Extended Poisson Theory (EPT): Sequential Development

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***Abstract*—The present article is a brief description of history of the development of Extended Poisson Theory (EPT) for preliminary analysis of primary problems of plates in bending, torsion, and extension. It is intended to rectify deficiencies in the classical plate theories like Kirchhoff theory within small deformation theory of elasticity.**

Keywords—Elasticity; Plates; Bending; Torsion; Extension

I. INTRODUCTION

It is with great pleasure and satisfaction to close the work on development of plate theories within small deformation theory of elasticity, originally intended to overcome lacuna in Kirchhoff's theory [1].

This phase of work was resumed by the author (after a gap of more than 12 years since retirement in the year 1995) mainly due to inducement from his Grand-children with their expertise in using lap-top. With their assistance and the graceful support from the associate Editor in-charge of publication, initial article entitled, "New Look at Kirchhoff's Theory of Plates", was published in the form of a Technical Note [2] in AIAA Journal.

A beginning towards the development of a plate theory was made years ago. It was, in fact, due to the attention drawn (by dearest colleague, K P Rao) to the Reissner's article [3]. Work at that time was confined only to the derivation of Reissner's expected sixth order equation of the form $D\nabla^2\nabla^2 w - C\,\nabla^2\nabla^2\nabla^2 w = q$ [4]. It was felt superfluous to dwell in detail about other aspects mentioned in the Reissner's article due to several investigators including scientists of great authority involved in the development of plate theories.

Basic interest is to find means of proper rectification of deficiencies in Kirchhoff's theory of bending of plates. Initially, auxiliary function $\varphi(x, y)$ introduced by Reissner as a 'stress function' is used in displacements to formulate a sixth order theory [2]. It could not resolve the Poisson-Kirchhoff boundary condition paradox in a satisfactory way. In this connection, it was noted from Lewinski's article [5] that estimated values of maximum vertical deflection from widely used First Order Shear Deformation Theory (FSDT) based on Hencky's work [6] are (rather unusually) higher than those from higher order shear deformation theories and other sophisticated theories.

II. SUCCESSIVE STAGES TOWARDS DEVELOPMENT OF EXTENDED POISSON THEORY

In a brief description of successive stages of author's development of recent plate theories, it is convenient, for simplicity in presentation, to consider a square plate bounded by $0 \le X, Y \le a$, $Z = \pm h$ planes with reference to Cartesian coordinate system. Material of the plate is homogeneous and isotropic with elastic constants E (Young's modulus), ν (Poisson's ratio) and G (Shear modulus) that are related to one other by $E = 2(1+\nu)\,G$. It is more than justified in view of the observation by Ghugal and Shimpi [7] in their review article that the development of refined structural theories for laminated plates (made up from advanced fiber reinforced composite materials) has their origins in the refined theories of isotropic plates. Moreover, illustrative example is confined to simply-supported square plate due to the data presented in [5]. In fact, analysis of plates with different geometries and material properties under different kinematic and loading conditions does not provide much scope for development of new theories other than those with the analysis of primary problems of a square plate. Here, it is more convenient to use coordinates X, Y, Z, and displacements (U, V, W) in non-dimensional form $x = X/a$, $y = Y/a$, $z = Z/h$, $(u, v, w) = (U, V, W)/h$ and half-thickness ratio $\alpha = (h/a)$.

With reference to the displacements $[w, u, v] = [w_0(x, y), z\, u_1(x, y), z\, v_1(x, y)]$ in the primary bending problems, displacements $u_1(x, y)$ and $v_1(x, y)$ are treated as two independent variables instead of gradients of a single variable $w_0(x, y)$ [2]. Displacement $w_0(x, y)$ is expressed from strain-displacement relations in the form

$$\alpha\, w_0(x, y) = -\int [u_1\, dx + v_1\, dy] \qquad (1)$$

so that one equation governing $[u_1, v_1]$ is $\omega_z = \alpha(v_{1,x} - u_{1,y}) = 0$ from integrability condition.

