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Generic Finiteness of the Solutions for the Stable Cahn-Hilliard Equation with fast Growing Nonlinearity

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***Abstract*—Consider the Neumman boundary value problem of the stable Cahn-Hilliard equation.Under an additional condition,we prove the problem has at least one solution. Moreover,the problem has at most finite number of solutions .**

Keywords—Stable Cahn-Hilliard equation; existence ; generic finiteness .

1. INTRODUCTION

We consider the following problem :

$$-\Delta K(u) = g \text{, in } \Omega \qquad (1.1)$$

$$\frac{\partial u}{\partial v} = \frac{\partial(\Delta u)}{\partial v} = 0 \text{, on } \partial\Omega \qquad (1.2)$$

$$m(u) = \alpha \qquad (1.3)$$

Where $\alpha \in R^1$ is a constant,

$$K(u) = -\Delta u + f(u) \qquad (1.4)$$

$$m(u) = \frac{1}{|\Omega|}\int_\Omega u \,\mathrm{d}\,x \qquad (1.5)$$

$\Omega \subset R^n (n \le 3)$ is a bounded open domain with sufficiently regular boundary $\partial\Omega$, $f \in C^2(R^1)$ is a real function which will be specified further below, v stands for the outward normal vector on $\partial\Omega$. The problem (1.1)-(1.3) is related to the stationary state of the well-known Cahn-Hilliard equation which has been extensively studued more recently by [1-6].

In this present work we prove some general properties for this problem. These properties are helpful for us to understand the long-time dynamics of the Cahn-Hilliard equation (of coure, our results are also of mathematical interests). First, we prove the existence result. Then we show a generic property of (1.1)-(1.3) ,i.e.,for every fixed $\alpha \in R^1$,we prove that there exists a dense open set G in the functional space described by g, such that for ever $g \in G$,(1.1)-(1.3) has only finite solutions . The reason why we are interested in the latter problem is that ,once we know that a system has at most finite stationary states, we can then give a precise description of the maximal global attractor of the system (see [5,Ch.Ⅶ].(For the Cahn-Hilliard equation which may possess a fast growing nonlinear term f as considered here ,the existence of the maximal global attractor has juxt been prover recently by Li and Zhong [6].))

Throughout this paper we assume that f satisfies the following structure conditions:

F_1) $f \in C^2(R^1)$, moreover , there exists a positive constant k_1 such that

$$f(s)s \ge -k_1 \quad \forall s \in R^1;$$

F_2) There exists a positive constant k_2 such that

$$f'(s) \ge -k_2 .$$

The Cahn-Hilliard equation arises as a model of plase transitions, Note that the typical function f arising in applications is a polynomial ,

$$f(s) = \sum_{j=1}^{2p-1} a_j s^j, a_{2p-1} > 0, \qquad (1.6)$$

and that this satisfies both F_1) and F_2). However,our f need not to be a polynomial function as those are considered in [1-4] and other references.

Throughout this paper, C denotes a general constant depending upon the constants that appear in $F_1) - F_2)$,the upper bounds of $|\alpha|$ (see (1.3) for the parameter α) and other quantities such as the Sobolev embedding constants etc.

2. SOME A PRIORO ESTIMATES

In this section we establish some a priori estimates for solutions of (1.1)-(1.3). Let $(*,*)$ and $|*|$ denote , respectively , the inner product and norm of $L^2(\Omega)$, we define the linear operator $A = -\Delta$ with domain of definition

$$D(A) = \{ u \in H^2(\Omega), \partial u/\partial v = 0 \text{,on } \partial\Omega \},$$

By spectral theory we can define the spaces $V_s = D\left(A^{\frac{s}{2}}\right)$ with seminorms $|u|_s = \left|A^{\frac{s}{2}}u\right|$ and norms $\|u\|_s = \left(|u|_s^2 + m(u)^2\right)^{\frac{1}{2}}$ for real s. (see [5,6], etc.) It is well known that when $s > 0, V_s$ is a subspace of $H^s(\Omega)$ and that $\|*\|_s$ is equivalent to the usual norm of $H^s(\Omega)$. Also , V_{-s} is the dual of V_s, Moreover, for $\forall s,s'\in R^1, s\le s'$,we have the following Poincare-type inequality

$$|u|_s \le \lambda_1^{-(s'-s)/2}|u|_{s'}, \forall u\in V_{s'} \qquad (2.1)$$

We also denote by H the space $L^2(\Omega)$,by H and V_s the spaces of definitions

