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Transfer-Function Approach To Modeling Systems With MATLAB

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***Abstract*— This article is about the study for modeling and control systems and the mathematical preliminaries necessary; first a synopsis is made of the important results of the Laplace transform, then is examined the expansion of partial fractions of a transfer function with MATLAB. Afterward we present two differential equations that model mechanical systems and are part of the usual literature in system dynamics engineering courses, we emphasize how the response of the pair of systems is obtained in an analytical and computational form, it is exhibited the complete solution of the two examples of mechanical systems where the response curves are generated. Our idea is to complement and improve the mathematical development with graphs and algebra avoided in the original source.**

Keywords: Partial fractions, Laplace transform, transfer function, Hooke's law

I. INTRODUCTION

The study of system dynamics has as its roots the transfer function, since it depends on modeling for effects of the control that is projected for purposes of performing an automation task. Mechanical, electrical, hydraulic or pneumatic systems or a combination of these can be represented with a system of ordinary differential equations, if we know the input and output of the system, we can form the transfer function. Therefore, this issue is an important part of the work in mechatronics engineering.

The main teaching literature is as follows K.Ogata book [1], which is the basis of our work is the most popular text and more commonly used. For applications focused on the use of MATLAB we have the Dukkipati [2] which can be complementary utility to Ogata, to adapt a bachelor's or master course, includes numerous worked examples. For graduate level, there is the book of Andrea-Novel and De Lara [3] with a great amount of demonstrations and mathematical developments. Finally, a book dedicated to the management of polynomials and optimization is the one of Pérez López [4] also with many exercises with MATLAB.

In research, we have the work of Sunz W. et al [5] for example, where is modeled and controlled a dc servomotor in the state space, the end proposes to design a graphical interface with MATLAB. Then we have the work of J.C. Moctezuma et al [6] where is implemented a hardware for neural network transfer functions using programmable logic and MATLAB. Another interesting work for the use of MATLAB is T. Romero et al [7] that suggests the modeling and simulation of an inverted pendulum design, useful for laboratory tests and that also is programmed with MATLAB. Therefore, this program has an important role in this research.

Our contribution is with the intention of obtaining a teaching lesson of the transfer function using the following configuration, which is organized according to their relevance: a) the solution of two mechanical systems determined by Ogata, b) the inclusion of a simple justifications set of the Laplace transform, c) to elaborate the calculations and graphs with MATLAB. In that sense our purpose is to reduce and simplify the calculations and to obtain teaching materials as an alternative to the existing literature. A tool that can be used to improve understanding of the transfer function in mechatronics engineering students.

II. LAPLACE TRANSFORM PROOF

Is originated from a defined series by [8]

$$\sum_{n=0}^{\infty} a(n)x^n = A(x). \tag{1}$$

For continuous variable is approximated to an integral

$$\int_0^{\infty} f(t)x^t dt. \tag{2}$$

X is defines as

$$x = e^{\ln x}. \tag{3}$$

By completing the power is had

$$x^t = \left[e^{\ln x}\right]^t. \tag{4}$$

This, to converge is required that $x<1$ then $0<x<1$

Whilst variable s is defined

$$-s = \ln x. \tag{5}$$

Therefore, the integral is the definition of the Laplace transform

$$\int_0^{\infty} f(t)e^{-st}dt = F(s) \quad . \tag{6}$$

The notation in this work is

$$L\left[f(t)\right] = \int_0^{\infty} f(t)e^{-st}dt \tag{7}$$

Also, it has to be considerate the inverse transform

$$L\left[f(t)\right] = F(s)$$
$$L^{-1}\left[F(s)\right] = f(t). \tag{8}$$

A. Important transforms

Next, the most usual functions in system dynamics are proven

The Laplace transform of the unit impulse function is

$$\int_0^{\infty} \delta(t)e^{-st}dt = 1. \tag{9}$$

To determine the Laplace transform of an exponential function, only consists in solving the following integral

$$\int_0^{\infty} e^{-\alpha t}e^{-st}dt = \int_0^{\infty} e^{-(\alpha+s)t}dt \,. \tag{10}$$

$$= \frac{-1}{(\alpha+s)} e^{-(\alpha+s)t}\Big|_0^{\infty} = \frac{1}{\alpha+s} \quad . \tag{11}$$