In Kirchhoff's theory, statically equivalent (reactive) transverse stresses $[\tau^*_{xz}, \tau^*_{yz}, \sigma^*_z]$ from z-integration of equilibrium equations are given by

$$[\tau^*_{xz}, \tau^*_{yz}] = f_2(z)\, E'\, [e_{1,x}, e_{1,y}] \quad (2)$$
$$\sigma^*_z = -f_3(z)\, E'\, \alpha^2\, \Delta e_1 \quad (3)$$

in which $f_2(z) = ½(1 - z^2)$, $f_3(z) = ½(z - z^3/3)$, $E' = E/(1 - \nu^2)$, $e_1 = \alpha\,(u_{1,x} + v_{1,y})$ and $\Delta = (\partial^2/\partial x^2 + \partial^2/\partial y^2)$. Face load condition gives the second equation governing $[u_1, v_1]$ in the form $E'\alpha^2\,\Delta e_1 + 3q_0/2 = 0$.

First order equation from integrability condition and second order equation from satisfaction of face load condition are not convenient to satisfy the following three conditions along x = constant edges (and analogous conditions along y = constant edges)

(i) $u = 0$ or $\sigma_x = zT_x(y)$ (4a)
(ii) $v = 0$ or $\tau_{xy} = zT_{xy}(y)$ (4b)
(iii) $w = 0$ or $\tau_{xz} = T_{xz}(y)$ (5)

Here, $[u_1, v_1]$ are expressed in terms of gradients of $w_0(x, y)$ and an auxiliary function $\varphi(x, y)$ in the form

$$[u_1, v_1] = -[(w_{0,x} - \varphi_{,y}), (w_{0,y} + \varphi_{,x})] \quad (6)$$

so that the equations governing u_1 and v_1 become with $D = (2/3)\,\alpha^4 E'$

$$\Delta\varphi = 0,\ \Delta\Delta w_0 = q/D \quad (7)$$

The functions φ and w_0 in the above sixth order system are coupled through edge conditions (i) and (ii) only. It provides the facility of satisfying prescribed τ_{xy} unlike in the Kirchhoff's theory. It is to be noted that τ_{xy} is associated with transverse shear in St. Venant's torsion problem and warping function u or v is harmonic. Here, none of the displacements are harmonic showing that bending and St. Venant's torsion problems are mutually exclusive to each other. Hence, it is not proper to use St. Venant's torsion problem to demonstrate the validity of any sixth order bending theory. Kirchhoff's theory is, in fact, a 0^{th} order shear deformation theory of associated torsion problem (in which normal strains are not zero) due to coupling of τ_{xy} with prescribed or reactive transverse shear along the edges of the plate.

If w_0 is zero all along the wall of the plate, $\varphi \equiv 0$ and the solution to $w_0(x, y)$ is from Kirchhoff's theory only. As such, resolution of Poisson-Kirchhoff boundary conditions paradox from the present Modified Kirchhoff Theory (MKT) is not satisfactory. In FSDT, face shear conditions are not satisfied and shear correction factor k^2 (=5/6) is introduced to account for shear energy due to parabolic z-distribution of reactive transverse shear stresses from Kirchhoff's theory. It is extensively used due to its simplicity in the application of numerical methods such as finite element methods. A novel concept of distribution correction factor β_0 (=5/2) is introduced to derive k^2 (=5/6) in the subsequent publication [8]. It is used in a later publication to derive exact solutions of equations in FSDT (but not equations of 3-D problem) by expanding k^2 in series form.

Next significant publication [9] is the one dedicated to Eric Reissner. Here, coordinate functions $f_n(z)$, n = 0, 1, 2, 3,... are generated for analysis of thick plates through recurrence relations with $f_0 = 1$, $f_{2n+1,z} = f_{2n}$, $f_{2n+2,z} = -f_{2n+1}$ such that $f_{2n+2}(\pm 1) = 0$. They are up to n = 6 given below which are used throughout the present phase of work

$$[f_0, f_1, f_2] = [1, z, ½(1 - z^2)] \quad (8a)$$
$$[f_3, f_4] = [½(z - z^3/3), (5-6z^2+z^4)/24] \quad (8b)$$
$$f_5 = z\,(25-10z^2+z^4)/120 \quad (8c)$$