$$\dot{H}=\{u\in H=L^2(\Omega), m(u)=0\}, \quad (2.2)$$

$$\dot{V}_s=\{u\in V_s, m(u)=0\}, \qquad (2.3)$$

Lemma 2.1 Assume that f satifies $F_1), F_2)$, $g\in H$,then there exists a positive constant C such that for every solution u of (1.1)-(1.3),

$$\|u\|_1 \le C(1+g) \qquad (2.4)$$

Proof The estimate (2.4) is obtained by multiplying (1.1) with $A^{-1}u$ and integrating over Ω .The argument in detail is similar to those in Lemma 1 of [3] and Lemma 3.1 of [5].we omit it.

Lemma 2.2 The assumptions are those in Lemma 2.1 ,Let u be a solution of (1.1)-(1.3),then

$$\|u\|_4 \le C(1+g) \qquad (2.5)$$

Proof We multiply (1.1) with $-K(u)$ and obtain (note that $\frac{\partial}{\partial v}K(u)=\frac{\partial}{\partial v}A^{-1}g=0$ on $\partial\Omega$)

$$|K(u)|_1^2=-(K(u),g)=-\left(K(u),AA^{-1}g\right)$$

$$\le \frac{1}{2}|K(u)|_1^2+\frac{1}{2}|g|_{-1}^2 \le (\text{by}(2.1))$$

$$\le \frac{1}{2}|K(u)|_1^2+C|g|^2$$

Hence $$|K(u)|_1 \le |g|^2 \qquad (2.6)$$

Now we multiply $K(u)$ by Au and integrate over Ω ,by $F_2)$,we have

$$(K(u),Au)=|u|_2^2+\int_\Omega f'(u)|\nabla u|^2\,dx \ge |u|_2^2-k_2|u|_1^2|u|_2^2$$

$$\le (K(u),Au)+k_2|u|_1^2$$

$$=\int_\Omega \nabla K(u)\cdot\nabla u\,dx+k_2|u|_1^2 \le C\left(|K(u)|_1^2+|u|_1^2\right)$$

By (2.6) and Lemma 2.1 ,we conclude that

$$\|u\|_2 \le C(1+g) \qquad (2.7)$$

By (1.1) ,we have

$$\Delta^2 u=\Delta f(u)+g \qquad (2.8)$$

Thanks to the Sobolev embedding $H^2(\Omega)\subset L^\infty(\Omega)$ ($n\le 3$),and the estimate (2.7) ,we get

$$|u|_{L^\infty(\Omega)} \le C \qquad (2.9)$$

Since f and its derivatives f', f'' are locally bounded,we deduce easily from (2.7)-(2.9) that the estimate (2.5) holds.

Remark 2.1 When $f(s)$ is a polynomial , $f(s)=\sum_{j=0}^{2p-1}a_j s^j$ ($a_{2p-1}>0$ and $g=0$,similar estimates for u are also obtained by Dlotko [3](see [3,Lemma 1])). However the author require that $p=2$ when the space dimension $n=3$,Moreover ,our methods here also differ signicantly from those used in [3].

Lemma 2.3 We assume that f satifies $F_1)$ and $F_2)$,Let $G\subset H$ and $I\subset R^1$ be compact ,then the set $U=\{u\in V_4, \exists(g,\alpha)\in G\times I$ such that (u,g,α) satisfies (1.1)-(1.3)$\}$ is compact in V_4.

Proof Since $H^4(\Omega)\subset C^2(\overline{\Omega})$ ($n\le 3$),we infer from Lemma 2.2 that any sequence $\{u_k\}\subset U$ has a subsequence (still denoted by $\{u_k\}$) coverging in $C^2(\overline{\Omega})$,Assume that

$$\Delta^2 u_k=\Delta f(u_k)+g_k \qquad (2.10)$$

Where $g_k\in G$ .By the compactness of $G,\{g_k\}$ has a subsequence $\{g_{k_j}\}$ which coverges in H .Therefore we deduce from (2.10) that $\{u_{k_j}\}$ coverging in V_4. It is easily seen that the limit function of $\{u_{k_j}\}$ belonge to U . The proof is completed.

3. Non empty and generic properties for the solution set

For given $(g,\alpha)\in H\times R^1$,we denote by $S(g,\alpha)$ the solution set of (1.1)~(1.3). Our aim in this part is two-fold. First,we prove that for every given $(g,\alpha)\in H\times R^1, S(g,\alpha)$ is nonempty,i.e.,the existence for (1.1)~(1.3);then we give a generic property for (1.1)~(1.3),move precisely,for every given $\alpha\in R^1$,we show that there exists a dense open set $G\subset H$ such that for every $g\in G, S(g,\alpha)$ is finite.

