 10 | $(n^2 + 51)/14 = (n^2 - 5)/14 + 4$ | As for $a$ = 4 above. |
| 11 | $(n^2 + 72)/14 = (n^2 + 2)/14 + 5$ | As for $a$ = 3 above. |
| 12 | $(n^2 + 95)/14 = (n^2 - 3)/14 + 7$ | As for $a$ = 2 above. |
| 13 | $(n^2 + 15)/14 = (n^2 - 6)/14 + 9$ | As for $a$ = 1 above. |
| 14 | $(n^2 + 147)/14 = (n^2 + 7)/14 + 10$ | $b$ is integer for $n$ = 7, 21, 35, 49, 63, 77, etc. |

Therefore, there hold such identities only when $a$ is an integral multiple of 7 giving the following theorems:

**Theorem 8.1.** For $a$ = 7, there hold the following identities for integral values of $n$, $b$.

| $n$ | $b$ | Identity | Equivalently | Ref. |
|---|---|---|---|---|
| 14 | 14 | $7^2 + 14^2 + 14^2 = 21^2$ | $1^2 + 2^2 + 2^2 = 3^2$ | Th. 2.1 |
| 28 | 56 | $7^2 + 28^2 + 56^2 = 63^2$ | $1^2 + 4^2 + 8^2 = 9^2$ | ,, |
| 42 | 126 | $7^2 + 42^2 + 126^2 = 133^2$ | $1^2 + 6^2 + 18^2 = 19^2$ | ,, |

| | | | | |
|---|---|---|---|---|
| 56 | 224 | $7^2 + 56^2 + 224^2 = 231^2$ | $1^2 + 8^2 + 32^2 = 33^2$ | ,, |
| 70 | 350 | $7^2 + 70^2 + 350^2 = 357^2$ | $1^2 + 10^2 + 50^2 = 51^2$ | ,, |
| 84 | 504 | $7^2 + 84^2 + 504^2 = 511^2$ | $1^2 + 12^2 + 72^2 = 73^2$ | ,, |

etc. //

**Theorem 8.2.** For $a = 14$, there hold the identities for integral values of $n$, $b$.

| *n* | *b* | Identity | Equivalently | Ref. |
|---|---|---|---|---|
| 7 | 14 | $14^2 + 7^2 + 14^2 = 21^2$ | $2^2 + 1^2 + 2^2 = 3^2$ | Th. 2.1 |
| 21 | 42 | $14^2 + 21^2 + 42^2 = 49^2$ | $2^2 + 3^2 + 6^2 = 7^2$ | Th. 2.2 |
| 35 | 98 | $14^2 + 35^2 + 98^2 = 105^2$ | $2^2 + 5^2 + 14^2 = 15^2$ | ,, |
| 49 | 182 | $14^2 + 49^2 + 182^2 = 189^2$ | $2^2 + 7^2 + 26^2 = 27^2$ | ,, |
| 63 | 294 | $14^2 + 63^2 + 294^2 = 301^2$ | $2^2 + 9^2 + 42^2 = 43^2$ | ,, |
| 77 | 434 | $14^2 + 77^2 + 434^2 = 441^2$ | $2^2 + 11^2 + 62^2 = 63^2$ | ,, |

etc. //

Similarly, there do not exist any such identities for $a = 15 – 20$. However, for $a = 21$, Eq. (8.1) yields

$$b = (n^2 + 392) / 14 \;=\; n^2 / 14 + 28,$$

which, in analogy with Theo. 8.1, take integral values when $n$ is any integral multiple of 14. Thus, there holds the theorem:

**Theorem 8.3.** For $a = 21$, we have the identities for integral values of $n$, $b$.

| *n* | *b* | Identity | Equivalently | Ref. |
|---|---|---|---|---|
| 14 | 42 | $21^2 + 14^2 + 42^2 = 49^2$ | $3^2 + 2^2 + 6^2 = 7^2$ | Th. 2.2 |
| 28 | 84 | $21^2 + 28^2 + 84^2 = 91^2$ | $3^2 + 4^2 + 12^2 = 13^2$ | Th. 2.3 |
| 42 | 154 | $21^2 + 42^2 + 154^2 = 161^2$ | $3^2 + 6^2 + 22^2 = 23^2$ | ,, |
| 56 | 252 | $21^2 + 56^2 + 252^2 = 259^2$ | $3^2 + 8^2 + 36^2 = 37^2$ | ,, |
| 70 | 378 | $21^2 + 70^2 + 378^2 = 385^2$ | $3^2 + 10^2 + 54^2 = 55^2$ | ,, |

etc. //

For $a = 22 – 27$ also there exist no identities but $a = 28$ yields

$$b = (n^2 + 735) / 14 \;=\; (n^2 + 7) / 14 + 52,$$

which, in analogy with Theo. 8.2, take integral values for $n = 7, 21, 35, 49$, etc. Thus, there holds the theorem:

**Theorem 8.4.** For $a = 28$, there hold the following identities for integral values of $n$, $b$.

| *n* | *b* | Identity | Equivalently | Ref. |
|---|---|---|---|---|
| 7 | 56 | $28^2 + 7^2 + 56^2 = 63^2$ | $4^2 + 1^2 + 8^2 = 9^2$ | Th. 2.1 |
| 21 | 84 | $28^2 + 21^2 + 84^2 = 91^2$ | $4^2 + 3^2 + 12^2 = 13^2$ | Th. 2.3 |
| 35 | 140 | $28^2 + 35^2 + 140^2 = 147^2$ | $4^2 + 5^2 + 20^2 = 21^2$ | Th. 2.4 |
| 49 | 224 | $28^2 + 49^2 + 224^2 = 231^2$ | $4^2 + 7^2 + 32^2 = 33^2$ | ,, |
| 63 | 336 | $28^2 + 63^2 + 336^2 = 343^2$ | $4^2 + 9^2 + 48^2 = 49^2$ | ,, |

etc. //

## 9. Identities of the type $a^2 + n^2 + b^2 = (b + 8)^2$

Above type of identities require:

$$b = (a^2 + n^2 – 64) / 16, \qquad (9.1)$$

where $a$ and $n$ assume some suitable integral values making $b$ integer. For $a = 1, 2, 3$ above relation yields values of $b$:

| *a* | *b* | Remark |
|---|---|---|
| 1 | $(n^2 – 63)/16 = (n^2 + 1)/16 – 4$ | $b$ is not an integer for any integer $n$. |
| 2 | $(n^2 – 60)/16 = (n^2 + 4)/16 – 4$ | ,, |
| 3 | $(n^2 – 55)/16 = (n^2 – 7)/16 – 3$ | ,, |
| 4 | $(n^2 – 48)/16 = n^2/16 – 3$ | $b$ is integer when $n$ is an integral multiple of 4 |
| 5 | $(n^2 – 39)/16 = (n^2 – 7)/16 – 2$ | As for $a = 3$ above. |
| 6 | $(n^2 – 28)/16 = (n^2 + 4)/16 – 2$ | As for $a = 2$ above. |
| 7 | $(n^2 – 15)/16 = (n^2 + 1)/16 – 1$ | As for $a = 1$ above. |
| 8 | $n^2/16$ | As for $a = 4$ above. |
| 9 | $(n^2 + 17)/16 = (n^2 + 1)/16 + 1$ | As for $a = 1$ above. |
| 10 | $(n^2 + 36)/16 = (n^2 + 4)/16 + 2$ | As for $a = 2$ above. |
| 11 | $(n^2 + 57)/16 = (n^2 – 7)/16 + 4$ | As for $a = 3$ above. |
| 12 | $(n^2 + 80)/16 = n^2/16 + 5$ | As for $a = 4$ above. |

Thus, there exist such identities only when $a$ is an integral multiple of 4 giving rise to the following theorems:

**Theorem 9.1.** For $a = 4$, there hold the identities for integral values of $n$, $b$.

| *n* | *b* | Identity | Equivalently | Ref. |
|---|---|---|---|---|

| | | | | |
|---|---|---|---|---|
| 8 | 1 | $4^2 + 8^2 + 1^2 = 9^2$ | | Th. 2.1 |
| 12 | 6 | $4^2 + 12^2 + 6^2 = 14^2$ | $2^2 + 6^2 + 3^2 = 7^2$ | Th. 2.2 |
| 16 | 13 | $4^2 + 16^2 + 13^2 = 21^2$ | | Th. 6.4 |
| 20 | 22 | $4^2 + 20^2 + 22^2 = 30^2$ | $2^2 + 10^2 + 11^2 = 15^2$ | Th. 5.1 |
| 24 | 33 | $4^2 + 24^2 + 33^2 = 41^2$ | | |
| 28 | 46 | $4^2 + 28^2 + 46^2 = 54^2$ | $2^2 + 14^2 + 23^2 = 27^2$ | Th. 5.1 |
| 32 | 61 | $4^2 + 32^2 + 61^2 = 69^2$ | | |

etc. //

**Theorem 9.2.** For $a$ = 8, there hold the identities for integral values of $n$, $b$.

| *n* | *b* | Identity | Equivalently | Ref. |
|---|---|---|---|---|
| 4 | 1 | $8^2 + 4^2 + 1^2 = 9^2$ | | Th. 2.1 |
| 8 | 4 | $8^2 + 8^2 + 4^2 = 12^2$ | $2^2 + 2^2 + 1^2 = 3^2$ | ” |
| 12 | 9 | $8^2 + 12^2 + 9^2 = 17^2$ | | Th. 6.8 |
| 16 | 16 | $8^2 + 16^2 + 16^2 = 24^2$ | $1^2 + 2^2 + 2^2 = 3^2$ | Th. 2.1 |
| 20 | 25 | $8^2 + 20^2 + 25^2 = 33^2$ | | |
| 24 | 36 | $8^2 + 24^2 + 36^2 = 44^2$ | $2^2 + 6^2 + 9^2 = 11^2$ | Th. 3.1 |
| 28 | 49 | $8^2 + 28^2 + 49^2 = 57^2$ | | |
| 32 | 64 | $8^2 + 32^2 + 64^2 = 72^2$ | $1^2 + 4^2 + 8^2 = 9^2$ | Th. 2.1 |

etc. //

**Theorem 9.3.** For $a$ = 12, there hold the identities for integral values of $n$, $b$.

| *n* | *b* | Identity | Equivalently | Ref. |
|---|---|---|---|---|
| 4 | 6 | $12^2 + 4^2 + 6^2 = 14^2$ | $6^2 + 2^2 + 3^2 = 7^2$ | Th. 2.2 |
| 8 | 9 | $12^2 + 8^2 + 9^2 = 17^2$ | | Th. 6.8 |
| 12 | 14 | $12^2 + 12^2 + 14^2 = 22^2$ | $6^2 + 6^2 + 7^2 = 11^2$ | Th. 5.3 |
| 16 | 21 | $12^2 + 16^2 + 21^2 = 29^2$ | | |
| 20 | 30 | $12^2 + 20^2 + 30^2 = 38^2$ | $6^2 + 10^2 + 15^2 = 19^2$ | Th. 5.3 |
| 24 | 41 | $12^2 + 24^2 + 41^2 = 49^2$ | | |
| 28 | 54 | $12^2 + 28^2 + 54^2 = 62^2$ | $6^2 + 14^2 + 27^2 = 31^2$ | Th. 5.3 |
| 32 | 69 | $12^2 + 32^2 + 69^2 = 77^2$ | | |

etc. //

**10. Identities of the type $a^2 + n^2 + b^2 = (b + 9)^2$**

Above type of identities require:

$$b = (a^2 + n^2 - 81) / 18, \qquad (10.1)$$

where $a$ and $n$ assume some suitable integral values making $b$ integer. For $a$ = 1, 2, 3 above relation yields values of $b$:

| *a* | *b* | Remark |
|---|---|---|
| 1 | $(n^2 - 80)/18 = (n^2 - 8)/18 - 4$ | $b$ is not an integer for any integer $n$. |
| 2 | $(n^2 - 77)/18 = (n^2 - 5)/18 - 4$ | ,, |
| 3 | $(n^2 - 72)/18 = n^2/18 - 4$ | $b$ is integer when $n$ is an integral multiple of 6. |
| 4 | $(n^2 - 65)/18 = (n^2 + 7)/18 - 4$ | $b$ is not an integer for any integer $n$. |
| 5 | $(n^2 - 56)/18 = (n^2 - 2)/18 - 3$ | ,, |
| 6 | $(n^2 - 45)/18 = (n^2 - 9)/18 - 2$ | $b$ is integer for $n$ = 3, 9, 15, 21, 27, 33, 39, etc. |
| 7 | $(n^2 - 32)/18 = (n^2 + 4)/18 - 2$ | $b$ is not an integer for any integer $n$. |
| 8 | $(n^2 - 17)/18 = (n^2 + 1)/18 - 1$ | ,, |
| 9 | $n^2/18$ | As for $a$ = 3 above. |
| 10 | $(n^2 + 19)/18 = (n^2 + 1)/18 + 1$ | As for $a$ = 8 above. |
| 11 | $(n^2 + 40)/18 = (n^2 + 4)/18 + 2$ | As for $a$ = 7 above. |
| 12 | $(n^2 + 63)/18 = (n^2 - 9)/18 + 4$ | As for $a$ = 6 above. |

etc. Hence, there exist such identities only when $n$ is an integral multiple of 3 giving rise to the following theorems:

**Theorem 10.1.** For $a$ = 3, there hold the identities for integral values of $n$, $b$.

| *n* | *b* | Identity | Ref. |
|---|---|---|---|
| 12 | 4 | $3^2 + 12^2 + 4^2 = 13^2$ | Th. 2.3 |
| 18 | 14 | $3^2 + 18^2 + 14^2 = 23^2$ | Th. 6.3 |
| 24 | 28 | $3^2 + 24^2 + 28^2 = 37^2$ | |
| 30 | 46 | $3^2 + 30^2 + 46^2 = 55^2$ | |
| 36 | 68 | $3^2 + 36^2 + 68^2 = 77^2$ | |
| 42 | 94 | $3^2 + 42^2 + 94^2 = 103^2$ | |

| | | | |
|---|---|---|---|
| 48 | 124 | $3^2 + 48^2 + 124^2 = 133^2$ | |

etc. //

**Theorem 10.2.** For $a$ = 6, there hold the following identities for integral values of $n$, $b$.

| *n* | *b* | Identity | Ref. |
|---|---|---|---|
| 9 | 2 | $6^2 + 9^2 + 2^2 = 11^2$ | Th. 3.1 |
| 15 | 10 | $6^2 + 15^2 + 10^2 = 19^2$ | Th. 5.3 |
| 21 | 22 | $6^2 + 21^2 + 22^2 = 31^2$ | |
| 27 | 38 | $6^2 + 27^2 + 38^2 = 47^2$ | |
| 33 | 58 | $6^2 + 33^2 + 58^2 = 67^2$ | |
| 39 | 82 | $6^2 + 39^2 + 82^2 = 91^2$ | |
| 45 | 110 | $6^2 + 45^2 + 110^2 = 119^2$ | |

etc. //

**Theorem 10.3.** For $a$ = 9, there hold the identities for integral values of $n$, $b$.

| *n* | *b* | Identity | Equivalently | Ref. |
|---|---|---|---|---|
| 6 | 2 | $9^2 + 6^2 + 2^2 = 11^2$ | | Th. 3.1 |
| 12 | 8 | $9^2 + 12^2 + 8^2 = 17^2$ | | Th. 6.8 |
| 18 | 18 | $9^2 + 18^2 + 18^2 = 27^2$ | $1^2 + 2^2 + 2^2 = 3^2$ | Th. 2.1 |
| 24 | 32 | $9^2 + 24^2 + 32^2 = 41^2$ | | |
| 30 | 50 | $9^2 + 30^2 + 50^2 = 59^2$ | | |
| 36 | 72 | $9^2 + 36^2 + 72^2 = 81^2$ | $1^2 + 4^2 + 8^2 = 9^2$ | Th. 2.1 |
| 42 | 98 | $9^2 + 42^2 + 98^2 = 107^2$ | | |
| 48 | 128 | $9^2 + 48^2 + 128^2 = 137^2$ | | |

etc. //

**Theorem 10.4.** For $a$ = 12, there hold the identities for integral values of $n$, $b$.

| *n* | *b* | Identity | Ref. |
|---|---|---|---|
| 3 | 4 | $12^2 + 3^2 + 4^2 = 13^2$ | Th. 2.3 |
| 9 | 8 | $12^2 + 9^2 + 8^2 = 17^2$ | Th. 6.8 |
| 15 | 16 | $12^2 + 15^2 + 16^2 = 25^2$ | |
| 21 | 28 | $12^2 + 21^2 + 28^2 = 37^2$ | |
| 27 | 44 | $12^2 + 27^2 + 44^2 = 53^2$ | |
| 33 | 64 | $12^2 + 33^2 + 64^2 = 73^2$ | |
| 39 | 88 | $12^2 + 39^2 + 88^2 = 97^2$ | |
| 45 | 116 | $12^2 + 45^2 + 116^2 = 125^2$ | |

etc. //

## 11. Identities of the type $a^2 + n^2 + b^2 = (b + 10)^2$

Above type of identities require:

$$b = (a^2 + n^2 - 100)/20 = (a^2 + n^2)/20 - 5, \qquad (11.1)$$

where $a$ and $n$ assume some suitable integral values making $b$ integer. For positive integral values of $a$ above relation yields the following values of $b$:

| *a* | *b* | Remarks |
|---|---|---|
| 1 | $(n^2 + 1) / 20 - 5$ | $b$ is not an integer for any integer $n$. |
| 2 | $(n^2 + 4) / 20 - 5$ | $b$ is an integer for $n$ = 4, 6, 14, 16, 24, 26, etc. ⇒ identities. |
| 3 | $(n^2 + 9) / 20 - 5$ | $b$ is not an integer for any integer $n$. |
| 4 | $(n^2 - 4) / 20 - 4$ | $b$ is an integer for $n$ = 2, 8, 12, 18, 22, 28, etc. ⇒ identities. |
| 5 | $(n^2 + 5) / 20 - 4$ | $b$ is not an integer for any integer $n$. |
| 6 | $(n^2 - 4) / 20 - 3$ | As for $a$ = 4 ⇒ identities. |
| 7 | $(n^2 + 9)/20 - 3$ | As for $a$ = 3 ⇒ no identities. |
| 8 | $(n^2 + 4)/20 - 2$ | As for $a$ = 2 ⇒ identities. |
| 9 | $(n^2 + 1)/20 - 1$ | As for $a$ = 1 ⇒ identities. |
| 10 | $n^2/20$ | $b$ is integer when $n$ is integral multiple of 10 ⇒ identities. |

etc. Conclusively, there exist identities for even values of $a$ giving rise to the following theorems:

**Theorem 11.1.** For $a$ = 2, there hold the identities for positive integral values of $n$, $b$.

| *n* | *b* | Identity | Equivalently | Ref. |
|---|---|---|---|---|
| 14 | 5 | $2^2 + 14^2 + 5^2 = 15^2$ | | Th. 2.2 |
| 16 | 8 | $2^2 + 16^2 + 8^2 = 18^2$ | $1^2 + 8^2 + 4^2 = 9^2$ | Th. 2.1 |
| 24 | 24 | $2^2 + 24^2 + 24^2 = 34^2$ | $1^2 + 12^2 + 12^2 = 17^2$ | Th. 6.1 |
| 26 | 29 | $2^2 + 26^2 + 29^2 = 39^2$ | | |

| | | | | |
|---|---|---|---|---|
| 34 | 53 | $2^2 + 34^2 + 53^2 = 63^2$ | | |
| 36 | 60 | $2^2 + 36^2 + 60^2 = 70^2$ | $1^2 + 18^2 + 30^2 = 35^2$ | Th. 6.1 |

etc. //

**Theorem 11.2.** For $a = 4$, there hold the following identities for positive integral values of $n$, $b$.

| ***n*** | ***b*** | **Identity** | **Equivalently** | **Ref.** |
|---|---|---|---|---|
| 12 | 3 | $4^2 + 12^2 + 3^2 = 13^2$ | | Th. 2.3 |
| 18 | 12 | $4^2 + 18^2 + 12^2 = 22^2$ | $2^2 + 9^2 + 6^2 = 11^2$ | Th. 3.1 |
| 22 | 20 | $4^2 + 22^2 + 20^2 = 30^2$ | $2^2 + 11^2 + 10^2 = 15^2$ | Th. 5.1 |
| 28 | 35 | $4^2 + 28^2 + 35^2 = 45^2$ | | |
| 32 | 47 | $4^2 + 32^2 + 47^2 = 57^2$ | | |
| 38 | 68 | $4^2 + 38^2 + 68^2 = 78^2$ | $2^2 + 19^2 + 34^2 = 39^2$ | Th. 6.2 |

etc. //

**Theorem 11.3.** For $a = 6$, there hold the following identities for positive integral values of $n$, $b$.

| ***n*** | ***b*** | **Identity** | **Equivalently** | **Ref.** |
|---|---|---|---|---|
| 12 | 4 | $6^2 + 12^2 + 4^2 = 14^2$ | $3^2 + 6^2 + 2^2 = 7^2$ | Th. 2.2 |
| 18 | 13 | $6^2 + 18^2 + 13^2 = 23^2$ | | |
| 22 | 21 | $6^2 + 22^2 + 21^2 = 31^2$ | | |
| 28 | 36 | $6^2 + 28^2 + 36^2 = 46^2$ | $3^2 + 14^2 + 18^2 = 23^2$ | Th. 6.3 |
| 32 | 48 | $6^2 + 32^2 + 48^2 = 58^2$ | $3^2 + 16^2 + 24^2 = 29^2$ | ,, |
| 38 | 69 | $6^2 + 38^2 + 69^2 = 79^2$ | | |

etc. //

**Theorem 11.4.** For $a = 8$, there hold the identities for positive integral values of $n$, $b$.

| ***n*** | ***b*** | **Identity** | **Equivalently** | **Ref.** |
|---|---|---|---|---|
| 14 | 8 | $8^2 + 14^2 + 8^2 = 18^2$ | $4^2 + 7^2 + 4^2 = 9^2$ | Th. 3.2 |
| 16 | 11 | $8^2 + 16^2 + 11^2 = 21^2$ | | |
| 24 | 27 | $8^2 + 24^2 + 27^2 = 37^2$ | | |
| 26 | 32 | $8^2 + 26^2 + 32^2 = 42^2$ | $4^2 + 13^2 + 16^2 = 21^2$ | Th. 6.4 |
| 34 | 56 | $8^2 + 34^2 + 56^2 = 66^2$ | $4^2 + 17^2 + 28^2 = 33^2$ | ,, |
| 36 | 63 | $8^2 + 36^2 + 63^2 = 73^2$ | | |

etc. //

**Theorem 11.5.** For $a = 10$, there hold the identities for integral values of $n$, $b$.

| ***n*** | ***b*** | **Identity** | **Equivalently** | **Ref.** |
|---|---|---|---|---|
| 10 | 5 | $10^2 + 10^2 + 5^2 = 15^2$ | $2^2 + 2^2 + 1^2 = 3^2$ | Th. 2.1 |
| 20 | 20 | $10^2 + 20^2 + 20^2 = 30^2$ | $1^2 + 2^2 + 2^2 = 3^2$ | ,, |
| 30 | 45 | $10^2 + 30^2 + 45^2 = 55^2$ | $2^2 + 6^2 + 9^2 = 11^2$ | Th. 3.1 |
| 40 | 80 | $10^2 + 40^2 + 80^2 = 90^2$ | $1^2 + 4^2 + 8^2 = 9^2$ | Th. 2.1 |
| 50 | 125 | $10^2 + 50^2 + 125^2 = 135^2$ | $2^2 + 10^2 + 25^2 = 27^2$ | Th. 3.1 |
| 60 | 180 | $10^2 + 60^2 + 180^2 = 190^2$ | $1^2 + 6^2 + 18^2 = 19^2$ | Th. 2.1 |

etc. //

## 12. Identities of the type $a^2 + n^2 + b^2 = (b + 11)^2$

Above type of identities require:

$$b = (a^2 + n^2 - 121) / 22, \qquad (12.1)$$

where $a$ and $n$ assume some suitable integral values making $b$ integer. For positive integral values of $a$ above relation yields the following values of $b$:

| ***a*** | ***b*** |
|---|---|
| 1 | $(n^2 - 120) / 22 = (n^2 - 10) / 22 - 5$ |
| 2 | $(n^2 - 117) / 22 = (n^2 - 7) / 22 - 5$ |
| 3 | $(n^2 - 112) / 22 = (n^2 - 2) / 22 - 5$ |
| 4 | $(n^2 - 105) / 22 = (n^2 + 5) / 22 - 5$ |
| 5 | $(n^2 - 96) / 22 = (n^2 - 8) / 22 - 4$ |
| 6 | $(n^2 - 85) / 22 = (n^2 + 3) / 22 - 4$ |
| 7 | $(n^2 - 72) / 22 = (n^2 - 6) / 22 - 3$ |
| 8 | $(n^2 - 57) / 22 = (n^2 + 9) / 22 - 3$ |
| 9 | $(n^2 - 40) / 22 = (n^2 + 4) / 22 - 2$ |
| 10 | $(n^2 - 21) / 22 = (n^2 + 1) / 22 - 1$ |
| 11 | $n^2 / 22$ |

**Case 12.1.** For odd values of $n$, $n^2 - 2$, $n^2 - 6$, $n^2 - 8$, $n^2 - 10$, $n^2 + 4$ are all odd; hence not divisible by 22. Eventually, there exist no integral values of $b$. On the other hand, $n^2 + 1$, $n^2 + 3$, $n^2 + 5$, $n^2 + 9$, $n^2 - 7$ are even but still not divisible by 22.

**Case 12.2.** For even values of $n$, $n^2 - 2$, $n^2 - 6$, $n^2 - 8$, $n^2 - 10$, $n^2 + 4$ are all even but not divisible by 22; while $n^2 + 1$, $n^2 + 3$, $n^2 + 5$, $n^2 + 9$, $n^2 - 7$ being odd are also not divisible by 22.