It is observed that plate element equilibrium equations from using polynomials in z (for example: power series, Taylor series, orthogonal polynomials, above mentioned $f_n(z)$ functions) in higher order theories other than one term representation of displacements in Kirchhoff's theory and FSDT become equivalent to one other with appropriate change of 2-D variables. Earlier proposed Iterative procedure [10] is used in satisfying point-wise equilibrium equations of 3-D infinitesimal element. It is observed that reactive σ_z is zero through thickness of the plate at locations of zeros of prescribed σ_z along faces of the plate. To overcome this limitation, σ_{z2n+3} ($n \geq 1$) is kept as a free variable by modifying f_{2n+3} in the form

$$f^*_{2n+3}(z) = f_{2n+3}(z) - \beta_{2n+1}\, f_{2n+1}(z) \quad (9)$$

in which $\beta_{2n+1} = [f_{2n+3}(1) / f_{2n+1}(1)]$ so that $f^*_{2n+3}(\pm 1) = 0$. In order to distinguish between vertical deflections of neutral and face planes, solution of a supplementary problem implied in Levy's work [11] is used. Corrective vertical displacement terms are from integration of ϵ_z obtained from constitutive relation using normal stresses determined from the analysis. In the illustrative example with ν = 0.3 and thickness ratio 2α = 1/3, maximum neutral and face deflections after first stage of iteration are

$$(E/2q)\ w\ (1/2, 1/2, 0) = 4.46 \quad (10a)$$
$$(E/2q)\ w\ (1/2, 1/2, 1) = 3.80 \quad (10b)$$

Exact solutions of 3-D equations in terms of displacements in which w(x, y, z) is a domain variable are also presented and the reported maximum neutral and face deflections are

$$(E/2q)\ w\ (1/2, 1/2, 0) = 4.487 \quad (11a)$$
$$(E/2q)\ w\ (1/2, 1/2, 0) = 4.166 \quad (11b)$$

It was presumed that the numerical results in eq. (11) correspond to the above mentioned exact solutions. It is, however, shown later that they correspond to the exact solutions with vertical deflection w as face variable.

The procedure presented in [9] is initially extended to the analysis of symmetric laminated plates with isotropic plies [12]. The coordinate $f_n(z)$ functions are modified to ply-wise functions such that the analysis of each ply is independent of lamination. Continuity of displacements and transverse stresses across interfaces are ensured through of solution of supplementary problem in the face ply and recurrence relations.

A theory [13] designated as "Poisson's theory of plates in bending" is presented for a proper resolution of *sixteen-decade-old* problem of Poisson-Kirchhoff boundary conditions paradox in the case of homogeneous isotropic plate. It is based on "assuming" zero transverse shear stresses instead of strains. Reactive (statically equivalent) transverse shear stresses are gradients of a function (in place of in-plane displacements as gradients of vertical deflection) so that reactive transverse stresses are independent of material constants in the preliminary solution. Equations governing in-plane displacements are independent of the vertical deflection $w_0\ (x,\ y)$. Coupling of these equations with w_0 is the root cause for the boundary conditions paradox. Edge support condition on w_0 does not play any role in obtaining in-plane displacements. Normally, solutions to the displacements are obtained from governing equations based on the stationary property of relevant total potential and reactive transverse shear stresses are expressed in terms of these displacements. In the present study, a reverse process in obtaining preliminary solution is adapted in which reactive transverse stresses are determined first and displacements are obtained in terms of these stresses. Equations governing second-order corrections to preliminary solutions of bending of anisotropic plates are derived through application of an iterative method.

Several articles submitted later have not been considered even for reviewing process saying that the contribution of each of them is either modest or out of the scope of the journal. In view of the difficulty in publishing the work mainly due to lack of detailed numerical study of any illustrative problem, the author sought suitable advice from Professor Satya N Atluri, UCI Distinguished professor. He was kind enough to advise the author to prepare a detailed paper on the analysis of laminated plates. With his highly appreciative encouragement and lot of patience shown with several revisions, the article, 'On uniform approximate solutions in bending of symmetric laminated plates', was published in "CMC: Computers, Materials, & Continua [14].

