# Dynamical Systems, Phase Plan Analysis-Linearization Method

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***Abstract*—Linear systems of differential equations are of interest because they play a crucial role in the classification of the fixed points of nonlinear systems. Most problems which arise in the real world are nonlinear, and in most cases nonlinear systems cannot be solved. One method to find approximate solution for the nonlinear systems is linearization method. Since most models reduce down to two or more such equations, and since only two variables can easily be drawn, we concentrate more on a system of two equations. The objective of this paper is to briefly describe the fundamental definitions of the differential equations systems about fixed points, stability, classification of fixed points, phase portrait plane, as well as some applications of those systems to population dynamics. Phase plane analysis is one of the most important techniques for studying the behaviour of nonlinear systems, since there is usually no analytical solution for a nonlinear system. Another approach shown in this paper is that of the geometric one which leads to the qualitative understanding of the behaviour, instead of detailed quantitative information. Consequently, the analysis of the nonlinear differential equations is much understandable than before.**

**Keywords — dynamical systems, fixed points, stability, linearization, phase plane**

## I. INTRODUCTION

A lot of phenomena of different fields of science and technology are modeled mathematically by linear and nonlinear differential equations. It is well-known that second-order linear differential equations can be solved analytically mainly those of constant coefficient, while the nonlinear differential equations can be solved analytically only in rare cases. For these systems we will see the linearization method. The aim of this paper is to learn how to interpret qualitatively the solutions of the equations of differential systems, without passing through the analytic process of finding the solutions. This is achieved by combing the analytic method and the geometric intuition. The object of this study is dynamical systems.

A dynamical system is any mathematical model which describes the state of a system in time.

For example, mathematical models which describe the oscillation of the mathematical pendulum, the flow of the water in the tube, the number of population in a metropolis, etc are dynamical systems.

In this paper we will treat two linear and nonlinear dimensional systems. Things have changed dramatically during the last three decades. We can easily find computers everywhere, and a lot of software packets are at our disposal. They can be used to approximate the solutions of the differential equations and see the results graphically. As a result, the analysis of the nonlinear differential equations is airily understood.

## II. GENERAL KNOWLEDGE

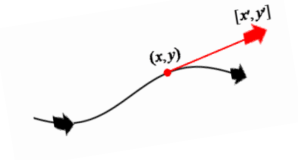
Let us investigate the dynamical system in the plane,

$$\begin{cases} x' = f(x,y) \\ y' = g(x,y) \end{cases} \tag{1}$$

We know the solution of the system (1), in the time interval $T$, is any pair of the functions $(x(t), y(t))$ for which the system equations (1) are transformed into identities $T$. [1], [2]

*The plot of any solution $(x(t), y(t))$ in the system $Oxy$ is called the trajectory of the system (1).*

At any point $(x,y)$ of the area $\omega \subset \Re^2$, where the functions $f$ and $g$ are determined, the system (1) defines a direction, set by the vector $[x', y'] = [f(x,y), g(x,y)]$.




Thus, in the area $\omega \subset \Re^2$, the system (1) defines a direction field (vectors).

*So, a trajectory of the system (1) is any line at w, which has the quality: at any point of its, whose direction of the tangent fits the direction field.*

Plane $\Re^2$ is the phase space of the system (1).

*Fixed Points*

*Definition.* The point $(x^*, y^*)$ is called a fixed point (equilibrium point) of the system (1) if

$$\begin{cases} f(x^*, y^*) = 0 \\ g(x^*, y^*) = 0 \end{cases}$$

Any fixed point $(x^*, y^*)$ is a 'trajectory'. If the phase point $(x, y)$ is found in a moment in the fixed point $(x^*, y^*)$ then it is left all the time at that point.[1]

## III. Linear Systems

The linear dynamical system in the plane is modelled by the linear differential system [1], [4],[5]

$$\begin{cases} x' = ax + by \\ y' = cx + dy \end{cases} \tag{2}$$

where $a$, $b$, $c$, and $d$ are real parameters.

Example. It is given the system

$$\begin{cases} x' = \alpha x \\ y' = -y \end{cases} \tag{3}$$

Plot the phase portrait when $\alpha$ moves from $-\infty$ to $+\infty$.

