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# Uniqueness of weak solutions of non-homogeneous A-harmonic equations with very weak boundary values

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***Abstract*—The uniqueness of weak solutions of non-homogeneous A-harmonic equations with very weak boundary values is obtained.**

***Keywords—uniqueness; weak solution; elliptic equation.***

## I. INTRODUCTION

The elliptic equation has a strong background, and has many applications in physics and engineering. For the recent developments of weak solutions of elliptic equations, we refer the reader to [1-3]. The aim of this present paper is to obtain the uniqueness of weak solutions of non-homogeneous A-harmonic equation with very weak boundary values. Our result is a generalization of the reference [4].

In this present paper, we consider the following non-homogeneous A-harmonic equation

$$\operatorname{div} A(x,\nabla u)=\operatorname{div} F(x,u) \qquad (1.1)$$

where $A:\Omega\times\mathbf{R}^n\to\mathbf{R}^n$ satisfies the conditions

(H1) $\langle A(x,\xi),\xi\rangle\ge\alpha|\xi|^p$,

(H2) $|A(x,\xi_1)-A(x,\xi_2)|\le\beta(|\xi_1|+|\xi_2|)^{p-2}|\xi_1-\xi_2|$,

(H3) $\langle A(x,\xi_1)-A(x,\xi_2),\xi_1-\xi_2\rangle>0$ whenever $\xi_1\ne\xi_2$.

for almost every $x\in\Omega$ and all $\xi,\xi_1,\xi_2\in\mathbf{R}^n$, $\alpha,\beta>0$ are constants, $1<p<n$ and $F(x,u)\in(L^{\tau}_{loc}(\Omega))^n$,

$$\tau\le\begin{cases}\dfrac{\lambda}{1+\lambda(p-2)/r}, & p>2\\ \lambda, & 1<p\le 2\end{cases}\quad \text{where } \lambda=\min\{s,\frac{nq}{n+q}\}.$$

Throughout this paper, $\Omega$ will denote bounded open set in $\mathbf{R}^n$, and $E\subset\partial\Omega$ is a closed set and small in an appropriate capacity sense. In order to avoid some technical difficulties related to the imbedding theorem we shall illustrate our approach only for $p$ smaller than the spatial dimension of $\Omega$.

The prototype of equation (1.1) is the homogeneous A-harmonic equation

$$\operatorname{div} A(x,\nabla u)=0. \qquad (1.2)$$

When the mapping $A(x,\xi)=|\xi|^{p-2}\xi$, (1.2) generates the $p$ - harmonic equation

$$\operatorname{div}|\nabla u|^{p-2}\nabla u=0 \qquad (1.3)$$

which satisfied the assumptions (H1)-(H3).

**Definition 1.1** A function $u\in W^{1,p}_{loc}(\Omega)$ is called a weak solution of A-harmonic equation (1.1) if $u$ satisfies

$$\int_\Omega\langle A(x,\nabla u),\nabla\varphi\rangle\,\mathrm{d}x=\int_\Omega\langle F(x,u),\nabla\varphi\rangle\,\mathrm{d}x \qquad (1.4)$$

for all $\varphi\in C_0^\infty(\Omega)$.

The solutions belong to the local Sobolev space $u\in W^{1,p}_{loc}(\Omega)$ and we prove a uniqueness result for solutions provided that they belong to $u\in W^{1,r}_{loc}(\Omega)$ for $r<p$ and that they take the same boundary values in $\partial\Omega\setminus E$ where $E\subset\partial\Omega$ is a closed set and small in an appropriate capacity sense.

In order to formulate our theorem we need to consider local boundary values. Let $F\subset\partial\Omega$ and $u\in W^{1,p}_{loc}(\Omega)$ .We say that $u$ has zero boundary values at $F$ in the $W^{1,p}$-sense, abbreviated $u\in W^{1,p}_{loc}(\Omega;F)$, if each $x\in F$ has a neighborhood $U$ and a function $\eta\in C_0^\infty(U)$ such that $\eta=1$ in some neighborhood of $x$ and $\eta u\in W_0^{1,p}(\Omega)$.