A layer-wise theory with the analysis of face ply independent of lamination is used in the bending of symmetric laminates with anisotropic plies [14]. Due to the presence of [τ_{xz0}, τ_{yz0}] in each ply in a layer-wise theory, more realistic and practical edge conditions are considered. The necessity of a solution of an auxiliary problem in the interior plies is explained and used in the generation of proper sequence of two dimensional problems. Displacements are expanded in terms of polynomials in thickness coordinate such that continuity of transverse stresses across interfaces is assured. Solution of a fourth order system of a supplementary problem in the face ply is necessary to ensure the continuity of in-plane displacements across interfaces and to rectify inadequacies of these polynomial expansions in the interior distribution of approximate solutions. Vertical deflection does not play any role in obtaining all six stress components and two in-plane displacements.

Solution from Kirchhoff's theory in a simple text book problem of bending of simply supported square plate under doubly sinusoidal vertical load is re-examined concerning exact solution of the three dimensional problem [15]. A sixth order system consisting of three second order equations is initially formulated with the aid of normal strain ignored in FSDT. These second order equations, uncoupled from transverse deflection in Kirchhoff's theory, govern in-plane displacements and transverse stresses. A supplementary problem consisting of a fourth order system of equations is included to rectify inadequacies of polynomial expansions in the thickness-wise distribution of approximate solutions. More realistic and practical edge conditions are considered. Solution of auxiliary problem for transverse stresses independent of material constants is used in the generation of proper sequence of two dimensional problems. (At the time of uploading the accepted manuscript, it was realized that the exact solution of 3-D problem with w(x, y, z) as domain variable corresponds to that of associated torsion problem. Moreover, estimated face deflection 2.37 in the example is not correct due to error in using β, unwittingly, in degrees instead of radians in evaluating hyperbolic terms.)

During the correspondence with Professor Lewinski, the author came to know about Jemielita's review article, 'On the winding paths of the theory of plates' [16]. In this article, he referred to an earlier article stating that the progress in the formulation of theories of plates made in 1789-1988 has been carefully reviewed in a 217 page survey [17] encompassing more than 3000 items, about 1500 of them being discussed. An attempt was made to answer the general question, *'To study or to create'*. The present author's recent investigations on plate theories formed the basis to review the development of plate theories [18]. It is shown that methods of analysis based on vertical displacement as domain variable deal with solution of associated torsion problem in bending of plates. It is essential to use vertical displacement as face variable instead of domain variable in the proper analysis of bending problems. Kirchhoff's theory is a 0^{th} order shear deformation theory. FSDT and higher order shear deformation theories with shear correction factors deal with artificial torsion problems. Poisson's theory and Extended Poisson's theory are based on satisfaction of both static and integrated equilibrium equations. Thickness-wise distribution of displacements in terms of polynomials $f_n(z)$ is not adequate for finding interior solutions of these displacements. Solutions of auxiliary and supplementary problems are necessary to rectify lacuna in Kirchhoff's theory.

Nature of solutions from different methods of analysis is re-examined with reference to exact solution of text book problem of bending of a simply supported square plate under bi-sinusoidal load [19]. The concept of shear correction factor in FSDT is extended to higher order terms. Higher order transverse shear terms with $f_{2k+2}(z)$ are expressed in terms of preceding shear terms with $f_{2k}(z)$ through distribution correction factors β_{2k} so that $[\tau_{xz}, \tau_{yz}] = \Sigma f_{2k} [\tau_{xz}, \tau_{yz}]_{2k}$ (using Strain-Displacement relations), and $[\tau_{xz}, \tau_{yz}] = \Sigma \beta_{2k} f_{2k+2} [\tau_{xz}, \tau_{yz}]_{2k}$ (based on shear correction factors*).* In such a case, solutions of plate element equations give shear strains $[u_1 + \alpha w_{0,x}, v_1 + \alpha w_{0,y}]$ tending to [0, 0] in the limit $k \rightarrow \infty$ due to $[\tau_{xz}, \tau_{yz}]$ in the first set but not zero due to stresses in the second set. Obviously, shear energy due to β_{2k} does not belong to the physical problem. It shows that the methods based on plate element equilibrium equations deal with solution of associated torsion problem instead of bending problem. Solution of bending problem is only through methods based on vertical displacement as a face variable.