*Solution*

For $\alpha \neq 0$, the system (2) has only one fixed point, which is the origin of the coordinates $O(0,0)$, while for $\alpha = 0$, any point of the $Ox$-axis's a fixed point.

The differential equations of the system are independent from each other, thus each equation can be solved separately. From their solution we have

$$x(t) = c_1 e^{\alpha t}, \quad y(t) = c_2 e^{-t}$$

The pair $\left(x(t), y(t)\right) = \left(x_0 e^{\alpha t}, y_0 e^{-t}\right)$ is the trajectory of the system (3) which passes through the arbitrary point $(x_0, y_0)$ at the initial point $t = 0$. The phase portraits for different values of the parameter $\alpha$ are shown in Figure 1.

The function $y(t)$ decays exponentially to zero when $t \to +\infty$, and tends to $\pm\infty$ when $t \to -\infty$.

While for the function $x(t)$ we distinguish some cases.

- When $\alpha < 0$, the function $x(t)$ decays exponentially, so all the trajectories $\left(x_0 e^{\alpha t}, y_0 e^{-t}\right)$ approach the origin of the coordinates $t \to +\infty$, move away indefinitely when $t \to -\infty$.
- When $\alpha > 0$, the function $x(t)$ changes exponentially tending to $\pm\infty$ (depending on the sign of $y_0$) when $t \to +\infty$, and tend to zero when $t \to -\infty$.

Thus, all the trajectories $\left(x_0 e^{\alpha t}, y_0 e^{-t}\right)$ approach the origin of the coordinates when $t \to +\infty$ and veer away endlessly when $t \to -\infty$. We should now show whether these trajectories are concave or convex. For this we only study the sign of the second derivative:

$$\frac{d^2 y}{dx^2} = \frac{y''(t)x'(t) - y'(t)x''(t)}{\left[x'(t)\right]^3} =$$

$$= \frac{(y_0 e^{-t})(\alpha x_0 e^{\alpha t}) - (-y_0 e^{-t})(\alpha^2 x_0 e^{\alpha t})}{\left[\alpha x_0 e^{\alpha t}\right]^3} = \frac{y_0(1+\alpha)e^{-t}}{\alpha^2 x_0^2 e^{2\alpha t}}$$

Notice that

$$\boxed{\operatorname{sgn} \frac{d^2 y}{dx^2} = \operatorname{sgn}(\alpha + 1) y_0}$$

which means

(1) for $y_0 > 0$ we have:

$$\frac{d^2 y}{dx^2}: \begin{cases} < 0, & for \quad \alpha < -1; \\ > 0, & for \quad -1 < \alpha < 0; \\ > 0, & for \quad \alpha > 0. \end{cases}$$

(2) for $y_0 < 0$ we have:

$$\frac{d^2 y}{dx^2}: \begin{cases} > 0, & for \quad \alpha < -1; \\ < 0, & for \quad -1 < \alpha < 0; \\ < 0, & for \quad \alpha > 0. \end{cases}$$

This is the way the phase portrait is explained in Figure 1, in the cases (a), (c) and (e).

- In the case $\alpha = 0$, the trajectories $\left(x_0 e^{\alpha t}, y_0 e^{-t}\right)$ have the form $\left(x_0, y_0 e^{-t}\right)$ which means they are perpendicular half lines $x = x_0$ (Figure 1/d), where $-\infty < x_0 < +\infty$.
- In the case $\alpha = -1$, the trajectories $\left(x_0 e^{\alpha t}, y_0 e^{-t}\right)$ have the form $\left(x_0 e^{-t}, y_0 e^{-t}\right)$. For

$x_0 = 0$, the trajectory is $(0, y_0 e^{-t})$ which for $y_0 > 0$ shows half $Oy^+$ - axis, and for $y_0 < 0$ shows $Oy^-$. For $x_0 \neq 0$, the trajectories $(x_0 e^{-t}, y_0 e^{-t})$ show the line of the form $y = kx$, where $k = y_0 / x_0$ (Figure 1/b). [1],[5]