Suppose that $\theta\in W^{1,p}_{loc}(E)$ .We say that $u$ has the boundary values $\theta$ at $F$ in the $W^{1,p}$ -senses if $u-\theta\in W^{1,p}_{loc}(\Omega;F)$ and if for each $x\in F$ there exists $\eta$ as above with $\eta\theta\in W^{1,p}(\Omega)$. Note that then also $\eta u$ belongs to $W^{1,p}(\Omega)$.

If $u$ has the boundary values $\theta$ at $F$ in the $W^{1,p}$ -sense, then $u$ has the boundary values $\theta$ at a neighborhood of $F$ . Hence we may always assume that $F$ is open relative to $\partial\Omega$.

For our main result we assume that the numbers $p\in(1,n)$, $q\in(1,\infty)$, $s\in(1,\infty)$ and $r\geq\max\{1,p-1\}$ satisfy

$$t=\frac{sq}{sq-s-q-\frac{sq(p-2)^{+}}{r}}>1. \tag{1.5}$$

Here $(p-2)^{+}=p-2$ if $p\geq 2$ and 0 otherwise.

**Note**. If $t>n$, then $cap_t E=0$ implies $E=\varnothing$. Hence only the values $t\leq n$ are of interest. Let $\mathscr{H}^{s}$ denote the $s$-dimensional Hausdorff measure. It is well known that $\mathscr{H}^{n-t}(E)<\infty$ implies $cap_t E=0$.

**Theorem 1.2** Suppose that $\theta\in W^{1,p}_{loc}(\Omega)$ and that $u_1,u_2$ are weak solutions of non-homogeneous A-harmonic equation (1.1) such that

(i) $u_1$ and $u_2$ have boundary values $\theta$ in the $W^{1,p}$-sense at $\partial\Omega\setminus E$;

(ii) $u_1-u_2\in L^{q}(\Omega)$;

(iii) $\nabla u_1-\nabla u_2\in L^{s}(\Omega)$;

(iv) $\nabla u_1,\nabla u_2\in L^{r}(\Omega)$ if $p>2$.

If $cap_t E=0$, then $u_1=u_2$ in $\Omega$.

**Note**. Let $\operatorname{div}F(x)=0$, then our main results is Theorem 1.1 in [4].

## II. Proof of Theorem 1.2

Let $u_1,u_2$ be weak solutions of non-homogeneous A-harmonic equation (1.1), $\Omega$ be a bounded open set. Condition (i) in Theorem 1.2 implies that each $y\in\partial\Omega\setminus E$ has a neighborhood $U=U(y)$ such that

$$u_i\in W^{1,p}(U\cap\Omega),\quad i=1,2. \tag{2.1}$$

Thus for each neighborhood $V$ of $E$ we have

$$u_i\in W^{1,p}(\Omega\setminus\overline{V}),\quad i=1,2. \tag{2.2}$$

Fix a ball $B\subset\subset\Omega$. Since $cap_t(E)=0$, we can choose an open set $D\subset\mathbf{R}^n$ such that $E\subset D$; $B\subset\mathbf{R}^n\setminus D$ and $cap_t(E;D)=0$. Here $cap_t(E;D)=0$ refers to the usual variational $t$-capacity of the condenser $(E;D)$ [2, Chapter 2]. Given $\varepsilon>0$ we can find an open set $U_\varepsilon$ and $\xi\in C_0^{\infty}(D)$ such that $E\subset U_\varepsilon\subset\subset D$, $0\leq\xi\leq 1$, $\xi=1$ on $\overline{U_\varepsilon}$ and

$$\int_D|\nabla\xi|^t dx<\varepsilon^t.$$

Hence

$$\|\nabla\xi\|_t<\varepsilon.$$

Write $\eta=1-\xi$. Then $\eta=0$ in $\overline{U_\varepsilon}$, $\eta=1$ in $\mathbf{R}^n\setminus D$, $0\leq\eta\leq 1$, $\eta\in C^{\infty}(\mathbf{R}^n)$, and

$$\|\nabla\eta\|_t<\varepsilon. \tag{2.3}$$

The above inequality has important role in the following proof. Note that here we get $\eta=1$ in $B$.