Theorem 3.1 Assume that f satisfies $F_1)$ and $F_2)$,then $S(g,\alpha)\ne\varnothing$ for $\forall(g,\alpha)\in H\times R^1$.

Proof Note that (1.1)-(1.3) is equivalent to the following equation:

$$A(Au+f(u+\alpha))=g, u\in V_4 \qquad (3.1)$$

Let $u\in V_2$. Since $\frac{\partial f(u+\alpha)}{\partial v}=f'(u+\alpha)\frac{\partial u}{\partial v}=0$ on $\partial\Omega$,we find that $Af(u+\alpha)\in H$.

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From the general theory for elliptic boundary-value problems,we deduce that there exists a unique $v \in \dot{V}_4$ satisfying

$$A^2 v = -Af(u+\alpha) + \mathrm{g} \qquad (3.2)$$

Thus we can define a mapping $T : \dot{V}_2 \to \dot{V}_2$,as follows:

$$Tu = v, \forall u \in \dot{V}_2 \qquad (3.3)$$

Where v is the solution of (3.2) with respect to u .It is easily seen that T is compact.

Suppose that $u \in \dot{V}_2$ satisfies: $u = \sigma Tu$ ($\sigma \in [0,1]$),then $u \in V_4$ and satisfies

$$A(Au + f(u+\alpha)) = \mathrm{g} \qquad (3.4)$$

Where $f = \sigma f$ .Obviously f satisfies F_1) and F_2) ,therefore by Lemma 2.2 ,there exists a con $C > 0$ (independent upon σ) such that $\|u\|_4$ is bounded by $C(1+|g|)$. In virtue of the classical Leray-Schauder fixed point theorem , we deduce that T has at least one ficed point which solves (1.1)-(1.3) .The proof is therefore complete.

Theorem 3.2 We assume that f satisfies F_1) and F_2). Then for every $\alpha \in R^1$,there exists a dense open set $G \subset \dot{H} = \dot{L}^2(\Omega)$ such that

1) for every $g \in G, S(g,\alpha)$ is finite;

2) for evefy connected compontent G_β of G ,the number of points of $S(g,\alpha)$ for $g \in G_\beta$ is constant.

Proof We will employ some technics used by Fioas and Temam [7]. Denote by $S(g,\alpha)$ the set of solutions of (3.1). It is clear that

$$S(g,\alpha) = S(g,\alpha) - \alpha = \{u - \alpha | u \in S(g,\alpha)\}$$

To prove Theorem 3.2 ,it is sufficient and convenient for us to consider the set $S(g,\alpha)$.

i) Let $F : \dot{V}_4 \to \dot{L}(\Omega)$ as follows

$$F(u) = \Delta^2 u - \Delta f(u+\alpha) \text{ ,for } u \in \dot{V}_4 .$$

Then $F \in C^1$ and

$$F'(u)U = \Delta^2 U - \Delta(f'(u+\alpha)U), \forall U \in \dot{V}_4$$

It is easily seen that the linear operator $K : \dot{V}_4 \to \dot{L}^2(\Omega)$ defined by

$$KU = -\Delta(f'(u+\alpha)U), \forall U \in \dot{V}_4$$

is compact . Since Δ^2 is an isomorphism from $\dot{V}_4$ to $\dot{L}^2(\Omega)$, we coclude that $F'(u)$ is a Fred-holm operator of index 0. We infer from the infinite dimensional version of Sard’s therorem (see Th. A in [7]) that the set G of regular values g of F is dence in $\dot{L}^2(\Omega)$,and $S(g,\alpha)$ is discrete in $\dot{V}_4$ for all $g \in G$ (α is given fixed). Since $S(g,\alpha)$ is compact in $\dot{V}_4$, this set is finite.

ii) The second step of the proof consists in showing that G is open.

Let $g_i \in \dot{L}^2(\Omega) - G (i = 1,2,\cdots)$ be a sequence converging in $\dot{V}_4$ to some limit g .We will prove that $g \in \dot{L}^2(\Omega) - G$ .Let $u_i \in S(g_i,\alpha)$,we assume that $F'(u_i)$ is not surjective .