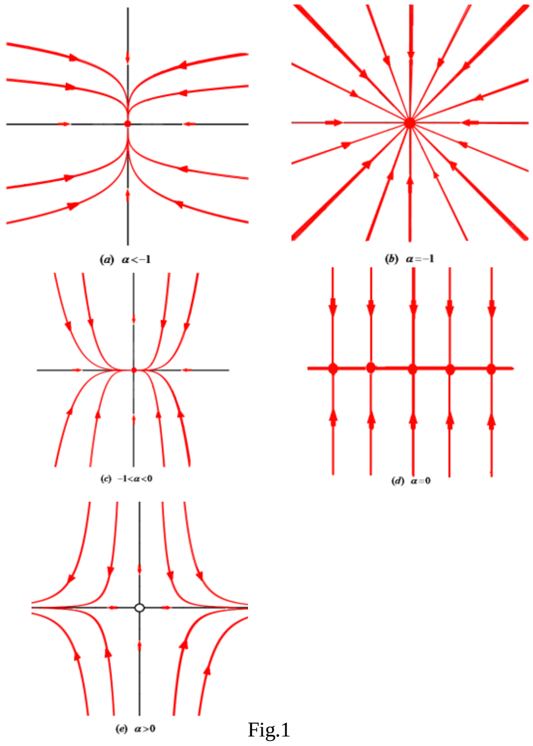



Fig.1

*Comments and explanations*

In the case(a), when $t \to +\infty$, the trajectories approach the origin $O(0;0)$, tangentially with the $Ox$-axis. On the other side, if we look *backwards*, which means $t \to -\infty$ ,the trajectories, moving away endlessly from $O(0;0)$, tend to become parallel to the $Ox$-axis.

In the case (c), the behaviour of the trajectories is the same with that of the case of (a), with the change that the place of the $Ox$-axis in this behaviour is substituted by the $Oy$-axis. In the cases(a) *and* (c)the fixed point $O(0;0)$ *is called' a stable node'.* In the case(b)the fixed point $O(0;0)$ *is called 'a symmetrical node' or "star".* In the cases (a), (b) and (c), the fixed point $O(0;0)$ is called *"an attracting point "or "a suction point ".*

*All the trajectories that come out of the point around the origin $O(0;0)$ ,approach it when $t \to +\infty$. In fact, the point $O(0;0)$ attracts all the trajectories of the phase plane, that is why this point is called globally attracting.*

In the case (d), i.e in the case when $\alpha = 0$, *there is an entire line of nonisolated fixed points in the $Ox$ - axis.* Every trajectory approaches these fixed points along vertical lines. In the case (e), most trajectories veer away endlessly from $O(0;0)$; there is an exception only for the trajectories which start on the points of the $Oy$-axis , so the fixed point $O(0;0)$ is unstable.

In the case (e), the fixed point $O(0;0)$ is called a saddle point and the $Ox$-axis is called an unstable manifold.

## IV. CLASSIFICATION OF THE TRAJECTORIES OF THE LINEAR SYSTEMS

Let us investigate the linear dynamical system

$$\begin{cases} x' = ax + by \\ y' = cx + dy \end{cases} \tag{3}$$

The origin $O(0;0)$ is a fixed point of (3), whatever the coefficients are $a$, $b$, $c$, $d$. But when

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = 0,$$

apart from the point $O$, the system (3) has a set of infinite other fixed points.

Let us see how the system (1) is solved using the matrix calculus. Let us write (3) in the form of differential matrix equation

$$X' = AX \tag{3'}$$

where

$$X = \begin{bmatrix} x \\ y \end{bmatrix}, \quad X' = \begin{bmatrix} x' \\ y' \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