Therefore, the transform is

$$L\left\{e^{\alpha t}\right\} = \frac{1}{\alpha+s} \,. \tag{12}$$

Another commonly used function in mechanical, electrical, pneumatic, and hydraulic system is the unit step function, provided that it is defined as constant

$$f(t) = A \quad . \tag{13}$$

Its integral and transform is

$$\int_0^{\infty} Ae^{-st}dt = A\int_0^{\infty} e^{-st}dt \,. \tag{14}$$

$$L\{A\} = \frac{1}{s}. \tag{15}$$

In the same way, the ramp function and its transform are

$$f(t) = At \,. \tag{16}$$

$$\int_0^{\infty} Ate^{-st}dt = A\int_0^{\infty} te^{-st}dt \,. \tag{17}$$

It can be visualized that it involves an integral by parts

$$\int udv = uv - \int vdu \tag{18}$$

$$\begin{aligned} u &= t \qquad & v &= -\frac{1}{s}e^{-st} \\ du &= dt & dv &= e^{-st}dt. \end{aligned} \tag{19}$$

By solving and evaluating it can be easily obtained the Laplace transform

$$-\frac{tA}{s}e^{-st}\Big|_0^{\infty} + \int_0^{\infty} \frac{A}{s}e^{-st}dt$$
$$L\{At\} = \frac{A}{s^2}. \tag{20}$$

To generate the transform of sine and cosine functions can be a little more sophisticated, since the calculation is very similar. It is only showed for the sine function, using Euler's identity

$$e^{i\omega t} = \cos(\omega t) + isen(\omega t)$$
$$e^{-i\omega t} = \cos(\omega t) - isen(\omega t) \,. \tag{21}$$

Which is presented next for including in the integral, defining the Laplace transform

$$sen(\omega t) = \frac{1}{2i}\left[e^{i\omega t} - e^{-i\omega t}\right] \tag{22}$$

$$\frac{A}{2i}\int_0^{\infty}\left[e^{i\omega t} - e^{-\omega t}\right]e^{-st}dt = F(s). \tag{23}$$

By solving and making algebra is obtained that

$$= \frac{A}{2i}\frac{1}{(s-i\omega)} - \frac{A}{2i}\frac{1}{(s+i\omega)}$$
$$L\{A\sin(\omega t)\} = \frac{A\omega}{s^2+\omega^2}. \tag{24}$$

A further important transform is the function multiplied by an exponential or the also named passage theorem [9]

$$L\left[e^{at}f(t)\right] = \int_o^{\infty} e^{at}f(t)e^{-st}dt = F(s+a). \tag{25}$$

$$L\left[e^{at}f(t)\right] = \int_0^{\infty} e^{at}e^{-st}f(t)dt \tag{26}$$

$$L\left[e^{at}f(t)\right] = \int_0^{\infty} e^{-(s-a)t}f(t)dt \tag{27}$$

$$L\left[e^{at}f(t)\right] = F(s+a) \tag{28}$$

However, in the construction of a transference function it is required manipulating ordinary differential equations. Thus, is demanded a simple proof of the first and second derivative and the integral of a function. So, first is necessary to prove that a derivative function has that

$$L\left[\frac{d}{dt}f(t)\right] = sF(s) - f(0) \;. \tag{29}$$

Now is leveraged a result of the integral by parts

$$\int_0^{\infty} f(t)e^{-st}dt = F(s). \tag{30}$$

$$\begin{aligned} u &= f(t) \qquad & v &= -\frac{1}{s}e^{-st} \\ du &= \frac{d}{dt}f(t) & dv &= e^{-st}dt. \end{aligned} \tag{31}$$

The presence of the derivative function is identified, so

$$f(t)\frac{e^{-st}}{-s}\bigg|_0^{\infty}-\int_0^{\infty}\left[\frac{d}{dt}f(t)\right]\frac{e^{-st}}{-s}dt \;. \tag{32}$$