Thus, there exist no integral values of $b$ for $a = 1 - 10$ implying no identities of above type. On the other hand, $n^2/22$ takes integral values when $n$ is an integral multiple of 22 giving rise to the following theorem:

**Theorem 12.1.** For $a = 11$, there hold the identities for integral values of $n$, $b$.

| $n$ | $b$ | Identity | Equivalently | Ref. |
|---|---|---|---|---|
| 22 | 22 | $11^2 + 22^2 + 22^2 = 33^2$ | $1^2 + 2^2 + 2^2 = 3^2$ | Th. 2.1 |
| 44 | 88 | $11^2 + 44^2 + 88^2 = 99^2$ | $1^2 + 4^2 + 8^2 = 9^2$ | ” |
| 66 | 198 | $11^2 + 66^2 + 198^2 = 209^2$ | $1^2 + 6^2 + 18^2 = 19^2$ | ” |
| 88 | 352 | $11^2 + 88^2 + 352^2 = 363^2$ | $1^2 + 8^2 + 32^2 = 33^2$ | ” |
| 110 | 550 | $11^2 + 110^2 + 550^2 = 561^2$ | $1^2 + 10^2 + 50^2 = 51^2$ | ” |

etc. //

Similarly, for $a = 12 - 21$, we get

$$b = (n^2 + 23) / 22 = (n^2 + 1) / 22 + 1, \dots ,$$
$$(n^2 + 320) / 22 = (n^2 - 10) / 22 + 15;$$

which also do not take integral values for any integer $n$. Hence, there exist no such identities for these values of $a$. However, $a = 22$ yields integral values of

$$b = (n^2 + 363) / 22 = (n^2 + 11) / 22 + 16$$

take for $n = 11, 33, 55, 77, 99$, etc. giving rise to the theorem:

**Theorem 12.2.** For $a = 22$, there hold the identities for integral values of $n$, $b$.

| $n$ | $b$ | Identity | Equivalently | Ref. |
|---|---|---|---|---|
| 11 | 22 | $22^2 + 11^2 + 22^2 = 33^2$ | $2^2 + 1^2 + 2^2 = 3^2$ | Th. 2.1 |
| 33 | 66 | $22^2 + 33^2 + 66^2 = 77^2$ | $2^2 + 3^2 + 6^2 = 7^2$ | Th. 2.2 |
| 55 | 154 | $22^2 + 55^2 + 154^2 = 165^2$ | $2^2 + 5^2 + 14^2 = 15^2$ | ” |
| 77 | 286 | $22^2 + 77^2 + 286^2 = 297^2$ | $2^2 + 7^2 + 26^2 = 27^2$ | ” |
| 99 | 462 | $22^2 + 99^2 + 462^2 = 473^2$ | $2^2 + 9^2 + 42^2 = 43^2$ | ” |

etc. //

For $a = 33$, $b = n^2/22 + 44$ takes integral values when $n$ is any integral multiple of 22 giving rise to the theorem:

**Theorem 12.3.** For $a = 33$, there hold the identities for integral values of $n$, $b$.

| $n$ | $b$ | Identity | Equivalently | Ref. |
|---|---|---|---|---|
| 22 | 66 | $33^2 + 22^2 + 66^2 = 77^2$ | $3^2 + 2^2 + 6^2 = 7^2$ | Th. 2.2 |
| 44 | 132 | $33^2 + 44^2 + 132^2 = 143^2$ | $3^2 + 4^2 + 12^2 = 13^2$ | Th. 2.3 |
| 66 | 242 | $33^2 + 66^2 + 242^2 = 253^2$ | $3^2 + 6^2 + 22^2 = 23^2$ | ” |
| 88 | 396 | $33^2 + 88^2 + 396^2 = 407^2$ | $3^2 + 8^2 + 36^2 = 37^2$ | ” |
| 110 | 594 | $33^2 + 110^2 + 594^2 = 605^2$ | $3^2 + 10^2 + 54^2 = 55^2$ | ” |

etc. //

For $a = 44$, $b = (n^2 + 1815) /22 = (n^2 + 11)/22 + 82$, which in analogy with Theo. 12.2, take integral values for $n = 11, 33, 55, 77, 99$, etc. Thus, we have the:

**Theorem 12.4.** For $a = 44$, there hold the identities for integral values of $n$, $b$.

| $n$ | $b$ | Identity | Equivalently | Ref. |
|---|---|---|---|---|
| 11 | 88 | $44^2 + 11^2 + 88^2 = 99^2$ | $4^2 + 1^2 + 8^2 = 9^2$ | Th. 2.1 |
| 33 | 132 | $44^2 + 33^2 + 132^2 = 143^2$ | $4^2 + 3^2 + 12^2 = 13^2$ | Th. 2.3 |
| 55 | 220 | $44^2 + 55^2 + 220^2 = 231^2$ | $4^2 + 5^2 + 20^2 = 21^2$ | Th. 2.4 |
| 77 | 352 | $44^2 + 77^2 + 352^2 = 363^2$ | $4^2 + 7^2 + 32^2 = 33^2$ | ” |
| 99 | 528 | $44^2 + 99^2 + 528^2 = 539^2$ | $4^2 + 9^2 + 48^2 = 49^2$ | ” |

etc. //

For $a = 55$, $b = (n^2 + 2904) /22 = n^2/22 + 132$ takes integral values when $n$ is any integral multiple of 22. Thus, we have the:

**Theorem 12.5.** For $a = 55$, there hold the identities for integral values of $n$, $b$.

| $n$ | $b$ | Identity | Equivalently | Ref. |
|---|---|---|---|---|
| 22 | 154 | $55^2 + 22^2 + 154^2 = 165^2$ | $5^2 + 2^2 + 14^2 = 15^2$ | Th. 2.2 |
| 44 | 220 | $55^2 + 44^2 + 220^2 = 231^2$ | $5^2 + 4^2 + 20^2 = 21^2$ | Th. 2.4 |
| 66 | 330 | $55^2 + 66^2 + 330^2 = 341^2$ | $5^2 + 6^2 + 30^2 = 31^2$ | Th. 2.5 |
| 88 | 484 | $55^2 + 88^2 + 484^2 = 495^2$ | $5^2 + 8^2 + 44^2 = 45^2$ | ” |
| 110 | 682 | $55^2 + 110^2 + 682^2 = 693^2$ | $5^2 + 10^2 + 62^2 = 63^2$ | ” |

etc. //

For $a = 66$, $b = (n^2 + 4235)/22 = (n^2 + 11) /22 + 192$, which in analogy with Theo. 12.2, take integral values for $n = 11, 33, 55, 77, 99$, etc. Thus, we have the:

**Theorem 12.6.** For $a$ = 66, there hold the following identities for integral values of $n$, $b$.

| *n* | *b* | Identity | Equivalently | Ref. |
|---|---|---|---|---|
| 11 | 198 | $66^2 + 11^2 + 198^2 = 209^2$ | $6^2 + 1^2 + 18^2 = 19^2$ | Th. 2.1 |
| 33 | 242 | $66^2 + 33^2 + 242^2 = 253^2$ | $6^2 + 3^2 + 22^2 = 23^2$ | Th. 2.3 |
| 55 | 330 | $66^2 + 55^2 + 330^2 = 341^2$ | $6^2 + 5^2 + 30^2 = 31^2$ | Th. 2.5 |
| 77 | 462 | $66^2 + 77^2 + 462^2 = 473^2$ | $6^2 + 7^2 + 42^2 = 43^2$ | Th. 2.6 |
| 99 | 638 | $66^2 + 99^2 + 638^2 = 649^2$ | $6^2 + 9^2 + 58^2 = 59^2$ | „ |

etc. //

## 13. Identities of the type $a^2 + n^2 + b^2 = (b + 12)^2$

Above type of identities require:

$$b = (a^2 + n^2 - 144) / 24 = (a^2 + n^2) / 24 - 6, \qquad (13.1)$$

where $a$ and $n$ assume some suitable integral values making $b$ integer. For positive integral values of $a$ above relation yields the following values of $b$:

| *a* | *b* | Remark |
|---|---|---|
| 1 | $(n^2 - 143)/24 = (n^2 + 1)/24 - 6$ | $b$ is not an integer for any integer $n$. |
| 2 | $(n^2 - 140)/24 = (n^2 + 4)/24 - 6$ | „ |
| 3 | $(n^2 - 135)/24 = (n^2 + 9)/24 - 6$ | „ |
| 4 | $(n^2 - 128)/24 = (n^2 - 8)/24 - 5$ | „ |
| 5 | $(n^2 - 119)/24 = (n^2 + 1)/24 - 5$ | „ |
| 6 | $(n^2 - 108)/24 = (n^2 + 12)/24 - 5$ | $b$ is integer when $n$ = 18,30,42,54,66, etc. |
| 7 | $(n^2 - 95)/24 = (n^2 + 1)/24 - 4$ | $b$ is not an integer for any integer $n$. |
| 8 | $(n^2 - 80)/24 = (n^2 - 8)/24 - 3$ | „ |
| 9 | $(n^2 - 63)/24 = (n^2 + 9)/24 - 3$ | „ |
| 10 | $(n^2 - 44)/24 = (n^2 + 4)/24 - 2$ | „ |
| 11 | $(n^2 - 23)/24 = (n^2 + 1)/24 - 1$ | „ |
| 12 | $n^2 / 24$ | $b$ is integer for $n$ = integral multiple of 12 $\Rightarrow$ identities |
| 18 | $(n^2 + 180)/24 = (n^2 + 12)/24 + 7$ | As for $a$ = 6. |
| 24 | $n^2/24 + 18$ | As for $a$ = 12. |
| 36 | $n^2/24 + 48$ | „ |

etc. Thus, we note that there exist such identities only when $a$ is an integral multiple of 6 giving rise to the following theorems:

**Theorem 13.1.** For $a$ = 6, there hold the identities for integral values of $n$, $b$.

| *n* | *b* | Identity | Equivalently | Ref. |
|---|---|---|---|---|
| 18 | 9 | $6^2 + 18^2 + 9^2 = 21^2$ | $2^2 + 6^2 + 3^2 = 7^2$ | Th. 2.2 |
| 30 | 33 | $6^2 + 30^2 + 33^2 = 45^2$ | $2^2 + 10^2 + 11^2 = 15^2$ | Th. 5.1 |
| 42 | 69 | $6^2 + 42^2 + 69^2 = 81^2$ | $2^2 + 14^2 + 23^2 = 27^2$ | „ |
| 54 | 117 | $6^2 + 54^2 + 117^2 = 129^2$ | $2^2 + 18^2 + 39^2 = 43^2$ | „ |
| 66 | 177 | $6^2 + 66^2 + 177^2 = 189^2$ | $2^2 + 22^2 + 59^2 = 63^2$ | „ |

etc. //

**Theorem 13.2.** For $a$ = 12, there hold the following identities for integral values of $n$, $b$.

| *n* | *b* | Identity | Equivalently | Ref. |
|---|---|---|---|---|
| 12 | 6 | $12^2 + 12^2 + 6^2 = 18^2$ | $2^2 + 2^2 + 1^2 = 3^2$ | Th. 2.1 |
| 24 | 24 | $12^2 + 24^2 + 24^2 = 36^2$ | $1^2 + 2^2 + 2^2 = 3^2$ | ” |
| 36 | 54 | $12^2 + 36^2 + 54^2 = 66^2$ | $2^2 + 6^2 + 9^2 = 11^2$ | Th. 3.1 |
| 48 | 96 | $12^2 + 48^2 + 96^2 = 108^2$ | $1^2 + 4^2 + 8^2 = 9^2$ | Th. 2.1 |
| 60 | 150 | $12^2 + 60^2 + 150^2 = 162^2$ | $2^2 + 10^2 + 25^2 = 27^2$ | Th. 3.1 |

etc. //

**Theorem 13.3.** For $a$ = 18, there hold the identities for integral values of $n$, $b$.

| *n* | *b* | Identity | Equivalently | Ref. |
|---|---|---|---|---|
| 6 | 9 | $18^2 + 6^2 + 9^2 = 21^2$ | $6^2 + 2^2 + 3^2 = 7^2$ | Th. 2.2 |
| 18 | 21 | $18^2 + 18^2 + 21^2 = 33^2$ | $6^2 + 6^2 + 7^2 = 11^2$ | Th. 5.3 |
| 30 | 45 | $18^2 + 30^2 + 45^2 = 57^2$ | $6^2 + 10^2 + 15^2 = 19^2$ | „ |
| 42 | 81 | $18^2 + 42^2 + 81^2 = 93^2$ | $6^2 + 14^2 + 27^2 = 31^2$ | „ |
| 54 | 129 | $18^2 + 54^2 + 129^2 = 141^2$ | $6^2 + 18^2 + 43^2 = 47^2$ | „ |
| 66 | 189 | $18^2 + 66^2 + 189^2 = 201^2$ | $6^2 + 22^2 + 63^2 = 67^2$ | „ |

etc. //

**Theorem 13.4.** For $a$ = 24, there hold the following identities for integral values of $n$, $b$.

| *n* | *b* | Identity | Equivalently | Ref. |
|---|---|---|---|---|
| 12 | 24 | $24^2 + 12^2 + 24^2 = 36^2$ | $2^2 + 1^2 + 2^2 = 3^2$ | Th. 2.1 |
| 24 | 42 | $24^2 + 24^2 + 42^2 = 54^2$ | $4^2 + 4^2 + 7^2 = 9^2$ | Th. 3.2 |
| 36 | 72 | $24^2 + 36^2 + 72^2 = 84^2$ | $2^2 + 3^2 + 6^2 = 7^2$ | Th. 2.2 |
| 48 | 114 | $24^2 + 48^2 + 114^2 = 126^2$ | $4^2 + 8^2 + 19^2 = 21^2$ | Th. 3.2 |
| 60 | 168 | $24^2 + 60^2 + 168^2 = 180^2$ | $2^2 + 5^2 + 14^2 = 15^2$ | Th. 2.2 |