Extended Poisson's theory presented in the earlier publications gives the wrong impression about systems of equations. It consists of a fourth order system to find $[u_1, v_1]_c$ due to $[\tau_{xz0}, \tau_{yz0}, z\,\sigma_{z1}]$ and another one to find $[u_1, v_1]_b$ due to prescribed in-plane bending load edge conditions ignoring their influence on transverse stresses. This impression is corrected in the article [20]. With priory known transverse stresses independent of material constants from auxiliary problem, this theory is, in fact, a sixth order theory in which $[u_1, v_1]$ and transverse shear stresses satisfy both static and integrated equilibrium equations and form the initial solutions in the iterative procedure adapted earlier in the CMC Journal. This theory is based on $[u, v] = z [u_1, v_1]$ like in FSDT but gives far superior solutions than even those from two term representation of displacements. In extension problems, however, two term representation of [u, v] is required to determine transverse stresses though their influence on primary variables $[u_0, v_0]$ is secondary but important in the analysis of laminates. Extended Poisson's theory is based on satisfaction of both static and integrated equilibrium equations. Solutions of auxiliary and supplementary problems are necessary to rectify lacuna in the classical theories. The functions $f_k(z)$ are chosen such that ply analysis is independent of lamination. A novel procedure is proposed for analysis of unsymmetrical laminates. More or less uniform accuracy of primary displacement variables $[u, v, w]_0$ along each normal of face parallel planes is achieved through the solution of a secondary problem. EPT is also extended in formulating a new smeared laminate theory [21]. By considering uncoupled 2-D equations through Fourier analysis of z-distributions, it was shown that the theories other than EPT give only the neglected primary σ_z ($= \sigma_{z0}/2$, $z\sigma_{z1}/2$ satisfying face load conditions in extension and bending problems, respectfully) in the constitutive relations.

Disadvantage in the application of EPT is in the development of software for generation of $f_k(z)$ functions of thickness coordinate z necessary for analysis of plates with thickness ratio varying up to unit value. It is shown in [22] that the initial solutions of primary problems are from EPT. Errors in transverse shear stress-strain relations are nullified through solutions of uncoupled 2-D problems from Fourier series expansion of z-distribution of these strains in appropriate sine and cosine functions. Earlier, such expansion was considered in Touratier's work [23] in rectifying lacuna in Kirchhoff's theory but used it in plate element equilibrium equations.

III. CONCLUDING REMARKS

Extended Poisson Theory (EPT) is a proper replacement of classical theories, viz., Kirchhoff Theory and First Order Shear Deformation Theory (FSDT) for obtaining initial solutions of primary bending and associated torsion problems of plates. It is used for proper second order corrections in the classical theory of extension problems.

Theories presented in [20], [21], and [22] are quite adequate and simple for analysis of homogeneous and laminated plates within the small deformation theory of elasticity with thickness ratio varying up to unit value.

REFERENCES

[1] G. Kirchhoff, "Über das Gleichgewicht und die Bewegung einer elastischen Scheibe". Journal für reins und angewandte Mathematik, Vol. 40, pp. 51–88, 1850.

[2] K. Vijayakumar, "New Look at Kirchhoff's Theory of Plates". AIAA Journal 04/2009; 47(4):1045-1046. DOI:10.2514/1.38471 ·

[3] E. Reissner, "Reflections on the Theory of Elastic Plates". Applied Mechanics Reviews (AMR), Vol. 38, pp. 1453-1464, 1985.

[4] K. Vijayakumar, "Poisson-Kirchhoff Paradox in Flexure of Plates". Technical Notes, AIAA Journal, issue February pp. 247-249, 1988.

[5] T. Lewinski, "On the Twelth-Order Theory of Elastic Plates". Mech. Res. Comm., Vol. 17(6), pp. 375-382, 1990.