Let $W_\varepsilon$ be a neighborhood of $E$ with $E\subset\subset W_\varepsilon\subset\subset U_\varepsilon$. Let

$$\varphi=\eta(u_1-u_2).$$

Since (3.2) holds and $u_1-u_2\in W_0^{1,p}(\Omega;\partial\Omega\setminus E)$ by Condition (i) in Theorem 1.2, then $\varphi\in W_0^{1,p}(\Omega)$, and the support of $\varphi$ stays away from $E$. Thus we can use $\varphi$ as a test function in Definition 1.1, i.e.,

$$\begin{aligned}&\int_{\Omega\setminus\overline{W_\varepsilon}}\left\langle A(x,\nabla u_i),\nabla\left(\eta(u_1-u_2)\right)\right\rangle \mathrm{d}x\\ &=\int_{\Omega\setminus\overline{W_\varepsilon}}\left\langle F(x,u_i),\nabla\left(\eta(u_1-u_2)\right)\right\rangle \mathrm{d}x,\quad i=1,2.\end{aligned}$$

Hence

$$\begin{aligned}&\int_{\Omega}\left\langle A(x,\nabla u_1)-A(x,\nabla u_2),\nabla\left(\eta(u_1-u_2)\right)\right\rangle \mathrm{d}x\\ &=\int_{\Omega}\left\langle F(x,u_1)-F(x,u_2),\nabla\left(\eta(u_1-u_2)\right)\right\rangle \mathrm{d}x.\end{aligned}$$

here we have used $\eta=0$ in $\overline{W_\varepsilon}$. Thus using the condition (H2) we obtain from the above formula

$$\begin{aligned}I&=\int_{\Omega}\eta\left\langle A(x,\nabla u_1)-A(x,\nabla u_2),\nabla(u_1-u_2)\right\rangle \mathrm{d}x\\ &=-\int_{\Omega}(u_1-u_2)\left\langle A(x,\nabla u_1)-A(x,\nabla u_2),\nabla\eta\right\rangle \mathrm{d}x\\ &\quad+\int_{\Omega}\left\langle F(x,u_1)-F(x,u_2),\nabla\left(\eta(u_1-u_2)\right)\right\rangle \mathrm{d}x\\ &\leq\int_{\Omega}|A(x,\nabla u_1)-A(x,\nabla u_2)||u_1-u_2||\nabla\eta|\,\mathrm{d}x\\ &\quad+\int_{\Omega}|F(x,u_1)-F(x,u_2)||u_1-u_2||\nabla\eta|\,\mathrm{d}x\\ &\quad+\int_{\Omega}|F(x,u_1)-F(x,u_2)||\nabla(u_1-u_2)|\,\mathrm{d}x\\ &\leq\beta\int_{\Omega}\left(|\nabla u_1|+|\nabla u_2|\right)^{p-2}|\nabla u_1-\nabla u_2||u_1-u_2||\nabla\eta|\,\mathrm{d}x\\ &\quad+\int_{\Omega}|F(x,u_1)-F(x,u_2)||u_1-u_2||\nabla\eta|\,\mathrm{d}x\\ &\quad+\int_{\Omega}|F(x,u_1)-F(x,u_2)||\nabla(u_1-u_2)||\eta|\,\mathrm{d}x\\ &=I_1+I_2+I_3\end{aligned} \tag{2.4}$$