The equation is attained (29)

$$F(s)=\frac{f(0)}{s}+\frac{1}{s}L\left[\frac{d}{dt}f(t)\right]. \tag{33}$$

In the same way is proceeded for the second derivative of a function

$$L\left[\frac{d^2}{dt^2}f(t)\right]=s^2F(s)-sf(0)-f(0). \tag{34}$$

A variable change convenes

$$\frac{d}{dt}f(t)=g(t)\;. \tag{35}$$

Then, is verified that

$$L\left[\frac{d^2}{dt^2}f(t)\right]=L\left[\frac{d}{dt}g(t)\right]. \tag{36}$$

Using the result of the first derivative, the result can be quickly achieved

$$=sL\left[g(t)\right]-g(0)\quad . \tag{37}$$

$$=sL\left[\frac{d}{dt}f(t)\right]-f(0)$$

$$L\left[\frac{d^2}{d^2t}f(t)\right]=s^2F(s)-sf(0)-f(0). \tag{38}$$

The transform of an integral is useful in electric systems, its justification is as the following

$$L\left[\int f(t)dt\right]=\int_0^{\infty}\left[\int f(t)\,dt\right]e^{-st}dt\,. \tag{39}$$

Other integral by parts

$$u=\int f(t)dt \qquad v=-e^{-st}\frac{1}{s}.$$

$$du=f(t)\,dt \qquad dv=e^{-st}dt \tag{40}$$

By replacing and identifying terms

$$=\left[\int f(t)\,dt\right]\frac{e^{-st}}{s}\bigg|_0^{\infty}+\int_0^{\infty}f(t)\frac{e^{-st}}{s}dt\,. \tag{41}$$

The transform definition of the second term can be observed, also is identified that in the first term when evaluating in infinite it becomes zero, and when evaluating zero is obtained

$$L\left[\int f(t)dt\right]=\frac{F(s)}{s}+\frac{f^{-1}(0)}{s}. \tag{42}$$

B. Transfer function

The transfer function of a time invariant ordinary differential equations linear system, is defined as the ratio of the Laplace transform of output between the Laplace transform of input, assuming that the initial conditions are zero.

$$TF=G(s)=\frac{\text{Laplace-tranformed output}}{\text{Laplace-transformed input}}.$$

In general, are polynomials such that

$$\frac{X(s)}{Y(s)}=\frac{b_0s^n+b_1s^{n-1}+\cdots+b_{m-1}s+b_m}{a_os^n+a_1s^{n-1}+\cdots+a_{n-1}s+a_n}. \tag{43}$$

C. Partial fractions with MATLAB

Further, the example of a transfer function developing in partial fractions

$$\frac{Y(s)}{U(s)}=\frac{25}{s^2+4s+25}. \tag{44}$$

The instructions are

$$num=[25]$$

$$den=[1..4...25]$$

$$residue(num,den)=[r,p,k]$$

The previous generates an equation so that

$$\frac{B(s)}{A(s)}=k(s)+\frac{r(1)}{s-p(1)}+\frac{r(2)}{s-p(2)}+....+\frac{r(n)}{s-p(n)}. \tag{45}$$

For this example, there is the next result

$$\frac{Y(s)}{U(s)}=\frac{2.727i}{s-(-2-4.58i)}-\frac{2.727i}{s-(-2+4.58i)}. \tag{46}$$

Completing the corresponding operations, is obtained the next equation (44)

Another example is the next transfer function with higher grades polynomials

$$\frac{B(s)}{A(s)}=\frac{s^4+8s^3+16s^2+9s+6}{s^3+6s^2+11s+6}. \tag{47}$$

Corresponding to the next expansion in partial fractions

$$\frac{B(s)}{A(s)}=s+2-\frac{6}{s+3}-\frac{4}{s+2}+\frac{3}{s+1}. \tag{48}$$

With function in the time domain

$$f(t)=1+2\delta(t)-6e^{-3t}-4e^{-2t}+3e^{-t}\,. \tag{49}$$

D. First example of a mechanical system

Described by the following differential equation

$$m\frac{d^2y}{dt^2}=-b\left[\frac{dy}{dt}-\frac{du}{dt}\right]-k(y-u). \tag{50}$$