The solution $(x(t), y(t))$ can be now written in the form of a vector or column matrix $X(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$. We search the solutions of the equation (3') in the form

$$X(t) = e^{\lambda t} V, \tag{4}$$

where $\lambda$ is a real or complex constant, and $V = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ a nonzero vector (column matrix), which does not depend on the time $t$. We can now find $\lambda$ and $V$. For this we substitute $X(t) = e^{\lambda t} V$ in (3') :

$$\left[e^{\lambda t}V\right]' = Ae^{\lambda t}V \Leftrightarrow \lambda e^{\lambda t}V = Ae^{\lambda t}V \Leftrightarrow \lambda V = AV$$

$$\Leftrightarrow (A-\lambda I)V = 0 \qquad (5)$$

where the matrix $I$ is a unit matrix.
From the algebra course we know that the values of $\lambda$ for which, the equation (5) has the solution $V \neq O$ are called eigenvalues, while the responsive solutions $V_\lambda$ are called eigenvectors.
In expanded form, the equation (5) shows the homogeneous with two unknowns. $\begin{cases}(a-\lambda)v_1 + bv_1 = 0 \\ cv_1 + (d-\lambda)v_2 = 0\end{cases}$

From the algebra course we know that the system (5) has a solution different from the zero one only when its determinant is equal to zero:

$$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0$$

based on which we have

$$\lambda^2 - (a+d)\lambda + (ad-bc) = 0 \qquad (6)$$

and it is called the characteristic equation of the $A$ matrix.
We write the equation (6) shortly in the form

$$\lambda^2 - \tau\lambda + \Delta = 0 \qquad (6')$$

where
$\tau = a+d$ (the trace of the $A$ matrix),
$\Delta = ad-bc$ (the determinant of $A$ matrix).[1]
The eigenvalues are the roots of the equation (6'):

$$\lambda_1 = \frac{\tau + \sqrt{\tau^2 - 4\Delta}}{2} \qquad \lambda_2 = \frac{\tau - \sqrt{\tau^2 - 4\Delta}}{2}$$

This way, after determining the eigenvalues and the eigenvectors, we find the trajectories of the system (3), as well as we study their behaviour around the fixed points.
It is obviously clear that the behaviour of the trajectories is defined by the numbers $\tau$ and $\Delta$; consequently even the numbers $\lambda_1$ and $\lambda_2$.

*The rules of reading the behaviour around the origin*

If the values $\tau$ and $\Delta$ are such that the point $(\tau, \Delta)$ is found in:

(1)the open area and restrained by the parabola $\Delta = \tau^2/4$ $O\tau^+$-axis (the first quadrant), then $O(0;0)$ is a *unstable node*;

(2) the open area, restrained by the parabola $\Delta = \tau^2/4$ and the $O\tau^-$-axis (the second quadrant), then $O(0;0)$ is a *stable node*;

(3) the open area, restrained by the parabola $\Delta = \tau^2/4$ and the $O\Delta^+$ - axis(the first quadrant), then $O(0;0)$ is *an unstable spiral*;

(4) the open area, restrained by the parabola $\Delta = \tau^2/4$ and the $O\Delta^-$ - axis (the second quadrant), then $O(0;0)$ is a *centre spiral*;

(5)the open area below the $O\tau$ - axis (the second and the fourth quadrants), then $O(0;0)$ is a *saddle* of one of its two forms;

(6)the parabola $\Delta = \tau^2/4$ (the first quadrant), then $O(0;0)$ is an *unstable star, when* $\lambda_1 = \lambda_2 = a = d > 0$ and $b = c = 0$ ,or *unstable node when* $\lambda_1 = \lambda_2 > 0$ and $b^2 + c^2 > 0$;

(7) the parabola $\Delta = \tau^2/4$ (the second quadrant), then $O(0;0)$ is a stable star when $\lambda_1 = \lambda_2 = a = d < 0$ *and* $b = c = 0$, or a stable node when $\lambda_1 = \lambda_2 < 0$ and $b^2 + c^2 > 0$;

(8) $O\Delta^+$ - axis, then the system $O(0;0)$ is a *centre*.

(9) $O\tau$ - axis i.e. $\Delta = 0$, then the system (3) has an infinite set of fixed points, nonisolated from each other, which fill a spoiling line, or one of the axes of the coordinates, where the trajectories behave towards as in (Figure 1/d).