**Case 1:** $p>2$. Using the Hölder inequality with

$$\frac{p-2}{r}+\frac{1}{s}+\frac{1}{q}+\frac{1}{t}=1,$$

we have

$$\begin{aligned}I_1&=\beta\int_{\Omega}(|\nabla u_1|+|\nabla u_2|)^{p-2}|\nabla u_1-\nabla u_2||u_1-u_2||\nabla\eta|\,\mathrm{d}x\\ &\leq\beta\left(\int_{\Omega}(|\nabla u_1|+|\nabla u_2|)^{r}\mathrm{d}x\right)^{\frac{p-2}{r}}\left(\int_{\Omega}|\nabla u_1-\nabla u_2|^{s}\mathrm{d}x\right)^{\frac{1}{s}}\\ &\quad\cdot\left(\int_{\Omega}|u_1-u_2|^{q}\mathrm{d}x\right)^{\frac{1}{q}}\left(\int_{\Omega}|\nabla\eta|^{t}\mathrm{d}x\right)^{\frac{1}{t}}.\end{aligned}$$

By (2.3) and the conditions (ii)-(iv) in Theorem 1.2 we have

$$I_1 \le C\varepsilon. \tag{2.5}$$

Similarly, using the Hölder inequality we have

$$\begin{aligned} I_2 &= \int_\Omega |F(x,u_1)-F(x,u_2)||u_1-u_2||\nabla\eta|\,\mathrm{d}x \\ &\le \left(\int_\Omega |F(x,u_1)-F(x,u_2)|^{\frac{s}{1+s(p-2)/r}}\mathrm{d}x\right)^{\frac{p-2}{r}+\frac{1}{s}} \\ &\cdot\left(\int_\Omega |u_1-u_2|^q \mathrm{d}x\right)^{\frac{1}{q}}\left(\int_\Omega |\nabla\eta|^t \mathrm{d}x\right)^{\frac{1}{t}} \end{aligned}$$

Noticing that $F(x,u)\in (L^\tau_{loc}(\Omega))^n$ and $\tau \le \dfrac{s}{1+s(p-2)/r}$ for $p>2$, by (2.3) and the conditions (ii) in Theorem 1.2 we have

$$I_2 \le C\varepsilon. \tag{2.6}$$

Similarly, Using the Hölder inequality with

$$\left(\frac{p-2}{r}+\frac{1}{n}+\frac{1}{q}\right)+\frac{1}{s}+\left(\frac{1}{t}-\frac{1}{n}\right)=1,$$

and poincaré inequality[5] we have

$$\begin{aligned} I_3 &= \int_\Omega |F(x,u_1)-F(x,u_2)||\nabla(u_1-u_2)||\eta|\,\mathrm{d}x \\ &\le \left(\int_\Omega |F(x,u_1)-F(x,u_2)|^{\frac{nq}{n+q+nq(p-2)/r}}\mathrm{d}x\right)^{\frac{p-2}{r}+\frac{1}{n}+\frac{1}{q}} \\ &\cdot\left(\int_\Omega |\nabla u_1-\nabla u_2|^s \mathrm{d}x\right)^{\frac{1}{s}}\left(\int_\Omega |\nabla\eta|^{\frac{nt}{n-t}} \mathrm{d}x\right)^{\frac{n-t}{nt}} \\ &\le \left(\int_\Omega |F(x,u_1)-F(x,u_2)|^{\frac{nqr}{nr+qr+nq(p-2)}}\mathrm{d}x\right)^{\frac{p-2}{r}+\frac{1}{n}+\frac{1}{q}} \\ &\cdot\left(\int_\Omega |\nabla u_1-\nabla u_2|^s \mathrm{d}x\right)^{\frac{1}{s}}\left(\int_\Omega |\nabla\eta|^t \mathrm{d}x\right)^{\frac{1}{t}} \end{aligned}$$