etc. //

**Theorem 13.5.** For $a$ = 36, there hold the following identities for integral values of $n$, $b$.

| *n* | *b* | Identity | Equivalently | Ref. |
|---|---|---|---|---|
| 12 | 54 | $36^2 + 12^2 + 54^2 = 66^2$ | $6^2 + 2^2 + 9^2 = 11^2$ | Th. 3.1 |
| 24 | 72 | $36^2 + 24^2 + 72^2 = 84^2$ | $3^2 + 2^2 + 6^2 = 7^2$ | Th. 2.2 |
| 36 | 102 | $36^2 + 36^2 + 102^2 = 114^2$ | $6^2 + 6^2 + 17^2 = 19^2$ | Th. 3.3 |
| 48 | 144 | $36^2 + 48^2 + 144^2 = 156^2$ | $3^2 + 4^2 + 12^2 = 13^2$ | Th. 2.3 |
| 60 | 198 | $36^2 + 60^2 + 198^2 = 210^2$ | $6^2 + 10^2 + 33^2 = 35^2$ | Th. 3.3 |

etc. //

## 14. Identities of the type $a^2 + n^2 + b^2 = (b + 13)^2$

Above type of identities require:

$$b = (a^2 + n^2 - 169) / 26, \qquad (14.1)$$

where $a$ and $n$ assume some suitable integral values making $b$ integer. For positive integral values of $a$ above relation yields the following values of $b$:

| *a* | *b* | Remark |
|---|---|---|
| 1 | $(n^2 - 168)/26 = (n^2 - 12)/26 - 6$ | $b$ is +ve integer for $n$ = 18, 34, 44, 60, 70, etc. |
| 2 | $(n^2 - 165)/26 = (n^2 - 9)/26 - 6$ | $b$ is +ve integer for $n$ = 23, 29, 49,55,75,81, etc. |
| 3 | $(n^2 - 160)/26 = (n^2 - 4)/26 - 6$ | $b$ is +ve integer for $n$ = 24, 28, 50, 54, 76, etc. |
| 4 | $(n^2 - 153)/26 = (n^2 + 3)/26 - 6$ | $b$ is +ve integer for $n$ = 19, 33, 45, 59, 71, etc. |
| 5 | $(n^2 - 144)/26 = (n^2 + 12)/26 - 6$ | $b$ is +ve integer for $n$ = 14, 38, 40, 64, 66, etc. |
| 6 | $(n^2 - 133)/26 = (n^2 - 3)/26 - 5$ | $b$ is +ve integer for $n$ = 17, 35, 43, 61, 69, etc. |
| 7 | $(n^2 - 120)/26 = (n^2 + 10)/26 - 5$ | $b$ is +ve integer for $n$ = 22, 30, 48, 56, 74, etc. |
| 8 | $(n^2 - 105)/26 = (n^2 - 1)/26 - 4$ | $b$ is +ve integer for $n$ = 25, 27, 51, 53, 77, etc. |
| 9 | $(n^2 - 88)/26 = (n^2 - 10)/26 - 3$ | $b$ is integer when $n$ = 20,32,46,58,72, etc. |
| 10 | $(n^2 - 69)/26 = (n^2 + 9)/26 - 3$ | $b$ is +ve integer for $n$ = 11, 15, 37, 41, 63, etc. |
| 11 | $(n^2 - 48)/26 = (n^2 + 4)/26 - 2$ | $b$ is +ve integer for $n$ = 10, 16, 36, 42, 62, etc. |
| 12 | $(n^2 - 25)/26 = (n^2 + 1)/26 - 1$ | $b$ is +ve integer for $n$ = 21, 31, 47, 57, 73, etc. |
| 13 | $n^2 / 26$ | $b$ is integer when $n$ is an integral multiple of 26 |

etc. Thus, we have the:

**Theorem 14.1.** For $a$ = 1, there hold the identities for integral values of $n$, $b$.

| *n* | *b* | Identity | Ref. |
|---|---|---|---|
| 18 | 6 | $1^2 + 18^2 + 6^2 = 19^2$ | Th. 2.1 |
| 34 | 38 | $1^2 + 34^2 + 38^2 = 51^2$ | |
| 44 | 68 | $1^2 + 44^2 + 68^2 = 81^2$ | |
| 60 | 132 | $1^2 + 60^2 + 132^2 = 145^2$ | |
| 70 | 182 | $1^2 + 70^2 + 182^2 = 195^2$ | |

etc. //

**Theorem 14.2.** For $a$ = 2, there hold the identities for integral values of $n$, $b$.

| *n* | *b* | Identity | Ref. |
|---|---|---|---|
| 23 | 14 | $2^2 + 23^2 + 14^2 = 27^2$ | Th.5.1 |
| 29 | 26 | $2^2 + 29^2 + 26^2 = 39^2$ | |
| 49 | 86 | $2^2 + 49^2 + 86^2 = 99^2$ | |
| 55 | 110 | $2^2 + 55^2 + 110^2 = 123^2$ | |
| 75 | 210 | $2^2 + 75^2 + 210^2 = 223^2$ | |
| 81 | 246 | $2^2 + 81^2 + 246^2 = 259^2$ | |

etc. //

**Theorem 14.3.** For $a$ = 3, there hold the identities for integral values of $n$, $b$.

| *n* | *b* | Identity | Ref. |
|---|---|---|---|
| 24 | 16 | $3^2 + 24^2 + 16^2 = 29^2$ | Th. 6.3 |
| 28 | 24 | $3^2 + 28^2 + 24^2 = 37^2$ | Th.10.1 |
| 50 | 90 | $3^2 + 50^2 + 90^2 = 103^2$ | |
| 54 | 106 | $3^2 + 54^2 + 106^2 = 119^2$ | |

| | | | |
|---|---|---|---|
| 76 | 216 | $3^2 + 76^2 + 216^2 = 229^2$ | |
| 80 | 240 | $3^2 + 80^2 + 240^2 = 253^2$ | |

etc. //

**Theorem 14.4.** For *a* = 4, there hold the identities for integral values of *n*, *b*.

| *n* | *b* | Identity | Ref. |
|---|---|---|---|
| 19 | 8 | $4^2 + 19^2 + 8^2 = 21^2$ | Th. 3.2 |
| 33 | 36 | $4^2 + 33^2 + 36^2 = 49^2$ | |
| 45 | 72 | $4^2 + 45^2 + 72^2 = 85^2$ | |
| 59 | 132 | $4^2 + 59^2 + 132^2 = 145^2$ | |
| 71 | 188 | $4^2 + 71^2 + 188^2 = 201^2$ | |

etc. //

**Theorem 14.5.** For *a* = 5, there hold the identities for integral values of *n*, *b*.

| *n* | *b* | Identity | Ref. |
|---|---|---|---|
| 14 | 2 | $5^2 + 14^2 + 2^2 = 15^2$ | Th. 2.2 |
| 38 | 50 | $5^2 + 38^2 + 50^2 = 63^2$ | |
| 40 | 56 | $5^2 + 40^2 + 56^2 = 69^2$ | |
| 64 | 152 | $5^2 + 64^2 + 152^2 = 165^2$ | |
| 66 | 162 | $5^2 + 66^2 + 162^2 = 175^2$ | |

etc. //

**Theorem 14.6.** For *a* = 6, there hold the identities for integral values of *n*, *b*.

| *n* | *b* | Identity | Ref. |
|---|---|---|---|
| 17 | 6 | $6^2 + 17^2 + 6^2 = 19^2$ | Th. 3.3 |
| 35 | 42 | $6^2 + 35^2 + 42^2 = 55^2$ | |
| 43 | 66 | $6^2 + 43^2 + 66^2 = 79^2$ | |
| 61 | 138 | $6^2 + 61^2 + 138^2 = 151^2$ | |
| 69 | 178 | $6^2 + 69^2 + 178^2 = 191^2$ | |

etc. //

**Theorem 14.7.** For *a* = 7, there hold the identities for integral values of *n*, *b*.

| *n* | *b* | Identity | Ref. |
|---|---|---|---|
| 22 | 14 | $7^2 + 22^2 + 14^2 = 27^2$ | Th. 6.7 |
| 30 | 30 | $7^2 + 30^2 + 30^2 = 43^2$ | |
| 48 | 84 | $7^2 + 48^2 + 84^2 = 97^2$ | |
| 56 | 116 | $7^2 + 56^2 + 116^2 = 129^2$ | |
| 74 | 206 | $7^2 + 74^2 + 206^2 = 219^2$ | |
| 82 | 254 | $7^2 + 82^2 + 254^2 = 267^2$ | |

etc. //

**Theorem 14.8.** For *a* = 8, there hold the identities for integral values of *n*, *b*.

| *n* | *b* | Identity | Ref. |
|---|---|---|---|
| 25 | 20 | $8^2 + 25^2 + 20^2 = 33^2$ | Th. 9.2 |
| 27 | 24 | $8^2 + 27^2 + 24^2 = 37^2$ | |
| 51 | 96 | $8^2 + 51^2 + 96^2 = 109^2$ | |
| 53 | 104 | $8^2 + 53^2 + 104^2 = 117^2$ | |
| 77 | 224 | $8^2 + 77^2 + 224^2 = 237^2$ | |
| 79 | 236 | $8^2 + 79^2 + 236^2 = 249^2$ | |

etc. //

**Theorem 14.9.** For *a* = 9, there hold the identities for integral values of *n*, *b*.

| *n* | *b* | Identity | Ref. |
|---|---|---|---|
| 20 | 12 | $9^2 + 20^2 + 12^2 = 25^2$ | Th. 6.9 |
| 32 | 36 | $9^2 + 32^2 + 36^2 = 49^2$ | |
| 46 | 78 | $9^2 + 46^2 + 78^2 = 91^2$ | |
| 58 | 126 | $9^2 + 58^2 + 126^2 = 139^2$ | |
| 72 | 196 | $9^2 + 72^2 + 196^2 = 209^2$ | |

etc. //

**Theorem 14.10.** For *a* = 10, there hold the identities for integral values of *n*, *b*.

| *n* | *b* | Identity | Ref. |
|---|---|---|---|
| 11 | 2 | $10^2 + 11^2 + 2^2 = 15^2$ | Th. 5.1 |

| | | | |
|---|---|---|---|
| 15 | 6 | $10^2 + 15^2 + 6^2 = 19^2$ | Th. 5.3 |
| 37 | 50 | $10^2 + 37^2 + 50^2 = 63^2$ | |
| 41 | 62 | $10^2 + 41^2 + 62^2 = 75^2$ | |
| 63 | 150 | $10^2 + 63^2 + 150^2 = 163^2$ | |
| 67 | 170 | $10^2 + 67^2 + 170^2 = 183^2$ | |

etc. //

**Theorem 14.11.** For $a$ = 11, there hold the identities for integral values of $n$, $b$.

| ***n*** | ***b*** | **Identity** | **Ref.** |
|---|---|---|---|
| 10 | 2 | $11^2 + 10^2 + 2^2 = 15^2$ | Th. 5.1 |
| 16 | 8 | $11^2 + 16^2 + 8^2 = 21^2$ | Th. 6.8 |
| 36 | 48 | $11^2 + 36^2 + 48^2 = 61^2$ | |
| 42 | 66 | $11^2 + 42^2 + 66^2 = 79^2$ | |
| 62 | 146 | $11^2 + 62^2 + 146^2 = 159^2$ | |
| 68 | 176 | $11^2 + 68^2 + 176^2 = 189^2$ | |

etc. //

**Theorem 14.12.** For $a$ = 12, there hold the identities for integral values of $n$, $b$.

| ***n*** | ***b*** | **Identity** | **Ref.** |
|---|---|---|---|
| 21 | 16 | $12^2 + 21^2 + 16^2 = 29^2$ | Th. 9.3 |
| 31 | 36 | $12^2 + 31^2 + 36^2 = 49^2$ | |
| 47 | 84 | $12^2 + 47^2 + 84^2 = 97^2$ | |
| 57 | 124 | $12^2 + 57^2 + 124^2 = 137^2$ | |
| 73 | 204 | $12^2 + 73^2 + 204^2 = 217^2$ | |
| 83 | 264 | $12^2 + 83^2 + 264^2 = 277^2$ | |

etc. //

**Theorem 14.13.** For $a$ = 13, there hold the identities for integral values of $n$, $b$.

| ***n*** | ***b*** | **Identity** | **Equivalently** | **Ref.** |
|---|---|---|---|---|
| 26 | 26 | $13^2 + 26^2 + 26^2 = 39^2$ | $1^2 + 2^2 + 2^2 = 3^2$ | Th. 2.1 |
| 52 | 104 | $13^2 + 52^2 + 104^2 = 117^2$ | $1^2 + 4^2 + 8^2 = 9^2$ | ” |
| 78 | 234 | $13^2 + 78^2 + 234^2 = 247^2$ | $1^2 + 6^2 + 18^2 = 19^2$ | ” |
| 104 | 416 | $13^2 + 104^2 + 416^2 = 429^2$ | $1^2 + 8^2 + 32^2 = 33^2$ | ” |
| 130 | 650 | $13^2 + 130^2 + 650^2 = 663^2$ | $1^2 + 10^2 + 50^2 = 51^2$ | ” |