All the bahaviours of the trajectories of the dynamical system (3) around the fixed point $O(0;0)$ as well as their classification, are summarized in the diagram of Figure2. The presentation of all this information in the plane $\tau\Delta$ gives us a visual summary of all different linear systems.
There are certain issues to be taken into consideration.
Firstly, the plane $\tau\Delta$ is a two- dimensional representative of that which is a four-dimensional space in reality, since the $2\times2$ matrices are determined by four parameters, the coefficients of the matrix. Thus there is an infinite number of different matrices which correspond to any point in the plan $\tau\Delta$. Although all these matrices have the same configuration of the eigenvalues, there may be delicate distinctions in the phase portraits, such as

centres and spirals, or the possibility of one or two independent eigenvectors in the case of repetitive eigenvectors.

We also think of the plane $\tau\Delta$ as an analogue of the diagram of bifurcation for the plane linear systems.

The parabola $\tau^2 - 4\Delta = 0$ of the phase portrait submits to a bifurcation: A huge change happens in the geometry of the phase portrait.

In conclusion, we notice that we can obtain a lot of information from $\Delta$ and $\tau$ concerning the system, without taking into account the eigenvalues. For example, if $\Delta < 0$, we know we have a saddle point.

If we make substitution in the system (7) $x = x^* + u$, $y = y^* + v$

It has the form

$$\begin{cases}(x^* + u)' = f(x^* + u,\ y^* + v) \\ (y^* + v)' = g(x^* + u,\ y^* + v)\end{cases} \tag{9}$$

Since $x^*$ and $y^*$ are constant, their derivatives are zero, so that the system (3) has the form

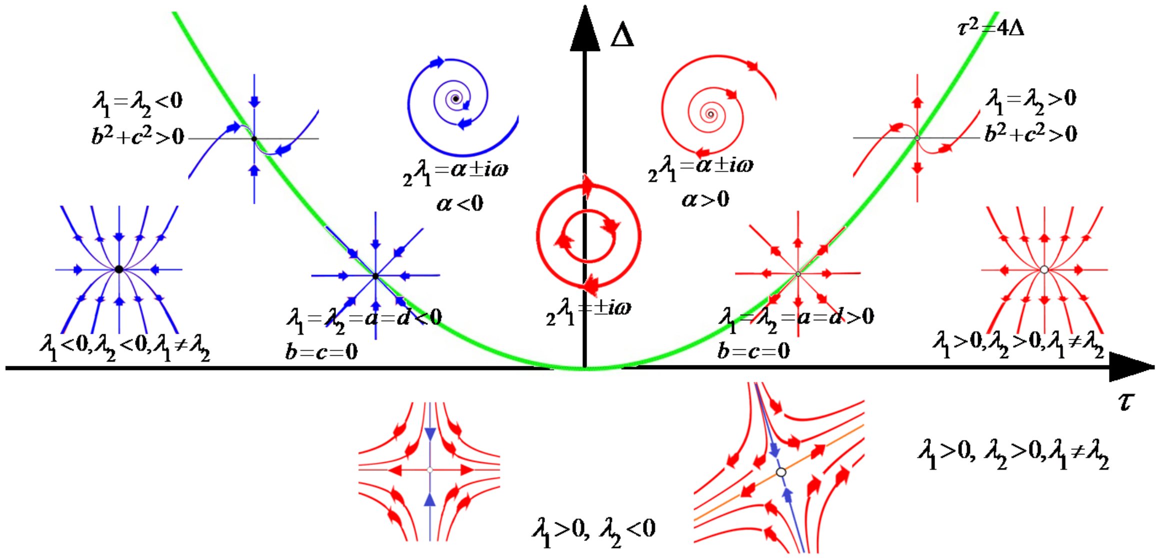



Fig. 2

## V. Nonlinear Dynamical Systems and the Method of Linearization

Consider the nonlinear dynamical system

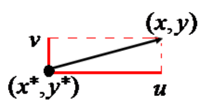
$$\begin{cases}x' = f(x,y) \\ y' = g(x,y)\end{cases} \tag{7}$$