Noticing that $F(x,u)\in (L^\tau_{loc}(\Omega))^n$ and $\tau \le \dfrac{nq}{n+q+nq(p-2)/r}$ for $p>2$, by (2.3) and the conditions (iii) in Theorem 1.2 we have

$$I_3 \le C\varepsilon. \tag{2.7}$$

Thus by (2.4)-(2.7) we have

$$I \le C\varepsilon. \tag{2.8}$$

**Case 2:** $p\le 2$. Let $M=\{x\in\Omega : |\nabla u_1|\le 1$ and $|\nabla u_2|\le 1\}$. Then

$$\begin{aligned} I_1 &= \beta\int_\Omega (|\nabla u_1|+|\nabla u_2|)^{p-2}|\nabla u_1-\nabla u_2||u_1-u_2||\nabla\eta|\,\mathrm{d}x \\ &\le \beta\int_M (|\nabla u_1|+|\nabla u_2|)^{p-2}|\nabla u_1-\nabla u_2||u_1-u_2||\nabla\eta|\,\mathrm{d}x \qquad (2.9) \\ &\quad +\beta\int_{\Omega\backslash M} (|\nabla u_1|+|\nabla u_2|)^{p-2}|\nabla u_1-\nabla u_2||u_1-u_2||\nabla\eta|\,\mathrm{d}x. \end{aligned}$$

To obtain (2.5) in this case, we estimate the first term on the right-hand of the above inequality. Using the Hölder inequality, we have

$$\begin{aligned} I_{11} &= \beta\int_M (|\nabla u_1|+|\nabla u_2|)^{p-2}|\nabla u_1-\nabla u_2||u_1-u_2||\nabla\eta|\,\mathrm{d}x \\ &\le \beta\int_M (|\nabla u_1|+|\nabla u_2|)^{p-1}|u_1-u_2||\nabla\eta|\,\mathrm{d}x \\ &\le 2\beta\int_M |u_1-u_2||\nabla\eta|\,\mathrm{d}x \\ &\le 2\beta\left(\int_M |u_1-u_2|^q\,\mathrm{d}x\right)^{\frac{1}{q}}\left(\int_M |\nabla\eta|^{q'}\,\mathrm{d}x\right)^{\frac{1}{q'}}. \end{aligned}$$

where $q'=\dfrac{q}{q-1}$. Noticing that $t=\dfrac{q}{q-1-q/s}$ for $p\le 2$, then $q'\le t$. Thus we obtain

$$\begin{aligned} I_{11} &\le 2\beta|\Omega|^{\frac{1}{q'}-\frac{1}{t}}\left(\int_\Omega |u_1-u_2|^q\,\mathrm{d}x\right)^{\frac{1}{q}}\left(\int_M |\nabla\eta|^t\,\mathrm{d}x\right)^{\frac{1}{t}} \qquad (2.10) \\ &\le C\varepsilon. \end{aligned}$$

Next we estimate the second term on the right-hand of the inequality (2.9). Using the Hölder inequality with

$$\frac{1}{s}+\frac{1}{q}+\frac{1}{t}=1,$$

we have

$$\begin{aligned} I_{12} &= \beta\int_{\Omega\backslash M} (|\nabla u_1|+|\nabla u_2|)^{p-2}|\nabla u_1-\nabla u_2||u_1-u_2||\nabla\eta|\,\mathrm{d}x \\ &\le \beta\int_\Omega |\nabla u_1-\nabla u_2||u_1-u_2||\nabla\eta|\,\mathrm{d}x \\ &\le \beta\left(\int_M |\nabla u_1-\nabla u_2|^s\,\mathrm{d}x\right)^{\frac{1}{s}}\left(\int_M |u_1-u_2|^q\,\mathrm{d}x\right)^{\frac{1}{q}} \\ &\cdot\left(\int_M |\nabla\eta|^t\,\mathrm{d}x\right)^{\frac{1}{t}} \qquad (2.11) \\ &\le C\varepsilon. \end{aligned}$$