## 15. Identities of the type $a^2 + n^2 + b^2 = (b + 14)^2$

Above type of identities require:

$$b = (a^2 + n^2 - 196)/28 = (a^2 + n^2)/28 - 7, \quad (15.1)$$

where $a$ and $n$ assume some suitable integral values making $b$ integer. For positive integral values of $a$ above relation yields the following values of $b$:

| ***a*** | ***b*** |
|---|---|
| 1 | $(n^2 + 1) / 28 - 7$ |
| 2 | $(n^2 + 4) / 28 - 7$ |
| 3 | $(n^2 + 9) / 28 - 7$ |
| 4 | $(n^2 - 180)/28 = (n^2 - 12)/28 - 6$ |
| 5 | $(n^2 - 171)/28 = (n^2 - 3)/28 - 6$ |
| 6 | $(n^2 - 160)/28 = (n^2 + 8)/28 - 6$ |
| 7 | $(n^2 - 147)/28 = (n^2 - 7)/28 - 5$ |
| 8 | $(n^2 - 132)/28 = (n^2 + 8)/28 - 5$ |
| 9 | $(n^2 - 115)/28 = (n^2 - 3)/28 - 4$ |
| 10 | $(n^2 - 96)/28 = (n^2 - 12)/28 - 3$ |
| 11 | $(n^2 - 75)/28 = (n^2 + 9)/28 - 3$ |
| 12 | $(n^2 - 52)/28 = (n^2 + 4)/28 - 2$ |
| 13 | $(n^2 - 27)/28 = (n^2 + 1)/28 - 1$ |
| 14 | $n^2 / 28$ |

**Note 15.1.** None of the values of $a$ = 1 – 13 yield integral values of $b$. However, $a$ = 14 makes $b$ integer when $n$ is an integral multiple of 14 yielding the:

**Theorem 15.1.** For $a$ = 14, there hold the identities for integral values of $n$, $b$.

| ***n*** | ***b*** | **Identity** | **Equivalently** | **Ref.** |
|---|---|---|---|---|
| 14 | 7 | $14^2 + 14^2 + 7^2 = 21^2$ | $2^2 + 2^2 + 1^2 = 3^2$ | Th. 2.1 |
| 28 | 28 | $14^2 + 28^2 + 28^2 = 42^2$ | $1^2 + 2^2 + 2^2 = 3^2$ | ” |

| | | | | |
|---|---|---|---|---|
| 42 | 63 | $14^2 + 42^2 + 63^2 = 77^2$ | $2^2 + 6^2 + 9^2 = 11^2$ | Th. 3.2 |
| 56 | 112 | $14^2 + 56^2 + 112^2 = 126^2$ | $1^2 + 4^2 + 8^2 = 9^2$ | Th. 2.1 |
| 70 | 175 | $14^2 + 70^2 + 175^2 = 189^2$ | $2^2 + 10^2 + 25^2 = 27^2$ | Th. 3.2 |

etc. //

Similarly, $a = 15 – 27$ also do not yield any integral values of $b$ for any integer $n$. In the following we check the situation when $a$ is an integral multiple of 14.

| ***a*** | ***b*** | **Remark** |
|---|---|---|
| 28 | $n^2 / 28 + 21$ | $b$ is +ve integer for $n$ = integral multiple of 14 |
| 42 | $n^2/28 + 56$ | ” |
| 56 | $n^2/28 + 105$ | ” |
| 70 | $n^2/28 + 168$ | ” |

etc. yielding the following theorems:

**Theorem 15.2.** For $a = 28$, there hold the identities for integral values of $n$, $b$.

| ***n*** | ***b*** | **Identity** | **Equivalently** | **Ref.** |
|---|---|---|---|---|
| 14 | 28 | $28^2 + 14^2 + 28^2 = 42^2$ | $2^2 + 1^2 + 2^2 = 3^2$ | Th. 2.1 |
| 28 | 49 | $28^2 + 28^2 + 49^2 = 63^2$ | $4^2 + 4^2 + 7^2 = 9^2$ | Th. 3.2 |
| 42 | 84 | $28^2 + 42^2 + 84^2 = 98^2$ | $2^2 + 3^2 + 6^2 = 7^2$ | Th. 2.2 |
| 56 | 133 | $28^2 + 56^2 + 133^2 = 147^2$ | $4^2 + 8^2 + 19^2 = 21^2$ | Th. 3.2 |
| 70 | 196 | $28^2 + 70^2 + 196^2 = 210^2$ | $2^2 + 5^2 + 14^2 = 15^2$ | Th. 2.2 |

etc. //

**Theorem 15.3.** For $a = 42$, there hold the identities for integral values of $n$, $b$.

| ***n*** | ***b*** | **Identity** | **Equivalently** | **Ref.** |
|---|---|---|---|---|
| 14 | 63 | $42^2 + 14^2 + 63^2 = 77^2$ | $6^2 + 2^2 + 9^2 = 11^2$ | Th. 3.1 |
| 28 | 84 | $42^2 + 28^2 + 84^2 = 98^2$ | $3^2 + 2^2 + 6^2 = 7^2$ | Th. 2.2 |
| 42 | 119 | $42^2 + 42^2 + 119^2 = 133^2$ | $6^2 + 6^2 + 17^2 = 19^2$ | Th. 3.3 |
| 56 | 168 | $42^2 + 56^2 + 168^2 = 182^2$ | $3^2 + 4^2 + 12^2 = 13^2$ | Th. 2.3 |
| 70 | 231 | $42^2 + 70^2 + 231^2 = 245^2$ | $6^2 + 10^2 + 33^2 = 35^2$ | Th. 3.3 |

etc. //

**Theorem 15.4.** For $a = 56$, there hold the identities for integral values of $n$, $b$.

| ***n*** | ***b*** | **Identity** | **Equivalently** | **Ref.** |
|---|---|---|---|---|
| 14 | 112 | $56^2 + 14^2 + 112^2 = 126^2$ | $4^2 + 1^2 + 8^2 = 9^2$ | Th. 2.1 |
| 28 | 133 | $56^2 + 28^2 + 133^2 = 147^2$ | $8^2 + 4^2 + 19^2 = 21^2$ | Th. 3.2 |
| 42 | 168 | $56^2 + 42^2 + 168^2 = 182^2$ | $4^2 + 3^2 + 12^2 = 13^2$ | Th. 2.3 |
| 56 | 217 | $56^2 + 56^2 + 217^2 = 231^2$ | $8^2 + 8^2 + 31^2 = 33^2$ | Th. 3.4 |
| 70 | 280 | $56^2 + 70^2 + 280^2 = 294^2$ | $4^2 + 5^2 + 20^2 = 21^2$ | Th. 2.4 |

etc. //

**Theorem 15.5.** For $a = 70$, there hold the identities for integral values of $n$, $b$.

| ***n*** | ***b*** | **Identity** | **Equivalently** | **Ref.** |
|---|---|---|---|---|
| 14 | 175 | $70^2 + 14^2 + 175^2 = 189^2$ | $10^2 + 2^2 + 25^2 = 27^2$ | Th. 3.1 |
| 28 | 196 | $70^2 + 28^2 + 196^2 = 210^2$ | $5^2 + 2^2 + 14^2 = 15^2$ | Th. 2.2 |
| 42 | 231 | $70^2 + 42^2 + 231^2 = 245^2$ | $10^2 + 6^2 + 33^2 = 35^2$ | Th. 3.3 |
| 56 | 280 | $70^2 + 56^2 + 280^2 = 294^2$ | $5^2 + 4^2 + 20^2 = 21^2$ | Th. 2.4 |
| 70 | 343 | $70^2 + 70^2 + 343^2 = 357^2$ | $10^2 + 10^2 + 49^2 = 51^2$ | Th. 3.5 |

etc. //

## 16. Identities of the type $a^2 + n^2 + b^2 = (b + 15)^2$

Above type of identities require:

$$b = (a^2 + n^2 – 225)/30 = (a^2 + n^2 – 15)/30 – 7, \quad (16.1)$$

where $a$ and $n$ assume some suitable integral values making $b$ integer. For positive integral values of $a$ above relation yields the following values of $b$:

| ***a*** | ***b*** | **Remark** |
|---|---|---|
| 1 | $(n^2 – 14) / 30 – 7$ | $b$ is not integer for any integer $n$. |
| 2 | $(n^2 – 11) / 30 – 7$ | ” |
| 3 | $(n^2 – 6) / 30 – 7$ | $b$ is +ve integer for $n$ = 24, 36, 54, 66, 84, 96, etc. |
| 4 | $(n^2 + 1) / 30 – 7$ | $b$ is not integer for any integer $n$. |
| 5 | $(n^2 + 10) / 30 – 7$ | ” |
| 6 | $(n^2 – 189)/30 = (n^2 – 9)/30 – 6$ | $b$ is +ve integer for $n$ = 27, 33, 57, 63, 87, 93, etc. |
| 7 | $(n^2 – 176)/30 = (n^2 + 4)/30 – 6$ | $b$ is not integer for any integer $n$. |
| 8 | $(n^2 – 161)/30 = (n^2 – 11)/30 – 5$ | ” |

| | | |
|---|---|---|
| 9 | $(n^2 - 144)/30 = (n^2 + 6)/30 - 5$ | *b* is +ve integer for *n* = 18, 42, 48, 72, 78, 102, etc. |
| 10 | $(n^2 - 125)/30 = (n^2 - 5)/30 - 4$ | *b* is not integer for any integer *n*. |
| 11 | $(n^2 - 104)/30 = (n^2 - 14)/30 - 3$ | „ |
| 12 | $(n^2 - 81)/30 = (n^2 + 9)/30 - 3$ | *b* is +ve integer for *n* = 21, 39, 51, 69, 81, 99, etc. |
| 13 | $(n^2 - 56)/30 = (n^2 + 4)/30 - 2$ | *b* is not integer for any integer *n*. |
| 14 | $(n^2 - 29)/30 = (n^2 + 1)/30 - 1$ | „ |
| 15 | $n^2 / 30$ | *b* is +ve integer for *n* = integral multiple of 30. |
| 18 | $(n^2 + 99)/30 = (n^2 + 9)/30 + 3$ | *b* is +ve integer for *n* = 9, 21, 39, 51, 69, 81, etc. |
| 21 | $(n^2 + 216)/30 = (n^2 + 6)/30 + 7$ | *b* is +ve integer for *n* = 12, 18, 42, 48, 72, 78, etc. |

etc. Conclusively, *b* takes integral values whenever *a* is an integral multiple of 3. Hence, there hold the following theorems:

**Theorem 16.1.** For *a* = 3, there hold the identities for integral values of *n*, *b*.

| *n* | *b* | Identity | Equivalently | Ref. |
|---|---|---|---|---|
| 24 | 12 | $3^2 + 24^2 + 12^2 = 27^2$ | $1^2 + 8^2 + 4^2 = 9^2$ | Th. 2.1 |
| 36 | 36 | $3^2 + 36^2 + 36^2 = 51^2$ | $1^2 + 12^2 + 12^2 = 17^2$ | Th. 6.1 |
| 54 | 90 | $3^2 + 54^2 + 90^2 = 105^2$ | $1^2 + 18^2 + 30^2 = 35^2$ | „ |
| 66 | 138 | $3^2 + 66^2 + 138^2 = 153^2$ | $1^2 + 22^2 + 46^2 = 51^2$ | „ |
| 84 | 228 | $3^2 + 84^2 + 228^2 = 243^2$ | $1^2 + 28^2 + 76^2 = 81^2$ | „ |

etc. //

**Theorem 16.2.** For *a* = 6, there hold the identities for integral values of *n*, *b*.

| *n* | *b* | Identity | Equivalently | Ref. |
|---|---|---|---|---|
| 27 | 18 | $6^2 + 27^2 + 18^2 = 33^2$ | $2^2 + 9^2 + 6^2 = 11^2$ | Th. 3.2 |
| 33 | 30 | $6^2 + 33^2 + 30^2 = 45^2$ | $2^2 + 11^2 + 10^2 = 15^2$ | Th. 5.1 |
| 57 | 102 | $6^2 + 57^2 + 102^2 = 117^2$ | $2^2 + 19^2 + 34^2 = 39^2$ | Th. 6.2 |
| 63 | 126 | $6^2 + 63^2 + 126^2 = 141^2$ | $2^2 + 21^2 + 42^2 = 47^2$ | „ |
| 87 | 246 | $6^2 + 87^2 + 246^2 = 261^2$ | $2^2 + 29^2 + 82^2 = 87^2$ | „ |
| 93 | 282 | $6^2 + 93^2 + 282^2 = 297^2$ | $2^2 + 31^2 + 94^2 = 99^2$ | „ |

etc. //

**Theorem 16.3.** For *a* = 9, there hold the identities for integral values of *n*, *b*.

| *n* | *b* | Identity | Equivalently | Ref. |
|---|---|---|---|---|
| 18 | 6 | $9^2 + 18^2 + 6^2 = 21^2$ | $3^2 + 6^2 + 2^2 = 7^2$ | Th. 2.2 |
| 42 | 54 | $9^2 + 42^2 + 54^2 = 69^2$ | $3^2 + 14^2 + 18^2 = 23^2$ | Th. 6.3 |
| 48 | 72 | $9^2 + 48^2 + 72^2 = 87^2$ | $3^2 + 16^2 + 24^2 = 29^2$ | „ |
| 72 | 168 | $9^2 + 72^2 + 168^2 = 183^2$ | $3^2 + 24^2 + 56^2 = 61^2$ | „ |
| 78 | 198 | $9^2 + 78^2 + 198^2 = 213^2$ | $3^2 + 26^2 + 66^2 = 71^2$ | „ |
| 102 | 342 | $9^2 + 102^2 + 342^2 = 357^2$ | $3^2 + 34^2 + 114^2 = 119^2$ | „ |