Let $(x^*, y^*)$ be its fixed point, i.e. it satisfies the conditions

$$f(x^*, y^*) = 0 \quad \text{dhe} \quad g(x^*, y^*) = 0 \tag{8}$$

Let

$$u = x - x^*$$
$$v = y - y^*$$




be the components of a small disturbance from the fixed point $(x^*, y^*)$ to the point $(x, y)$. [1]

$$\begin{cases}u' = f(x^* + u,\ y^* + v) \\ v' = g(x^* + u,\ y^* + v)\end{cases} \tag{10}$$

If we expand the functions $f(x^* + u,\ y^* + v)$ and $g(x^* + u,\ y^* + v)$ in the Taylor series with the centre at the point $(x^*, y^*)$, the system (10) takes the form:

$$\begin{cases}u' = f(x^*,y^*) + \dfrac{\partial f}{\partial x}(x^*,y^*)\cdot u + \dfrac{\partial f}{\partial y}(x^*,y^*)\cdot v + O(u^2) + O(v^2) + O(uv) \\ v' = g(x^*,y^*) + \dfrac{\partial g}{\partial x}(x^*,y^*)\cdot u + \dfrac{\partial g}{\partial y}(x^*,y^*)\cdot v + O(u^2) + O(v^2) + O(uv)\end{cases}$$

where $O(u^2)$, $O(v^2)$, $O(uv)$ are the sums of the terms of the Taylor series of the same order of the quantity respectively $u^2$, $v^2$, $uv$.

Since $f(x^*, y^*) = 0$ and $g(x^*, y^*) = 0$ (see (8)), the above system takes the form

$$\begin{cases} u' = \dfrac{\partial f}{\partial x}(x^*, y^*)\cdot u + \dfrac{\partial f}{\partial y}(x^*, y^*)\cdot v + O(u^2) + O(v^2) + O(uv) \\ v' = \dfrac{\partial g}{\partial x}(x^*, y^*)\cdot u + \dfrac{\partial g}{\partial y}(x^*, y^*)\cdot v + O(u^2) + O(v^2) + O(uv) \end{cases} \quad (11)$$

In order to understand what happens to a trajectory which starts in a point $(x, y)$ around the fixed point $(x^*, y^*)$ ,we take $x$ extremely nearby $x^*$ and $y$ extremely nearby $y^*$ , which means $u$ and $v$ are extremely small.

Since $u$ and $v$ are extremely small, the terms $O(u^2)$ , $O(v^2)$ and $O(uv)$ don't have to be considered (are neglected), so the system (11) can be approximated to the linear dynamical system [1], [5]

$$\begin{cases} u' = \dfrac{\partial f(x^*, y^*)}{\partial x}u + \dfrac{\partial f(x^*, y^*)}{\partial y}v \\ v' = \dfrac{\partial g(x^*, y^*)}{\partial x}u + \dfrac{\partial g(x^*, y^*)}{\partial y}v \end{cases} \quad (12)$$

which has $u$ and $v$ as its dynamical terms.

The matrix

$$J(x^*, y^*) = \begin{bmatrix} \dfrac{\partial f(x^*, y^*)}{\partial x} & \dfrac{\partial f(x^*, y^*)}{\partial y} \\ \dfrac{\partial g(x^*, y^*)}{\partial x} & \dfrac{\partial g(x^*, y^*)}{\partial y} \end{bmatrix}$$

is called a *Jacobian matrix* in the fixed point $(x^*, y^*)$ .

*The essence of the linearization method*

*In order to study the behaviour of the trajectories of the nonlinear system (7) nearby the fixed point* $(x^*, y^*)$ *, it is fairly enough to study the behaviour of the trajectories of the linear system (12) around its fixed point* $(0;0)$ .

*The classification of the fixed point* $(0;0)$ *the system (12)is done by the Jacobian matrix* $J(x^*, y^*)$ .

*The behaviour which the trajectories of the system (12) have towards the point* $(0;0)$ *is the same as the behaviour of the trajectories of the dynamical system (7) have towards the point* $(x^*, y^*)$ .

## VI. APPLICATION

Consider the competitive model between two species without overcrowding (Lotka- Volterra) [2]

Fig. 3

Let's denote the population of the two species $x(t)$ and $y(t)$ respectively. Remember $x(t)$ shows the population of the present prey at time $t$ and $y(t)$ shows the population of the predator at time $t$ . Suppose both $x(t)$ and $y(t)$ are nonnegative.