Similarly, using the Hölder inequality with

$$\frac{1}{s}+\frac{1}{q}+\frac{1}{t}=1,$$

we have

$$\begin{aligned} I_2 &= \int_\Omega |F(x,u_1)-F(x,u_2)||u_1-u_2||\nabla\eta|\,\mathrm{d}x \\ &\le \left(\int_\Omega |F(x,u_1)-F(x,u_2)|^s \mathrm{d}x\right)^{\frac{1}{s}} \\ &\cdot\left(\int_\Omega |u_1-u_2|^q \mathrm{d}x\right)^{\frac{1}{q}}\left(\int_\Omega |\nabla\eta|^t \mathrm{d}x\right)^{\frac{1}{t}} \end{aligned}$$

Noticing that $F(x,u)\in (L^\tau_{loc}(\Omega))^n$ and $\tau\le s$ for $p\le 2$, by (2.3) and the conditions (ii) in Theorem 1.2 we have

$$I_2 \le C\varepsilon. \tag{2.12}$$

Similarly, Using the Hölder inequality with

$$(\frac{1}{n}+\frac{1}{q})+\frac{1}{s}+(\frac{1}{t}-\frac{1}{n})=1,$$

and poincaré inequality we have

$$\begin{aligned}I_3 &= \int_\Omega |F(x,u_1)-F(x,u_2)||\nabla(u_1-u_2)||\eta|\,dx \\ &\le \left(\int_\Omega |F(x,u_1)-F(x,u_2)|^{\frac{nq}{n+q}}dx\right)^{\frac{1}{n}+\frac{1}{q}} \\ &\cdot\left(\int_\Omega |\nabla u_1-\nabla u_2|^s dx\right)^{\frac{1}{s}}\left(\int_\Omega |\nabla\eta|^{\frac{nt}{n-t}}dx\right)^{\frac{n-t}{nt}} \\ &\le \left(\int_\Omega |F(x,u_1)-F(x,u_2)|^{\frac{nq}{n+q}}dx\right)^{\frac{1}{n}+\frac{1}{q}} \\ &\cdot\left(\int_\Omega |\nabla u_1-\nabla u_2|^s dx\right)^{\frac{1}{s}}\left(\int_\Omega |\nabla\eta|^t dx\right)^{\frac{1}{t}}\end{aligned}$$

Noticing that $F(x,u)\in(L^\tau_{loc}(\Omega))^n$ and $\tau\le\frac{nq}{n+q}$ for $p\le 2$, by (2.3) and the conditions (iii) in Theorem 1.2 we have

$$I_3\le C\varepsilon. \quad (2.13)$$

Thus by (2.4),(2.10)-(2.13) we have

$$I\le C\varepsilon. \quad (2.14)$$

Hence in both cases we have the estimate

$$I\le C\varepsilon, \quad (2.15)$$

where $C<\infty$ is independent of $\varepsilon$. This estimate together with the condition (H3) yields

$$0\le\int_B \langle A(x,\nabla u_1)-A(x,\nabla u_2),\nabla(u_1-u_2)\rangle\,dx\le C\varepsilon. \quad (2.16)$$

Note that here we have used $\eta=1$ in $B$. Letting $\varepsilon\to 0$ we obtain $\nabla u_1=\nabla u_2$ a.e. in $B$. Since $B\subset\subset\Omega$ was arbitrary, $\nabla u_1=\nabla u_2$ in $\Omega$ and hence $u_1=u_2=C=const.$ in each component of $\Omega$.

Now $cap_t(E)=0$ and since the boundary of an arbitrary bounded domain cannot be of $t$-capacity zero, in each component $V$ of $\Omega$ the condition $u_1-u_2\in W_0^{1,p}(V;\partial V\setminus E)$ implies $C=0$. Thus $u_1=u_2$ in $\Omega$ and the theorem follows. This completes the proof of Theorem 1.2.□

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