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# Ancient Egyptian Quadrature Executed Using A Set Square

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***Abstract*— In ancient Egypt approximation of the area of a circle by a square based on diameter of the circle. It's not know how this method of squaring the circle was obtained. A geometrical construction of Egyptian quadrature needs division of the diameter into 9 equal parts. In this paper a set square is proposed to perform this quadrature without this requirement.**

***Keywords: Ahmes, Bing, circle, number pi, Egypt, square, polygon, set square, squaring, onscribed polygon***

## I. INTRODUCTION

In ancient Egypt the area of a circle was calculated according to the method recorded on the papyrus by a scribe Ahmes. The Ahmes (Rhind) papyrus was written in hieratic, and probably originated from the Middle Kingdom: 2000-1800 BC. There are a few mathematical problems in the papyrus related to the circle and its area. Our concern here is with the problem 50 in the Ahmes papyrus. The problem states: "Shorten the diameter of the circle by 1/9 to get the side of the square". This rather simple recipe results in the following relations for the area A of the circle with the radius r and diameter d, Thus we have the following relation: $A = \pi r^2 = \pi (\frac{d}{2})^2 \approx (8\frac{d}{9})^2 = \left(d - \frac{1}{9}d\right)^2$. This squaring of the circle is realized with the approximation of $\pi \approx 3.160493827160494$ ….… It's astonishing accuracy with error only 0.0189.

We don't know exactly how the mathematicians in Egypt developed such approach to realize the approximate quadrature of the circle. Engels in his paper suggested that the Egyptian mathematicians in their method based on orthogonal nets [1]. Engels stated: "If one attempts to draw a circle and a square intersecting this circle and having equal area, then nearly everybody intuitively gives a solution something like Figure 1" [1]. According to Engels' words this technique seems to be fundamental to develop the Egyptian construction of the square for a giving circle.

The geometrical construction of the ancient Egyptian quadrature requires to divide a given diameter of the circle into 9 equal parts. This task needs some knowledge on proportion of the segments (Thales' theorem) [2]. Is it possible that Egyptian mathematician used another approach or a specific tool? In this paper we proposed and developed a set square. The tool allows us easily perform the task: squaring the circle. Of course it's only an approximate solution.

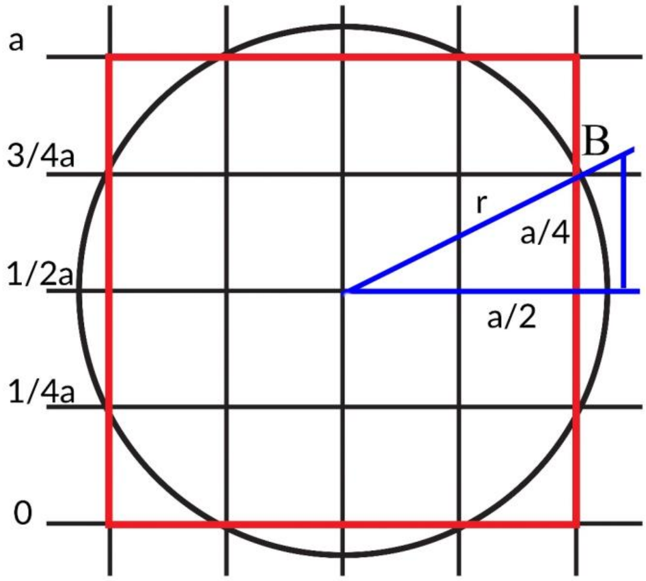


Figure 1. Two different points of view. Engels [1]: The circle on orthogonal nets. Szyszkowicz [4]: The onscribed square on the circle.

## II. METHOD

The author of this paper defined and introduced a class of regular polygons related to the circle [3, 4]. They are called the onscribed polygons [4]. The onscribed polygons are located between the circumscribed and inscribed polygons in the circle with their center in the circle center. Their main property is that exactly one half of their perimeter is in the circle and another half is out of the circle. Such constructed polygons realize minimum distance from the circle. In this case, the distance is defined as an area of the symmetric difference (Δ) of two figures; here a circle and regular polygon. For example, for a unit circle and a square placed on the circle in such way, that it cuts its circumference in 8 points and these points determine regular octagon, the distance is $2 - \sqrt{2} = 0.58578643762 \ldots$ . The square proposed by Engels and illustrated on Figure 1 is the onscribed square. Its side is $\frac{4}{\sqrt{5}}$ and the area is 3.2 (for a unit circle). The distance to the unit circle is $3.2 - \pi = 0.0584073464102071 \ldots$ and this distance is minimal in sense of the used definition.

Engels divided the square presented on Figure 1 into 256 (16 by 16) sub-squares (Figure 3 in Engels' paper [1]). He observed that$a = \left(\frac{8}{9}\right) d$, while $a = (2/\sqrt{5})$d is correct. The area of the circle passing through B (Figure 1) is $\pi(\sqrt{5}/4a)^2 \approx 0.9817a^2 < a^2$. Using 8/9 instead $2/\sqrt{5}$ improves the result and gives more accurate value for the area of the circle $(9a/16)^2 \approx 0.9940a^2$.