One system of differential equations which can model the changes in the population of these two species is

$$\frac{dx}{dt} = x' = 2x - xy$$
$$\frac{dy}{dt} = y' = -y + xy$$

The term $2x$ in the equation for $\frac{dx}{dt}$ represents the exponential growth of the prey in the absence of the predators, and the term $-xy$ corresponds to the negative effect over the prey of the interaction predator- prey. The term $-y$ in $\frac{dy}{dt}$ corresponds to the assumption that the predators die out if there is no prey to eat, and the term $xy$ corresponds to the pozitive effect over the predators of the interaction predator-prey.

The coefficients 2,-1,-1 and 1 depend on the included species.

We solve this system by taking $\begin{cases} x' = 2x - xy = 0 \\ y' = -y + xy = 0 \end{cases}$ and find the fixed points $(x_1^*, y_1^*) = (0,0)$ and $(x_2^*, y_2^*) = (1,2)$ .The fixed point $(x_1^*, y_1^*) = (0,0)$ has a perfect meaning -if the two populations predator and prey extinct, we surely do not expect the population to grow at anylater time.

The other solution $(x_2^*, y_2^*) = (1, 2)$, means that, if the population of the prey is 1and the population of the predator is 2, the system is in perfect equilibrum. There is enough prey to support a constant population of predators of 2, at the same time there aren't neither many predators (which would cause the extinction of the population of the prey) nor less (in which case the number of prey would increase).

The birth rate of each specie is exactly the same as the its death rate, and these populations are kept at an indefinite time. The system is in *equilibrum*. If $x = 0$, the first equation in this system vanishes. That is why the function $x(t) = 0$ satisfies this differential equation without considering what initial condition we have chosen for $y$. In this case the second differential equation is reduced to

$$\frac{dy}{dt} = -y$$

which we know as an exponential model of the extinction for the population of the predators.From this equation we know that the population of the predators tends to zero in an exponential manner.

This entire scenario for $x = 0$ is reasonable, because if there is no prey for some time, then there will never be any prey despite the number of the predators there may be. Furthermore, without food supply the predators would die out.

In the same way, notice that the equation for $\frac{dy}{dt}$ vanishes if $y = 0$, and the equation for $\frac{dx}{dt}$ reduces to

$$\frac{dx}{dt} = 2x$$

which is a model of exponential growth. This means that any nonzero prey population grows without bound under these assumptions. Once more, these conclusions make sense because there are no predators to control the growth of the prey population.

In order to understand all the solutions of this predator-prey system

$$\frac{dx}{dt} = x' = 2x - xy$$

$$\frac{dy}{dt} = y' = -y + xy$$

it is important to note that the change of either population depends both on $x(t)$ and on $y(t)$.

The phase portrait for this system for a special solution is [2], [3], [6]

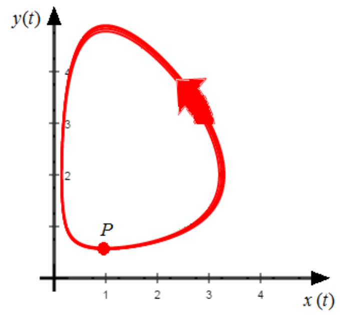



Fig. 4

It is often helpful to view a solution curve for a system of differential equations not merely as a set of points in the plane but, rather in a more dynamic way, as a point following a curve which is determined by the solution of the differential equation.

In Figure 4 we show a special solutionwhich starts at the point $P$. As $t$ increases, $x(t)$ increases at first, while $y(t)$ stays relatively constant. Near $x = 3.3$, the solution curve turns significantly upward. Thus the predator population $y(t)$ starts increasing significantly. As $y(t)$ nears $y = 2$, the curve starts heading to the left. Thus $x(t)$ has reached a maximum and has started to decrease. As $t$ increases, the values of $x(t)$ and $y(t)$ change as indicated by the shape of the solution curve. Eventually the solution curve returns to its starting point $P$ and begins its cycle again.