etc. //

**Theorem 16.4.** For *a* = 12, there hold the identities for integral values of *n*, *b*.

| *n* | *b* | Identity | Equivalently | Ref. |
|---|---|---|---|---|
| 21 | 12 | $12^2 + 21^2 + 12^2 = 27^2$ | $4^2 + 7^2 + 4^2 = 9^2$ | Th. 3.4 |
| 39 | 48 | $12^2 + 39^2 + 48^2 = 63^2$ | $4^2 + 13^2 + 16^2 = 21^2$ | Th. 6.4 |
| 51 | 84 | $12^2 + 51^2 + 84^2 = 99^2$ | $4^2 + 17^2 + 28^2 = 33^2$ | „ |
| 69 | 156 | $12^2 + 69^2 + 156^2 = 171^2$ | $4^2 + 23^2 + 52^2 = 57^2$ | „ |
| 81 | 216 | $12^2 + 81^2 + 216^2 = 231^2$ | $4^2 + 27^2 + 72^2 = 77^2$ | „ |
| 99 | 324 | $12^2 + 99^2 + 324^2 = 339^2$ | $4^2 + 33^2 + 108^2 = 113^2$ | „ |

etc. //

**Theorem 16.5.** For *a* = 15, there hold the identities for integral values of *n*, *b*.

| *n* | *b* | Identity | Equivalently | Ref. |
|---|---|---|---|---|
| 30 | 30 | $15^2 + 30^2 + 30^2 = 45^2$ | $1^2 + 2^2 + 2^2 = 3^2$ | Th. 2.1 |
| 60 | 120 | $15^2 + 60^2 + 120^2 = 135^2$ | $1^2 + 4^2 + 8^2 = 9^2$ | „ |
| 90 | 270 | $15^2 + 90^2 + 270^2 = 285^2$ | $1^2 + 6^2 + 18^2 = 19^2$ | „ |
| 120 | 480 | $15^2 + 120^2 + 480^2 = 495^2$ | $1^2 + 8^2 + 32^2 = 33^2$ | „ |
| 150 | 750 | $15^2 + 150^2 + 750^2 = 765^2$ | $1^2 + 10^2 + 50^2 = 51^2$ | „ |

etc. //

**Theorem 16.6.** For *a* = 18, there hold the identities for integral values of *n*, *b*.

| *n* | *b* | Identity | Equivalently | Ref. |
|---|---|---|---|---|

| | | | | |
|---|---|---|---|---|
| 9 | 6 | $18^2 + 9^2 + 6^2 = 21^2$ | $6^2 + 3^2 + 2^2 = 7^2$ | Th. 2.2 |
| 21 | 18 | $18^2 + 21^2 + 18^2 = 33^2$ | $6^2 + 7^2 + 6^2 = 11^2$ | Th. 5.3 |
| 39 | 54 | $18^2 + 39^2 + 54^2 = 69^2$ | $6^2 + 13^2 + 18^2 = 23^2$ | Th. 6.6 |
| 51 | 90 | $18^2 + 51^2 + 90^2 = 105^2$ | $6^2 + 17^2 + 30^2 = 35^2$ | ,, |
| 69 | 162 | $18^2 + 69^2 + 162^2 = 177^2$ | $6^2 + 23^2 + 54^2 = 59^2$ | ,, |
| 81 | 222 | $18^2 + 81^2 + 222^2 = 237^2$ | $6^2 + 27^2 + 74^2 = 79^2$ | ,, |

etc. //

**Theorem 16.7.** For $a$ = 21, there hold the identities for integral values of $n$, $b$.

| *n* | *b* | Identity | Equivalently | Ref. |
|---|---|---|---|---|
| 12 | 12 | $21^2 + 12^2 + 12^2 = 27^2$ | $7^2 + 4^2 + 4^2 = 9^2$ | Th. 3.4 |
| 18 | 18 | $21^2 + 18^2 + 18^2 = 33^2$ | $7^2 + 6^2 + 6^2 = 11^2$ | Th. 5.3 |
| 42 | 66 | $21^2 + 42^2 + 66^2 = 81^2$ | $7^2 + 14^2 + 22^2 = 27^2$ | Th. 6.7 |
| 48 | 84 | $21^2 + 48^2 + 84^2 = 99^2$ | $7^2 + 16^2 + 28^2 = 33^2$ | ,, |
| 72 | 180 | $21^2 + 72^2 + 180^2 = 195^2$ | $7^2 + 24^2 + 60^2 = 65^2$ | ,, |
| 78 | 210 | $21^2 + 78^2 + 210^2 = 225^2$ | $7^2 + 26^2 + 70^2 = 75^2$ | ,, |

etc. //

## 17. Identities of the type $a^2 + n^2 + b^2 = (b + 16)^2$

Above type of identities require:

$$b = (a^2 + n^2 - 256) / 32 = (a^2 + n^2) / 32 - 8, \quad (17.1)$$

where $a$ and $n$ assume some suitable integral values making $b$ integer. For positive integral values of $a$ above relation yields the following values of $b$:

| *a* | *b* | Remark |
|---|---|---|
| 1 | $(n^2 + 1) / 32 - 8$ | $b$ is not integer for any integer $n$. |
| 2 | $(n^2 + 4) / 32 - 8$ | ,, |
| 3 | $(n^2 + 9) / 32 - 8$ | ,, |
| 4 | $(n^2 - 16) / 32 - 7$ | $b$ is integer for $n$ = 4, 12, 20, 28, 36, 44, 52, 60, etc. |
| 5 | $(n^2 - 231)/32 = (n^2 - 7)/32 - 7$ | $b$ is not integer for any integer $n$. |
| 6 | $(n^2 - 220)/32 = (n^2 + 4)/32 - 7$ | ,, |
| 7 | $(n^2 - 207)/32 = (n^2 - 15)/32 - 6$ | ,, |
| 8 | $(n^2 - 192)/32 = n^2 / 32 - 6$ | $b$ is +ve integer for $n$ is any integral multiple of 8. |
| 9 | $(n^2 - 175)/32 = (n^2 - 15)/32 - 5$ | $b$ is not integer for any integer $n$. |
| 10 | $(n^2 - 156)/32 = (n^2 + 4)/32 - 5$ | ,, |
| 11 | $(n^2 - 135)/32 = (n^2 - 7)/32 - 4$ | ,, |
| 12 | $(n^2 - 112)/32 = (n^2 - 16)/32 - 3$ | As for $a$ = 4. |
| 13 | $(n^2 - 87)/32 = (n^2 + 9)/32 - 3$ | $b$ is not integer for any integer $n$. |
| 14 | $(n^2 - 60)/32 = (n^2 + 4)/32 - 2$ | ,, |
| 15 | $(n^2 - 31)/32 = (n^2 + 1)/32 - 1$ | ,, |
| 16 | $n^2 / 32$ | As for $a$ = 8. |
| 17 | $(n^2 + 33)/32 = (n^2 + 1)/32 + 1$ | As for $a$ = 15. |
| 18 | $(n^2 + 68)/32 = (n^2 + 4)/32 + 2$ | As for $a$ = 14. |
| 19 | $(n^2 + 105)/32 = (n^2 + 9)/32 + 3$ | As for $a$ = 13. |
| 20 | $(n^2 + 144)/32 = (n^2 - 16)/32 + 5$ | As for $a$ = 4. |

etc. Conclusively, $b$ takes integral values whenever $a$ is an integral multiple of 4. Hence, there hold the following theorems:

**Theorem 17.1.** For $a$ = 4, there hold the identities for integral values of $n$, $b$.

| *n* | *b* | Identity | Ref. |
|---|---|---|---|
| 20 | 5 | $4^2 + 20^2 + 5^2 = 21^2$ | Th. 2.4 |
| 28 | 17 | $4^2 + 28^2 + 17^2 = 33^2$ | Th. 6.4 |
| 36 | 33 | $4^2 + 36^2 + 33^2 = 49^2$ | Th. 14.4 |
| 44 | 53 | $4^2 + 44^2 + 53^2 = 69^2$ | |
| 52 | 77 | $4^2 + 52^2 + 77^2 = 93^2$ | |
| 60 | 105 | $4^2 + 60^2 + 105^2 = 121^2$ | |

etc. //

**Theorem 17.2.** For $a$ = 8, there hold the identities for integral values of $n$, $b$.

| *n* | *b* | Identity | Equivalently | Ref. |
|---|---|---|---|---|
| 16 | 2 | $8^2 + 16^2 + 2^2 = 18^2$ | $4^2 + 8^2 + 1^2 = 9^2$ | Th. 2.1 |
| 24 | 12 | $8^2 + 24^2 + 12^2 = 28^2$ | $2^2 + 6^2 + 3^2 = 7^2$ | Th. 2.2 |

| | | | | |
|---|---|---|---|---|
| 32 | 26 | $8^2 + 32^2 + 26^2 = 42^2$ | $4^2 + 16^2 + 13^2 = 21^2$ | Th. 6.4 |
| 40 | 44 | $8^2 + 40^2 + 44^2 = 60^2$ | $2^2 + 10^2 + 11^2 = 15^2$ | Th. 5.1 |
| 48 | 66 | $8^2 + 48^2 + 66^2 = 82^2$ | $4^2 + 24^2 + 33^2 = 41^2$ | Th. 9.1 |
| 56 | 92 | $8^2 + 56^2 + 92^2 = 108^2$ | $2^2 + 14^2 + 23^2 = 27^2$ | Th. 5.1 |

etc. //

**Theorem 17.3.** For $a$ = 12, there hold the identities for integral values of $n$, $b$.

| *n* | *b* | **Identity** | **Ref.** |
|---|---|---|---|
| 12 | 1 | $12^2 + 12^2 + 1^2 = 17^2$ | Th. 6.1 |
| 20 | 9 | $12^2 + 20^2 + 9^2 = 25^2$ | Th. 6.9 |
| 28 | 21 | $12^2 + 28^2 + 21^2 = 37^2$ | Th. 10.4 |
| 36 | 37 | $12^2 + 36^2 + 37^2 = 53^2$ | |
| 44 | 57 | $12^2 + 44^2 + 57^2 = 73^2$ | |
| 52 | 81 | $12^2 + 52^2 + 81^2 = 97^2$ | |

etc. //

**Theorem 17.4.** For $a$ = 16, there hold the identities for integral values of $n$, $b$.

| *n* | *b* | **Identity** | **Equivalently** | **Ref.** |
|---|---|---|---|---|
| 8 | 2 | $16^2 + 8^2 + 2^2 = 18^2$ | $8^2 + 4^2 + 1^2 = 9^2$ | Th. 2.1 |
| 16 | 8 | $16^2 + 16^2 + 8^2 = 24^2$ | $2^2 + 2^2 + 1^2 = 3^2$ | ” |
| 24 | 18 | $16^2 + 24^2 + 18^2 = 34^2$ | $8^2 + 12^2 + 9^2 = 17^2$ | Th. 6.8 |
| 32 | 32 | $16^2 + 32^2 + 32^2 = 48^2$ | $1^2 + 2^2 + 2^2 = 3^2$ | Th. 2.1 |
| 40 | 50 | $16^2 + 40^2 + 50^2 = 66^2$ | $8^2 + 20^2 + 25^2 = 33^2$ | Th. 9.2 |
| 48 | 72 | $16^2 + 48^2 + 72^2 = 88^2$ | $2^2 + 6^2 + 9^2 = 11^2$ | Th. 3.1 |

etc. //

**Theorem 17.5.** For $a$ = 20, there hold the identities for integral values of $n$, $b$.

| *n* | *b* | **Identity** | **Ref.** |
|---|---|---|---|
| 4 | 5 | $20^2 + 4^2 + 5^2 = 21^2$ | Th. 2.4 |
| 12 | 9 | $20^2 + 12^2 + 9^2 = 25^2$ | Th. 2.1 |
| 20 | 17 | $20^2 + 20^2 + 17^2 = 33^2$ | § 14 |
| 28 | 29 | $20^2 + 28^2 + 29^2 = 45^2$ | ” |
| 36 | 45 | $20^2 + 36^2 + 45^2 = 61^2$ | ” |
| 44 | 65 | $20^2 + 44^2 + 65^2 = 81^2$ | ” |

etc. //

**18. Identities of the type $a^2 + n^2 + b^2 = (b + 17)^2$**

Above type of identities require:

$$b = (a^2 + n^2 - 289)/34 = (a^2 + n^2 - 17)/34 - 8, \quad (18.1)$$

where $a$ and $n$ assume some suitable integral values making $b$ integer. For positive integral values of $a$ above relation yields the following values of $b$:

| *a* | *b* | **Remark** |
|---|---|---|
| 1 | $(n^2 - 16)/34 - 8$ | $b$ is +ve integer for $n$ = 30, 38, 64, 72, 98, etc. |
| 2 | $(n^2 - 13)/34 - 8$ | $b$ is +ve integer for $n$ = 25, 43, 59,77, 93, etc. |
| 3 | $(n^2 - 8)/34 - 8$ | $b$ is +ve integer for $n$ = 22, 46, 56, 80, 90, etc. |
| 4 | $(n^2 - 1)/34 - 8$ | $b$ is +ve integer for $n$ = 33,35,67,69,101,103etc. |
| 5 | $(n^2 + 8)/34 - 8$ | $b$ is +ve integer for $n$ = 20, 48,54,82,88,116, etc. |
| 6 | $(n^2 - 253)/34 = (n^2 - 15)/34 - 7$ | $b$ is +ve integer for $n$ = 27,41,61,75, 95,109, etc. |
| 7 | $(n^2 - 240)/34 = (n^2 - 2)/34 - 7$ | $b$ is +ve integer for $n$ = 28, 40,62,74,96,108, etc. |
| 8 | $(n^2 - 225)/34 = (n^2 + 13)/34 - 7$ | $b$ is +ve integer for $n$ = 19,49,53, 83,87,117, etc. |
| 9 | $(n^2 - 208)/34 = (n^2 - 4)/34 - 6$ | $b$ is integer when $n$ = 32,36,66,70,100,104, etc. |
| 10 | $(n^2 - 189)/34 = (n^2 + 15)/34 - 6$ | $b$ is +ve integer for $n$ = 23,45,57,79,91,113, etc. |
| 11 | $(n^2 - 168)/34 = (n^2 + 2)/34 - 5$ | $b$ is +ve integer for $n$ = 24,44,58,78,92,112, etc. |
| 12 | $(n^2 - 145)/34 = (n^2 - 9)/34 - 4$ | $b$ is +ve integer for $n$ = 31,37,65,71,99,105, etc. |
| 13 | $(n^2 - 120)/34 = (n^2 + 16)/34 - 4$ | $b$ is +ve integer for $n$ = 16, 18, 50, 52, 84, 86, etc. |
| 14 | $(n^2 - 93)/34 = (n^2 + 9)/34 - 3$ | $b$ is +ve integer for $n$ = 29,39,63,73,97,107, etc. |
| 15 | $(n^2 - 64)/34 = (n^2 + 4)/34 - 2$ | $b$ is +ve integer for $n$ = 26,42,60,76,94,110, etc. |
| 16 | $(n^2 - 33)/34 = (n^2 + 1)/34 - 1$ | $b$ is +ve integer for $n$ = 13, 21, 47, 55, 81, 89, etc. |
| 17 | $n^2 / 34$ | $b$ is integer when $n$ is an integral multiple of 34 |

etc. Thus, we have the:

**Theorem 18.1.** For $a$ = 1, there hold the identities for integral values of $n$, $b$.

| *n* | *b* | Identity | Ref. |
|---|---|---|---|
| 30 | 18 | $1^2 + 30^2 + 18^2 = 35^2$ | Th. 6.1 |
| 38 | 34 | $1^2 + 38^2 + 34^2 = 51^2$ | Th. 14.1 |
| 64 | 112 | $1^2 + 64^2 + 112^2 = 129^2$ | |
| 72 | 144 | $1^2 + 72^2 + 144^2 = 161^2$ | |
| 98 | 174 | $1^2 + 98^2 + 174^2 = 191^2$ | |
| 106 | 322 | $1^2 + 106^2 + 322^2 = 339^2$ | |

etc. //

**Theorem 18.2.** For $a = 2$, there hold the identities for integral values of $n$, $b$.

| *n* | *b* | Identity | Ref. |
|---|---|---|---|
| 25 | 10 | $2^2 + 25^2 + 10^2 = 27^2$ | Th.3.1 |
| 43 | 46 | $2^2 + 43^2 + 46^2 = 63^2$ | |
| 59 | 94 | $2^2 + 59^2 + 94^2 = 111^2$ | |
| 77 | 166 | $2^2 + 77^2 + 166^2 = 183^2$ | |
| 93 | 246 | $2^2 + 93^2 + 246^2 = 263^2$ | |
| 111 | 354 | $2^2 + 111^2 + 354^2 = 371^2$ | |

etc. //

**Theorem 18.3.** For $a = 3$, there hold the identities for integral values of $n$, $b$.

| *n* | *b* | Identity | Ref. |
|---|---|---|---|
| 22 | 6 | $3^2 + 22^2 + 6^2 = 33^2$ | Th. 2.3 |
| 46 | 54 | $3^2 + 46^2 + 54^2 = 71^2$ | |
| 56 | 84 | $3^2 + 56^2 + 84^2 = 101^2$ | |
| 80 | 180 | $3^2 + 80^2 + 180^2 = 197^2$ | |
| 90 | 230 | $3^2 + 90^2 + 230^2 = 247^2$ | |
| 114 | 374 | $3^2 + 114^2 + 374^2 = 391^2$ | |

etc. //

**Theorem 18.4.** For $a = 4$, there hold the identities for integral values of $n$, $b$.

| *n* | *b* | Identity | Ref. |
|---|---|---|---|
| 33 | 24 | $4^2 + 33^2 + 24^2 = 41^2$ | Th. 9.1 |
| 35 | 28 | $4^2 + 35^2 + 28^2 = 45^2$ | Th. 11.2 |
| 67 | 124 | $4^2 + 67^2 + 124^2 = 141^2$ | |
| 69 | 132 | $4^2 + 69^2 + 132^2 = 149^2$ | |
| 101 | 292 | $4^2 + 101^2 + 292^2 = 309^2$ | |
| 103 | 304 | $4^2 + 103^2 + 304^2 = 321^2$ | |

etc. //

**Theorem 18.5.** For $a = 5$, there hold the identities for integral values of $n$, $b$.

| *n* | *b* | Identity | Ref. |
|---|---|---|---|
| 20 | 4 | $5^2 + 20^2 + 4^2 = 21^2$ | Th. 2.4 |
| 48 | 60 | $5^2 + 48^2 + 60^2 = 77^2$ | |
| 54 | 78 | $5^2 + 54^2 + 78^2 = 95^2$ | |
| 82 | 190 | $5^2 + 82^2 + 190^2 = 207^2$ | |
| 88 | 220 | $5^2 + 88^2 + 220^2 = 237^2$ | |
| 116 | 388 | $5^2 + 116^2 + 388^2 = 405^2$ | |

etc. //

**Theorem 18.6.** For $a = 6$, there hold the identities for integral values of $n$, $b$.

| *n* | *b* | Identity | Ref. |
|---|---|---|---|
| 27 | 14 | $6^2 + 27^2 + 14^2 = 31^2$ | Th. 5.3 |
| 41 | 42 | $6^2 + 41^2 + 42^2 = 59^2$ | |
| 61 | 102 | $6^2 + 61^2 + 102^2 = 119^2$ | |
| 75 | 158 | $6^2 + 75^2 + 158^2 = 175^2$ | |
| 95 | 258 | $6^2 + 95^2 + 258^2 = 275^2$ | |
| 109 | 342 | $6^2 + 109^2 + 342^2 = 359^2$ | |

etc. //

**Theorem 18.7.** For $a = 7$, there hold the identities for integral values of $n$, $b$.

| *n* | *b* | Identity | Ref. |
|---|---|---|---|
| 28 | 16 | $7^2 + 28^2 + 16^2 = 33^2$ | Th. 6.7 |
| 40 | 40 | $7^2 + 40^2 + 40^2 = 57^2$ | |

| | | | |
|---|---|---|---|
| 62 | 106 | $7^2 + 62^2 + 106^2 = 123^2$ | |
| 74 | 154 | $7^2 + 74^2 + 154^2 = 171^2$ | |
| 96 | 264 | $7^2 + 96^2 + 264^2 = 281^2$ | |
| 108 | 336 | $7^2 + 108^2 + 336^2 = 353^2$ | |

etc. //

**Theorem 18.8.** For $a = 8$, there hold the identities for integral values of $n$, $b$.

| *n* | *b* | Identity | Ref. |
|---|---|---|---|
| 19 | 4 | $8^2 + 19^2 + 4^2 = 21^2$ | Th. 3.2 |
| 49 | 64 | $8^2 + 49^2 + 64^2 = 81^2$ | |
| 53 | 74 | $8^2 + 53^2 + 74^2 = 91^2$ | |
| 83 | 196 | $8^2 + 83^2 + 196^2 = 213^2$ | |
| 87 | 216 | $8^2 + 87^2 + 216^2 = 233^2$ | |
| 117 | 396 | $8^2 + 117^2 + 396^2 = 413^2$ | |

etc. //

**Theorem 18.9.** For $a = 9$, there hold the identities for integral values of $n$, $b$.

| *n* | *b* | Identity | Ref. |
|---|---|---|---|
| 32 | 24 | $9^2 + 32^2 + 24^2 = 41^2$ | Th. 10.3 |
| 36 | 32 | $9^2 + 36^2 + 32^2 = 49^2$ | Th. 14.9 |
| 66 | 122 | $9^2 + 66^2 + 122^2 = 139^2$ | |
| 70 | 138 | $9^2 + 70^2 + 138^2 = 155^2$ | |
| 100 | 288 | $9^2 + 100^2 + 288^2 = 305^2$ | |
| 104 | 318 | $9^2 + 104^2 + 318^2 = 335^2$ | |

etc. //

**Theorem 18.10.** For $a = 10$, there hold the identities for integral values of $n$, $b$.

| *n* | *b* | Identity | Ref. |
|---|---|---|---|
| 23 | 10 | $10^2 + 23^2 + 10^2 = 27^2$ | Th. 5.5 |
| 45 | 54 | $10^2 + 45^2 + 54^2 = 71^2$ | |
| 57 | 90 | $10^2 + 57^2 + 90^2 = 107^2$ | |
| 79 | 178 | $10^2 + 79^2 + 178^2 = 195^2$ | |
| 91 | 238 | $10^2 + 91^2 + 238^2 = 255^2$ | |
| 113 | 370 | $10^2 + 113^2 + 370^2 = 387^2$ | |

etc. //

**Theorem 18.11.** For $a = 11$, there hold the identities for integral values of $n$, $b$.

| *n* | *b* | Identity | Ref. |
|---|---|---|---|
| 24 | 12 | $11^2 + 24^2 + 12^2 = 29^2$ | § 6 |
| 44 | 52 | $11^2 + 44^2 + 52^2 = 69^2$ | |
| 58 | 94 | $11^2 + 58^2 + 94^2 = 111^2$ | |
| 78 | 174 | $11^2 + 78^2 + 174^2 = 191^2$ | |
| 92 | 244 | $11^2 + 92^2 + 244^2 = 261^2$ | |
| 112 | 364 | $11^2 + 112^2 + 364^2 = 381^2$ | |

etc. //

**Theorem 18.12.** For $a = 12$, there hold the identities for integral values of $n$, $b$.

| *n* | *b* | Identity | Ref. |
|---|---|---|---|
| 31 | 24 | $12^2 + 31^2 + 24^2 = 41^2$ | § 11 |
| 37 | 36 | $12^2 + 37^2 + 36^2 = 53^2$ | Th. 17.3 |
| 65 | 120 | $12^2 + 65^2 + 120^2 = 137^2$ | |
| 71 | 144 | $12^2 + 71^2 + 144^2 = 161^2$ | |
| 91 | 284 | $12^2 + 91^2 + 284^2 = 301^2$ | |
| 105 | 320 | $12^2 + 105^2 + 320^2 = 337^2$ | |

etc. //

**Theorem 18.13.** For $a = 13$, there hold the identities for integral values of $n$, $b$.

| *n* | *b* | Identity | Ref. |
|---|---|---|---|
| 16 | 4 | $13^2 + 16^2 + 4^2 = 21^2$ | Th. 6.4 |
| 18 | 6 | $13^2 + 18^2 + 6^2 = 23^2$ | Th. 6.5 |
| 50 | 70 | $13^2 + 50^2 + 70^2 = 87^2$ | |
| 52 | 76 | $13^2 + 52^2 + 76^2 = 93^2$ | |

| | | | |
|---|---|---|---|
| 84 | 204 | $13^2 + 84^2 + 204^2 = 221^2$ | |
| 86 | 214 | $13^2 + 86^2 + 214^2 = 231^2$ | |

etc. //

**Theorem 18.14.** For $a$ = 14, there hold the identities for integral values of $n$, $b$.

| $n$ | $b$ | Identity | Ref. |
|---|---|---|---|
| 29 | 22 | $14^2 + 29^2 + 22^2 = 39^2$ | § 11 |
| 39 | 42 | $14^2 + 39^2 + 42^2 = 59^2$ | |
| 63 | 114 | $14^2 + 63^2 + 114^2 = 131^2$ | |
| 73 | 154 | $14^2 + 73^2 + 154^2 = 171^2$ | |
| 97 | 274 | $14^2 + 97^2 + 274^2 = 291^2$ | |
| 107 | 318 | $14^2 + 107^2 + 318^2 = 335^2$ | |

etc. //

**Theorem 18.15.** For $a$ = 15, there hold the identities for integral values of $n$, $b$.

| $n$ | $b$ | Identity | Ref. |
|---|---|---|---|
| 26 | 18 | $15^2 + 26^2 + 18^2 = 35^2$ | § 10 |
| 42 | 50 | $15^2 + 42^2 + 50^2 = 67^2$ | |
| 60 | 104 | $15^2 + 60^2 + 104^2 = 121^2$ | |
| 76 | 168 | $15^2 + 76^2 + 168^2 = 185^2$ | |
| 94 | 258 | $15^2 + 94^2 + 258^2 = 275^2$ | |
| 110 | 354 | $15^2 + 110^2 + 354^2 = 371^2$ | |