With the next datum

$$m=10kg \qquad b=20N-\frac{s}{m} \qquad k=100N/m$$

The differential equation can be described as

$$m\frac{d^2y}{dt^2}+b\frac{dy}{dt}+ky=b\frac{du}{dt}+ku. \tag{51}$$

By effecting the Laplace transform

$$\left(ms^2+bs+k\right)Y(s)=(bs+k)U(s). \quad (52)$$

The input and output are identified and generates the transfer function

$$TU=\frac{Y(s)}{U(s)}=\frac{bs+k}{ms^2+bs+k}. \quad (53)$$

By substituting the respective values

$$\frac{Y(s)}{U(s)}=\frac{20s+100}{10s^2+20s+100}=\frac{2s+10}{s^2+2s+10}. \quad (54)$$

Considering the unit step function

$$U(s)=\frac{1}{s}. \quad (55)$$

$$Y(s)=\frac{2s+10}{s^3+2s^2+10s}. \quad (56)$$

Promptly, applying MATLAB commands is created

$$Y(s)=\frac{-0.5-0.1667i}{s+1-3i}+\frac{-0.5+0.1667i}{s+1+3i}-\frac{1}{s}. \quad (57)$$

By developing algebra is obtained

$$Y(s)=\frac{1}{s}-\frac{s}{(s+1)^2+3^2}. \quad (58)$$

By using passage theorem, is rewritten as

$$Y(s)=\frac{1}{s}-\frac{s-1+1}{(s+1)^2+3^2}. \quad (59)$$

By separating the numerator is attained

$$Y(s)=\frac{1}{s}-\frac{s+1}{(s+1)^2+3^2}+\frac{1}{3}\frac{3}{(s+1)^2+3^2}. \quad (60)$$

Is identified the possibility for using the passage theorem and can be obtained the function in the time domain

$$y(t)=1-e^{-t}\cos(3t)+\frac{1}{3}sen(3t). \quad (61)$$

Whose graph is shown in Fig. 1

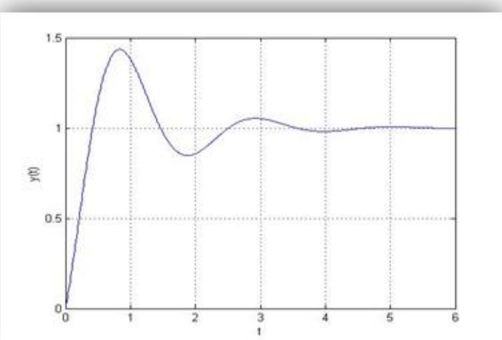

Fig 1. *Mechanical system of the first example*

E. Second example of a mechanical system

Represented by the next pair of differential equations

$$mx''+k_1x+k_2x+k_2(x-y)=p(t). \quad (62)$$

$$k_2(x-y)=b_2y'. \quad (63)$$

With the constants given by

$$b_2=0.4N\frac{s}{m} \qquad k_1=6N/m \qquad k_2=4N/m$$

$$m=0.1kg \qquad p(t)=10N$$

By executing path Laplace transforms

$$\left(ms^2+k_1+k_2\right)X(s)=k_2Y(s)+P(s). \quad (64)$$

$$k_2X(s)=(k_2+b_2s)Y(s). \quad (65)$$

Solving for one and substituting in the other

$$\left[\left(ms^2+k_1+k_2\right)(k_2+b_2s)-k_2\right]X(s)=(k_2+b_2s)P(s). \quad (66)$$

The transfer function is attained

$$\frac{X(s)}{P(s)}=\frac{b_2s+k_2}{mb_2s^3+mk_2s^2+(k_1+k_2)b_2s+k_1k_2}. \quad (67)$$