[6] H. Hencky, “Über die Berücksichtigung der Schubverzerrung in ebenen Platten”. Ingenieur-Archiv, Vol. 16, pp. 72-76, 1947. DOI: 10.1007/bf00534518

[7] Y. M. Ghugal, R. P. Shimpi, “A Review of Refined Shear Deformation Theories of Isotropic and Anisotropic Laminated Plates”. Journal of Reinforced Plastics and Composites 2002; 21; 775
DOI: 10.1177/073168402128988481

[8] K. Vijayakumar, “Modified Kirchhoff's theory of plates including transverse shear deformations”. Mechanics Research Communications 04/2011; 38(3):211-213. DOI: 10.1016/j.mechrescom.2011.02.007

[9] K. Vijayakumar, “A relook at Reissner’s theory of plates in bending”. Archive of Applied Mechanics 11/2011; 81(11):1717-1724. DOI: 10.1007/s00419-011-0513-4 ·

[10] K Vijayakumar, A V Krishnamurty, “Iterative modeling for stress analysis off composite laminates”. Composites Science and Technology 01/1988; 32(3):165-181. DOI: 10.1016/0266-3538(88)90018-8

[11] M. Levy, “Memoire sur la Theorie des Plaques Elastiques Planes”. Journal des Mathematiques Pures et Appliquees, vol. 30, pp. 219–306, 1877.

[12] K. Vijayakumar, “Layerwise Theory of Bending of Symmetric Laminates with Isotropic Plies”. AIAA Journal 09/2011; 49(9):2073-2076. DOI:10.2514/1.J051080 ·

[13] K. Vijayakumar, “Poisson’s Theory for Analysis of Bending of Isotropic and Anisotropic Plates” ISRN Civil Engineering 2013 562482 8 pages; http://dx.doi.org/10.1155/2013/562482

[14] K. Vijayakumar, “On Uniform Approximate Solutions in Bending of Symmetric Laminated Plates”. CMC: Computers, Materials & Continua 34 (1), 1-25, 2013

[15] K. Vijayakumar, “On a sequence of approximate solutions: bending of a simply supported square plate”. International Journal of Advanced Structural Engineering, December 5:18, 9 pages, First online: 23 August 2013. DOI: 10.1186/2008-6695-5-18

[16] G. Jemielita, “On the winding paths of the theory of plates”. MechanicaTeoretyczna I stosowana, Journal of Theoretical and Applied Mechanics, 2, Vol. 31, pp. 317-327, 1993.

[17] G. Jemielita, “On the winding paths of the theory of plates”. (In Polish) PraceNaukowePoltechniki Warszawskiej, Budownictwo, 117: 217 pages.

[18] K. Vijayakumar, “Review of a few selected theories of plates in bending”. Hindawi Publishing Corporation, International Scholarly Research Notices. Vol. 2014, Article ID 29147 http://dx.doi.org/10.1155/2014/291478

[19] K. Vijayakumar, “Extended Poisson’s Theory for Analysis of Bending of a Simply Supported Square Plate”. SSRG International Journal of Civil Engineering (SSRG-IJCE), Vol. 2 Issue 11, November 2015.

[20] K. Vijayakumar, “Extended Poisson theory for analysis of laminated plates”. Journal of Multidisciplinary Engineering Science and Technology (JMEST), Vol. 3, Issue 2, 17 pages, February 2016.

[21] K. Vijayakumar, “Smeared Laminate Theory of Unsymmetrical Laminated Plates: use of Extended Poisson Theory”. JMEST, Volume 2, Issue 11, pp. 3202-3211, November, 2015. ISSN: 3159-0040

[22] K. Vijayakumar, “Modifications of Extended Poisson Theory of Plates”. Journal of Multidisciplinary Engineering Science and Technology (JMEST) ISSN: 3159-0040 Vol. 2 Issue 12, December – 2015. www.jmest.org JMESTN42351279 3516

[23] M. Touratier, “An efficient standard plate theory”. International Journal of Engineering science, Vol. 29 (8), pp. 901-916, 1991.