Since

$$\dim\ker F'(u_i) = \dim \operatorname{coker} F'(u_i) > 0 \qquad (3.5)$$

there exists $v_i \in V_4, \|v_i\|_4 = 1$,such that $F'(u_i)v_i = 0$ i.e.,

$$\Delta^2 v_i - \Delta(f'(u_i+\alpha)v_i) = 0 \qquad (3.6)$$

Since $\{g_i\}$ is bounded in $L^2(\Omega)$,in virtue of the a priri estmate obtained in Section 2 , we know that $\{u_i\}$ is bounded in $\dot{V}_4$.

The above properties of $\{u_i\}$ and $\{v_i\}$ enables us to extract subsequences (still denoted by $\{u_i\}$ and $\{v_i\}$) such that

$$u_i \to u, v_i \to v \text{ weakly in } \dot{V}_4 \qquad (3.7)$$

Since $H^4(\Omega) \subset C^2(\overline{\Omega})$,we deduce by (3.7) that $u_i \to u, v_i \to v$ in $C^2(\overline{\Omega})$,Now we can pass to the limit in (3.6) and $\Delta^2 u_i - \Delta f'(u_i+\alpha) = g_i$,we have

$$\Delta^2 u - \Delta f(u+\alpha) = g \,;\ \Delta^2 v - \Delta(f'(u+\alpha)v) = 0$$

i.e. , $u \in S(g,\alpha)$ and $v \in \ker F'(u)$. Since $\|v_i\|_4 = 1, v \neq 0$. Recalling that $F'(u)$ is a Fredholm operater of index 0,we conclude that u is a critical point of F . Therefore

$$g \in \dot{L}^2(\Omega) - G$$

iii) Let $G_1, G_2, \cdots, G_j, \cdots$ be the connected components of G (G_i is open), and let g_0, g_1 be two points in G_j for same j . Let $u_0 \in F^{-1}(g_0) = S(g_0,\alpha)$ .Since G_j is connected,There exists a continuous curve $g(t) : [0,1] \to G_j$ such that $g(0) = g_0, g(1) = g_1$ and the implicit function theorem implies that there exists a unique curve $u(t)$ which satisfies

$$F(u(t)) = g(t), u(0) = u_0 .$$

Since $g(t)$ is a regular value for every $t \in [0,1], u(t)$ is defined on the whole interval $[0,1]$,therefore

$u(1) \in F^{-1}(g_1) = S(g_1, \alpha)$,Such a curve can be constructed from any point $u_k \in S(g_0, \alpha)$. Also two different curves cannot reach the same point u_* of $S(g_1, \alpha)$,Otherwise this would be contradicts to the implicit funmction theorem around u_*.

Hence there are at least as many points in $S(g_1, \alpha)$ as in $S(g_0, \alpha)$ .By symmetry the number of the points in $S(g_0, \alpha)$ .and that of the point in $S(g_1, \alpha)$ are the same.

Let $S(G_j) = \bigcup_{g \in G_j} S(g, \alpha)$ .then we infer frome the above analysis that $S(G_j)$ can be divided into several parts and each part is a connected component of $S(G_j)$.

Let S_k be such a part of $S(G_j)$.then it is easily seen that the restriction of F on S_k is a C^1 differential isomorphism from S_k to G_j.

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References

[1] N.ALKAKOS P.W.BATES,AND G, FUSCO,Slow motion for the Cahn-Hilliard equation in one spacedimension [J].Diff.Eqns.1991,90:81～135.

[2] J.W.CHOLEWA AND T.DLOTKO,Global attrctor of the Cahn-Hilliard system [J].Bull.Austral.Math. Soc, 1994,49:177～202.

[3] T.DLOTKO,Global attractor for the Cahn-Hilliard equation in H^2 and H^3 [J].Diff. Eqns.1994, 113:381～393.

[4] B.NICOLAENKO B.SHEURER AND R.TEMAM, Some global dynamical properties of a class of pattern formation equations [J].Comm.Partial Diff.Eqns. 1989,14(2):245～297.

[5] R.TEMAM,Infinite Dimensional Dynamical Systems in Mechanics and Physics [M].Springer-Verlag,New York,1988.

[6] LI DESHENG AND ZHONG CHENGKUI,Global attractor for the Chan-Hilliard system with fast growing nonlinearity [J].Diff.Eqns.1998,149:191～210.

[7] C.FOIAS AND R.TEMAM,Structure of the set of stationary solutions of the Navier-Stokes equations [J].Comm.Puer Appl.Math.,1977,30:149～368.