By substituting the values and executing the corresponding algebra is proven that

$$X(s)=\frac{-1.369(s+1.2898)-3.9743}{(s+1.2898)^2+8.899^2}-\frac{0.2977}{s+7.420}+\frac{1.6667}{s}. \quad (68)$$

With the function in the time domain

$$x(t)=-1.369e^{-1.2898t}\cos(8.899t)-0.446e^{-1.2898t}sen(8.899t)-0.2977e^{-7.420t}+1.6667. \quad (69)$$

With the following graph of Fig. 2

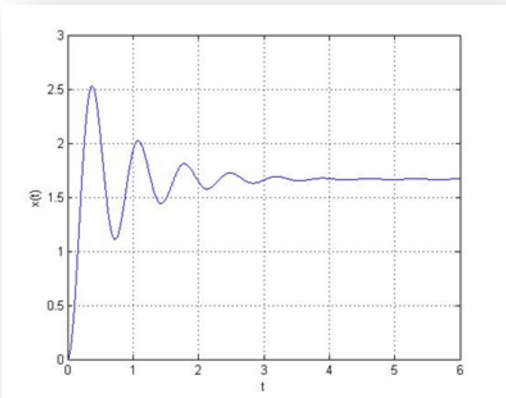

Fig 2. *Mechanical system of the second example*

III RESULTS DISCUSSION

We have reproduced the calculations of Ogata in the explanation of the transfer function, however, we want to emphasize the following: first, is right to include the transform origin, as are the equations (1) to (8). Books usually omit such reasoning in the explanation of the Laplace transform, it was included for a good start of the subject. Likewise, the equation integral (9) which is an impulse function, is not well discussed in Ogata's dissertation and must be assumed that its clear and precise use is of favor. On the other hand, for the Laplace transform of an integral function, must be emphasized in the evaluation of the results of the integral by parts involucrate. Other key progress is conceiving the equation (44) from (46), that work must be an essential exercise. Similarly, generate from the expression of the equation (57) to (58) merit longer algebraic develop that clarify the understanding for the mathematical tool. Idem, the equation expression (68). Lastly, should be clarified that technical details of the determined systems are omitted to exclusively focus on the exercise's mathematics.

IV CONCLUSION

The study of the model and control of mechatronics systems requires great abilities in effecting the Laplace transform y constructing transfer functions, also demands certain qualification in ordinary differential equations. Our job is about reconsidering the lessons in Ogata and discern in the steps that merit the mathematical development. Our job has validity in the didactic field of the subject and also in result of the importance that acquire the use of software, such as MATLAB in the study objective of this important domain of the mechatronic engineering. The next challenge is to extend it to electric, hydraulic and pneumatic systems in just one lesson. We believe that would be a great contribution to improve the teaching and learning in system dynamics.

REFERENCES

[1] K. Ogata, System Dynamics, fourth edition, Edit. Prentice Hall, USA, pp 112-119, 2004.

[2] R. V.Dukkipati, Analysis and Design of Control Systems using MATLAB, Edit. New Age International, New Delhi pp 72-112, 2006.

[3] B. Andrea-Novel and M De Lara, Control Theory of Engineering. Edit Springer pp 233-238, 2013.

[4] Cesar Pérez Lopez, MATLAB Optimization Techniques, Edit. Springer pp 217-222, 2010.

[5] W. Sunz et al, Modelación y control en espacio de estados de un servomecanismo usando MATLAB. Revista ingeniería UC, Vol.8, No.1, pp 1-21, 2001.

[6] J.C. Moctezuma, Implementación de hardware de funciones de transferencia para redes neuronales artificiales. XII workshop, Costa Rica, 2006.

[7] T. Romero, Modelado, simulación y control de un péndulo invertido usando componentes análogos, 8 Congreso iberoamericano de ingeniería mecánica, 2007.

[8] Carlos Zelada, transformada de Laplace, www.academatica.com.

[9] S. Campbell and R. Haberman, Introducción a las ecuaciones diferenciales, Edit. McGraw Hill, PP. 227- 242, 1998.