We can plot simultaneously a lot of solution curves on the phase plane using Maple Software.

In Figure 5 we see the complete phase plane for our predator- prey system. Nevertheless, we restrict our attention to the first quadrant since it does not make sense to talk about the negative populations. [3]

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>with(DEtools); phaseportrait([(D(x))(t) = x(t)*(2-y(t)), (D(y))(t) = (x(t)-1)*y(t)], [x(t), y(t)], t = -10 .. 10, [[x(0) = 1, y(0) = 2], [x(0) = 2, y(0) = 1], [x(0) = 4, y(0) = 1], [x(0) = 1.1, y(0) = 3.1],[x(0) = 2.1, y(0) = 4.1],[x(0) = 2.19, y(0) = 4.19],[x(0) = 2.199, y(0) = 9.199]], x = 0 .. 5, y = 0 .. 5, stepsize = 0.5e-1, linecolour = blue, arrows = SLIM, thickness = 2);
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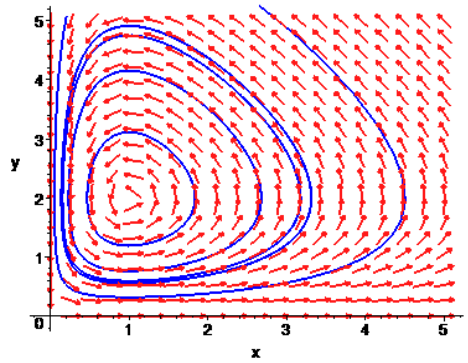

Fig. 5

In this predator-prey system, all other solutions for which $x_0 > 0$ and $y_0 > 0$ yeild other curves that loop around the

equilibrum point $(x_2^*, y_2^*) = (1,2)$, in a counterclockwise manner.

In the end, they return to their initial points, and hence this model forecasts that apart from the equilibrum solution, both

$x(t)$ and $y(t)$ rise and fall in a periodic manner.

If we use the linearization method to classify the fixed points, we calculate the Jacobian matrix $J$ :

$$J = \begin{pmatrix} 2-y & -x \\ y & -1+x \end{pmatrix}$$

For the fixed point $(x_1^*, y_1^*) = (0,0)$ we have

$$J = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \;,\quad \Delta = \begin{vmatrix} 2 & 0 \\ 0 & -1 \end{vmatrix} = -2 < 0 \;,\quad \tau = -1+2 = 1 \;,$$

$\tau^2 - 4\Delta = 9 > 0$ , $\lambda_1 = -1,\ \lambda_2 = 2$

This fixed point is a saddle node.

For the fixed point $(x_2^*, y_2^*) = (1,2)$ we have

$$J = \begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix} ,\quad \Delta = \begin{vmatrix} 0 & -1 \\ 2 & 0 \end{vmatrix} = 2 > 0 \;,\; \tau = 0 \;,\; \tau^2 - 4\Delta < 0$$

${}_2\lambda_1 = \pm i\sqrt{2}$

This fixed point is a centre.

## IV. Conclusions

There are general formulas for solutions of linear systems. These general formulas include all solutions. Unfortunately, for nonlinear systems no such general formulas exist. This means that it is very difficult, or not even possible, to establish properties of solutions.

Nonlinear systems can be investigated with qualitative methods. In two- dimensional systems, the analysis of fixed points and linear approximation near the fixed points in general allows to understand the system. The linearization technique is used to find approximate solutions for the nonlinear systems, but is it important to emphasize that linearization is valid only in a neighborhood of the fixed point. In nonlinear systems, stability near a fixed point is dependent on initial conditions, and only in specific cases local analysis also give insight in the global behaviour of the model, that is, the model dynamics far away from the fixed point.

The study of the competition among the living species through dynamical systems is the best method ever existing. This influences us on the change of the behaviour towards them in maintaining the equilibrium and the biological ecosystems created during the long process of natural evolution, realized from the competition within species and among species.

Nowadays pace of development of the computer graphics enables the numerical graphical presentation of the phase portrait of the two – dimensional nonlinear dynamical system.

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