The method described by Engels has a few issues. It's relatively easy, for a given orthogonal nets, to determine the corresponding circle and the square (see Figure 1). This circle is related to this specific nets. Usually, we have rather invers problem: for a given circle we have to determine the square (net-lines). As was already mentioned the onscribed square realizes minimum of the distance. Consider two situations presented on Figure 2. Let z determines half of the size of the square side. We consider two squares S with z and z + h and the circle C. Their corresponding differences between their distances to the circle are: $Area\ (S_{z+h}\Delta C) - Area(S_z\Delta C) = 8qh - 8ph = 8h(q-p) < 0$ and similarly $Area\ (S_{z+h}\Delta C) - Area(S_z\Delta C) = 8qh - 8ph = 8h(q-p) > 0$. Thus the equality holds for p=q, [3]. The symbol Δ was used here to indicate symmetrical difference operation for two figures; a circle and square.

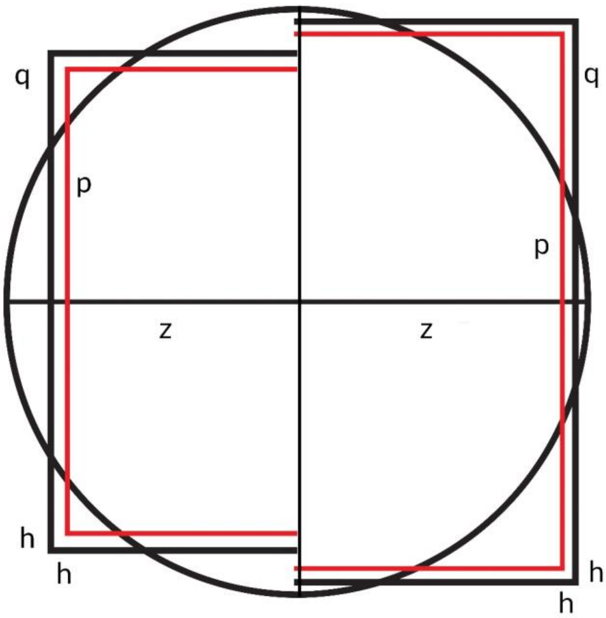


Figure 2. Two situations with p and q; p (inside of the circle, small z) and q (outside of the circle, large z). Left: q<p, right: q>p.

## III. SET SQUARE

In addition, Figure 1 also shows a set square (a blue triangle) with its sides in ratio 2:1. This simple tool allows to find the onscribed square for any circle without an orthogonal nets. Thus the triangle performs the squaring of the circle with $\pi = 3.2$. Of course, we can improve this tool to obtain better accuracy. Consider the set square with the corresponding sides in ratio 9:8 as it was proposed by Ahmes (Figure 3, left panel).

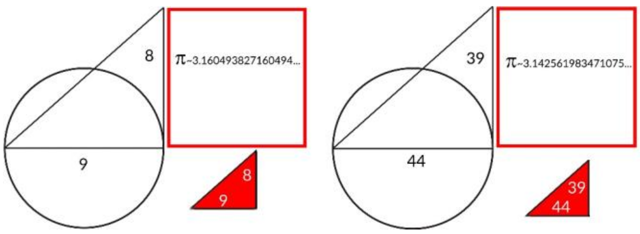


Figure 3. Two Szyszkowicz's set squares with different accuracy. For the sides in ratio 9642: 8545, $\pi$ =3.141592642401758… .

Thus now the triangle allows us to execute Ahmes' quadrature for a given circle. Using the relation between the scaled diameter and the area of the circle, we can determine the corresponding exact ratio as $\frac{\sqrt{\pi}}{2} = 0.886226925452758\ldots$.(It is derived from the following relations: $\pi r^2 = \pi(d/2)^2 = (xd)^2, x = \frac{\sqrt{\pi}}{2}$, [2]). We can determine the best rational approximation to this irrational and transcendental number. The program in R language presented here allows to find these rational approximations. The program uses continued fractions to determine the best approximations of x [2]. Figure 3 (right panel) shows the squaring of the circle with the set square of the side 44:39. The set square with the ratio 9642: 8545 results in the approximation to the number $\pi$ equals 3.141592642401758… .

## IV. CONLUSIONS

It's possible that the mathematicians in ancient Egypt used a right triangle with sides in the proportion 9:8. Such device allows to find the corresponding square. Another and similar approach to construct a square set to perform the quadrature of the circle was proposed by a Russian engineer Edward Bing [5]. The idea of using a set square to squaring the circle according Ahmes' approach was developed by the author of this paper. The author also proposed another quadrature [6, 7]. In addition, we can determine the corresponding angle in the proposed set square. The angle 41.5482262125792 is an approximation as its tangent should be √π/2. In practice, we don't need to determine this angle, if the goal is to build the set square in wood, made in plastic or meal.

Listing of the program in R. The program generates nine (n=9) terms of the continued fraction.

```
#Rational approximation to sqrt(pi)/2.

# Author: M. Szyszkowicz

library(contfrac); options(digits=15)

x=sqrt(pi)*0.5; print(x) #0.886226925452758

tana=as_cf(x, n = 9)
```

```
print(tana)  #Continued fraction

#x=[0; 1, 7, 1, 3, 1, 2, 1, 57]

frac = convergents(tana)

print(frac$A)

#Numerator: 0, 1, 7, 8, 31, 39, 109, 148, 8545

print(frac$B)

#Denominator:1, 1, 8, 9, 35, 44, 123, 167, 9642

for (k in 1:9){

print(frac$A[k]/frac$B[k])  } # end of loop

# The results of rational approximation to

# x= Numerator /Denominator.

# (0, 1, 0.875, 0.888888888888889,
0.885714285714286, 0.886363636363636,

# 0.886178861788618, 0.88622754491018,
0.886226923874715)

# Tangent of the angle 41.5482262125792 (in degree)
= 0.886226925452758 ##The end##
```

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