etc. //

**Theorem 18.16.** For $a$ = 16, there hold the identities for integral values of $n$, $b$.

| $n$ | $b$ | Identity | Ref. |
|---|---|---|---|
| 13 | 4 | $16^2 + 13^2 + 4^2 = 21^2$ | Th. 6.4 |
| 21 | 12 | $16^2 + 21^2 + 12^2 = 29^2$ | Th. 9.3 |
| 47 | 64 | $16^2 + 47^2 + 64^2 = 81^2$ | |
| 55 | 88 | $16^2 + 55^2 + 88^2 = 105^2$ | |
| 81 | 192 | $16^2 + 81^2 + 192^2 = 209^2$ | |
| 89 | 232 | $16^2 + 89^2 + 232^2 = 249^2$ | |

etc. //

**Theorem 18.17.** For $a$ = 17, there hold the identities for integral values of $n$, $b$.

| $n$ | $b$ | Identity | Equivalently | Ref. |
|---|---|---|---|---|
| 34 | 34 | $17^2 + 34^2 + 34^2 = 51^2$ | $1^2 + 2^2 + 2^2 = 3^2$ | Th. 2.1 |
| 68 | 136 | $17^2 + 68^2 + 136^2 = 153^2$ | $1^2 + 4^2 + 8^2 = 9^2$ | ” |
| 102 | 306 | $17^2 + 102^2 + 306^2 = 323^2$ | $1^2 + 6^2 + 18^2 = 19^2$ | ” |
| 136 | 544 | $17^2 + 136^2 + 544^2 = 561^2$ | $1^2 + 8^2 + 32^2 = 33^2$ | ” |
| 170 | 850 | $17^2 + 170^2 + 850^2 = 867^2$ | $1^2 + 10^2 + 50^2 = 51^2$ | ” |
| 204 | 1224 | $17^2 + 204^2 + 1224^2 = 1241^2$ | $1^2 + 12^2 + 72^2 = 73^2$ | ” |

etc. //

## 19. Identities of complex nature

**Theorem 19.1.** Interestingly, there holds the following identity:

$$a^2 + (a + 1)^2 + \{a\,(a + 1)\}^2 = (a^2 + a + 1)^2, \quad (16.1)$$

where $a$ is any real number.

**Proof.** Left hand expression expands as

$$a^2 + (a^2 + 2a + 1) + (a^4 + 2a^3 + a^2) = a^4 + 2a^3 + 3a^2 + 2a + 1,$$

and so is the right hand expression for any real value of $a$. However, confining to only positive integral values of $a$, one may derive the following identities:

| $a$ | Identity | Ref. |
|---|---|---|
| 1 | $1^2 + 2^2 + 2^2 = 3^2$ | Th. 2.1 |
| 2 | $2^2 + 3^2 + 6^2 = 7^2$ | Th. 2.2 |
| 3 | $3^2 + 4^2 + 12^2 = 13^2$ | Th. 2.3 |
| 4 | $4^2 + 5^2 + 20^2 = 21^2$ | Th. 2.4 |
| 5 | $5^2 + 6^2 + 30^2 = 31^2$ | Th. 2.5 |
| 6 | $6^2 + 7^2 + 42^2 = 43^2$ | Th. 2.6 |

etc. //

**Theorem 19.2.** Another set of identity satisfied by some integers could be of the type:

$$a^2 + (2n)^2 + b^2 = (a + b)^2, \qquad (19.2)$$

when $b$ must be equal to $2n^2 / a$.

**Proof.** Expanding the right hand expression and cancelling the common terms on either side one easily derives the value of $b$ as above. Giving different positive integral values to $a$ and $n$, we thus derive the following identities:

**Theorem 19.3.** Choosing $a = 1 \Rightarrow b = 2n^2$, there hold the identities

| ***n*** | ***b*** | **Identity** | **Ref.** |
|---|---|---|---|
| 1 | 2 | $1^2 + 2^2 + 2^2 = 3^2$ | Th. 2.1 |
| 2 | 8 | $1^2 + 4^2 + 8^2 = 9^2$ | ” |
| 3 | 18 | $1^2 + 6^2 + 18^2 = 19^2$ | ” |
| 4 | 32 | $1^2 + 8^2 + 32^2 = 33^2$ | ” |
| 5 | 50 | $1^2 + 10^2 + 50^2 = 51^2$ | ” |
| 6 | 72 | $1^2 + 12^2 + 72^2 = 73^2$ | ” |

etc. //

**Theorem 19.4.** Choosing $a = 2 \Rightarrow b = n^2$, there hold the identities

| ***n*** | ***b*** | **Identity** | **Equivalently** | **Ref.** |
|---|---|---|---|---|
| 1 | 1 | $2^2 + 2^2 + 1^2 = 3^2$ | | Th. 2.1 |
| 2 | 4 | $2^2 + 4^2 + 4^2 = 6^2$ | $1^2 + 2^2 + 2^2 = 3^2$ | ” |
| 3 | 9 | $2^2 + 6^2 + 9^2 = 11^2$ | | Th. 3.1 |
| 4 | 16 | $2^2 + 8^2 + 16^2 = 18^2$ | $1^2 + 4^2 + 8^2 = 9^2$ | Th. 2.1 |
| 5 | 25 | $2^2 + 10^2 + 25^2 = 27^2$ | | Th. 3.1 |
| 6 | 36 | $2^2 + 12^2 + 36^2 = 38^2$ | $1^2 + 6^2 + 18^2 = 19^2$ | Th. 2.1 |

etc. //

**Theorem 19.5.** Choosing $a = 3 \Rightarrow b = 2n^2/3$, there hold the identities

| ***n*** | ***b*** | **Identity** | **Equivalently** | **Ref.** |
|---|---|---|---|---|
| 3 | 6 | $3^2 + 6^2 + 6^2 = 9^2$ | $1^2 + 2^2 + 2^2 = 3^2$ | Th. 2.1 |
| 6 | 24 | $3^2 + 12^2 + 24^2 = 27^2$ | $1^2 + 4^2 + 8^2 = 9^2$ | ” |
| 9 | 54 | $3^2 + 18^2 + 54^2 = 57^2$ | $1^2 + 6^2 + 18^2 = 19^2$ | ” |
| 12 | 96 | $3^2 + 24^2 + 96^2 = 99^2$ | $1^2 + 8^2 + 32^2 = 33^2$ | ” |
| 15 | 150 | $3^2 + 30^2 + 150^2 = 153^2$ | $1^2 + 10^2 + 50^2 = 51^2$ | ” |
| 18 | 216 | $3^2 + 36^2 + 216^2 = 219^2$ | $1^2 + 12^2 + 72^2 = 73^2$ | ” |

etc. //

Unlike to (19.2) next set of identities may be considered as

$$a^2 + (2n + 1)^2 + b^2 = (1 + b)^2, \qquad (19.3)$$

when $b$ must be equal to $a^2/2 + 2n\,(n + 1)$.

**Proof.** Expanding the right hand expression and cancelling the common terms on either side we easily derive the value of $b$ as above. In order to make $b$ whole number we choose even integral values of $a$ and give different positive integral values to $n$. We thus derive the following identities:

**Theorem 19.6.** Choosing $a = 2 \Rightarrow b = 2 + 2n\,(n + 1)$, there hold the identities

| ***n*** | ***n (n + 1)*** | ***b*** | **Identity** | **Ref.** |
|---|---|---|---|---|
| 1 | 2 | 6 | $2^2 + 3^2 + 6^2 = 7^2$ | Th. 2.2 |
| 2 | 6 | 14 | $2^2 + 5^2 + 14^2 = 15^2$ | ” |
| 3 | 12 | 26 | $2^2 + 7^2 + 26^2 = 27^2$ | ” |
| 4 | 20 | 42 | $2^2 + 9^2 + 42^2 = 43^2$ | ” |
| 5 | 30 | 62 | $2^2 + 11^2 + 62^2 = 63^2$ | ” |
| 6 | 42 | 86 | $2^2 + 13^2 + 86^2 = 87^2$ | ” |

etc. //

**Theorem 19.7.** Choosing $a = 4 \Rightarrow b = 8 + 2n\,(n + 1)$, there hold the identities

| ***n*** | ***n (n + 1)*** | ***b*** | **Identity** | **Ref.** |
|---|---|---|---|---|
| 1 | 2 | 12 | $4^2 + 3^2 + 12^2 = 13^2$ | Th. 2.3 |
| 2 | 6 | 20 | $4^2 + 5^2 + 20^2 = 21^2$ | Th. 2.4 |
| 3 | 12 | 32 | $4^2 + 7^2 + 32^2 = 33^2$ | ” |
| 4 | 20 | 48 | $4^2 + 9^2 + 48^2 = 49^2$ | ” |
| 5 | 30 | 68 | $4^2 + 11^2 + 68^2 = 69^2$ | ” |
| 6 | 42 | 92 | $4^2 + 13^2 + 92^2 = 93^2$ | ” |

etc. //

Exploring possibilities so that the identity

$$a^2 + (n + 2)^2 + b^2 = (2 + b)^2$$

holds for some integral values of $a$ and $n$, we must have

$$b = (a^2 + n^2) / 4 + n.$$

**Proof.** Expanding the terms on both sides and cancelling the common terms one easily derives the value of $b$.

**Theorem 19.8.** Choosing $a = 2 \Rightarrow b = n^2/4 + n + 1$, there hold the identities:

| ***n*** | ***b*** | **Identity** | **Equivalently** | **Ref.** |
|---|---|---|---|---|
| 2 | 4 | $2^2 + 4^2 + 4^2 = 6^2$ | $1^2 + 2^2 + 2^2 = 3^2$ | Th. 2.1 |
| 4 | 9 | $2^2 + 6^2 + 9^2 = 11^2$ | | Th. 3.1 |
| 6 | 16 | $2^2 + 8^2 + 16^2 = 18^2$ | $1^2 + 4^2 + 8^2 = 9^2$ | Th. 2.1 |
| 8 | 25 | $2^2 + 10^2 + 25^2 = 27^2$ | | Th. 3.1 |
| 10 | 36 | $2^2 + 12^2 + 36^2 = 38^2$ | $1^2 + 6^2 + 18^2 = 19^2$ | Th. 2.1 |

etc. It may be noted that these identities have also been found in Theo. 19.4. //

**Theorem 19.9.** Choosing $a = 4 \Rightarrow b = n^2/4 + n + 4$. There hold the identities:

| ***n*** | ***b*** | **Identity** | **Equivalently** | **Ref.** |
|---|---|---|---|---|
| 2 | 7 | $4^2 + 4^2 + 7^2 = 9^2$ | | Th. 3.2 |
| 4 | 12 | $4^2 + 6^2 + 12^2 = 14^2$ | $2^2 + 3^2 + 6^2 = 7^2$ | Th. 2.2 |
| 6 | 19 | $4^2 + 8^2 + 19^2 = 21^2$ | | Th. 3.2 |
| 8 | 28 | $4^2 + 10^2 + 28^2 = 30^2$ | $2^2 + 5^2 + 14^2 = 15^2$ | Th. 2.2 |
| 10 | 39 | $4^2 + 12^2 + 39^2 = 41^2$ | | Th. 3.2 |

etc. //

**Theorem 19.10.** Choosing $a = 6 \Rightarrow b = n^2/4 + n + 9$, there hold the identities:

| ***n*** | ***b*** | **Identity** | **Equivalently** | **Ref.** |
|---|---|---|---|---|
| 2 | 12 | $6^2 + 4^2 + 12^2 = 14^2$ | $2^2 + 3^2 + 6^2 = 7^2$ | Th. 2.2 |
| 4 | 17 | $6^2 + 6^2 + 17^2 = 19^2$ | | Th. 3.3 |
| 6 | 24 | $6^2 + 8^2 + 24^2 = 26^2$ | $3^2 + 4^2 + 12^2 = 13^2$ | Th. 2.3 |
| 8 | 33 | $6^2 + 10^2 + 33^2 = 35^2$ | | Th. 3.3 |
| 10 | 44 | $6^2 + 12^2 + 44^2 = 46^2$ | $3^2 + 6^2 + 22^2 = 23^2$ | Th. 2.3 |

etc. //

**Theorem 19.11.** Choosing $a = 8 \Rightarrow b = n^2/4 + n + 16$, there hold the identities:

| ***n*** | ***b*** | **Identity** | **Equivalently** | **Ref.** |
|---|---|---|---|---|
| 2 | 19 | $8^2 + 4^2 + 19^2 = 21^2$ | | Th. 3.2 |
| 4 | 24 | $8^2 + 6^2 + 24^2 = 26^2$ | $4^2 + 3^2 + 12^2 = 13^2$ | Th. 2.3 |
| 6 | 31 | $8^2 + 8^2 + 31^2 = 33^2$ | | Th. 3.4 |
| 8 | 40 | $8^2 + 10^2 + 40^2 = 42^2$ | $4^2 + 5^2 + 20^2 = 21^2$ | Th. 2.4 |
| 10 | 51 | $8^2 + 12^2 + 51^2 = 53^2$ | | Th. 3.4 |

etc. //

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## References

[1] S. Alvarez, "Note on an *n*-dimensional Pythagorean theorem", http://www.cs.bc.edu/ alvarez/NDPyt.pdf

[2] K. Czyzewska, "Generalization of the Pythagorean theorem", Demonstratio Math., **24** (nos. 1-2**)**, 1991, pp. 305 – 310.

[3] P. Oliverio, "Self-generating Pythagorean quadruples and *n*-tuples". Fibonacci Quarterly **34** (no. 2), 1996, pp. 98 – 101.