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Extended Poisson Theory For Analysis Of Laminated Plates

Running Title: Extended Poisson Theory of plates

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***Abstract*—Author's recently proposed extended Poisson theory is presented here in a more precise and concise form and used for analysis of bending, torsion, and extension problems of symmetric and unsymmetrical laminated plates with anisotropic plies. Proper polynomial functions in thickness coordinate are used so as to satisfy both static and integrated equilibrium equations. In bending and extension problems, solutions for in-plane displacements and transverse stresses are independent of vertical displacement. However, polynomials in thickness coordinate are not adequate for proper solutions of 3-D problems. Solution of supplementary problem in the face plies is required for obtaining interior distribution of displacements. It is also required in adjacent plies of the reference plane in the analysis of unsymmetrical laminates to ensure continuity of displacements and transverse stresses. Methods of analysis of bending, torsion, and extension problems are mutually exclusive to one other. Traditional methods based on plate element equilibrium equations through stationary property of relevant total potentials are not suitable for generation of proper sequence of 2-D problems though may not assume to be of significant consideration for the practicing engineers.**

Keywords—Elasticity, Plates, Laminates, Bending, Torsion, Extension

I. INTRODUCTION

Analysis of plates within the classical small deformation theory of elasticity is concerned with problems of flexure, extension, and torsion. It is generally based on making suitable assumptions about thickness-wise distribution functions $f_n(z)$ of displacements and/or stresses (or strains) to derive two-dimensional plate equations. In discussing nature of solutions from these 2-D equations with reference to the exact solutions of 3-D problems, it is necessary to consider displacements from thickness-wise integration of strains. The displacements independent of thickness coordinate are used as domain variables in most of the theories reported in the literature. Out of these three variables denoted by $[u, v, w]_0$, in-plane displacements are basic variables in extension problems. Solutions for these displacements satisfy both static and integrated equilibrium equations.

Basic variable $w_0(x, y)$ in the bending problems is a domain variable and governed by a fourth order equation in Kirchhoff's theory [1] based on plate element equilibrium equations (PEEES). It is also used as domain variable indirectly in Reissner's theory [2] and directly in First Order Shear Deformation Theory (FSDT) based on Hencky's work [3]. Higher order theories (based on either stationary property of relevant total potential or plate element equations like in asymptotic methods with $w_0(x, y)$ as a domain variable) are intended for further rectification of lacuna in Kirchhoff's theory though it is not proper for this purpose to use St.Venant's torsion problem in which normal strains are zero to justify these theories. Associated torsion problem in the presence of normal strains is different from each other of bending and St. Venant's torsion problems.

In the literature, one finds vast amount of investigations and several review articles, in particular, on laminated composite plates (to cite a few of them, [4-9]. These studies are mainly in the analysis of plates with different geometries and material properties under different kinematic and loading conditions. They provide useful design data for the practicing engineer and the present work is not intended to validate such data from these theories based on PEEES. It is to show inadequacy of these theories in providing sequence of 2D problems converging to 3D problem.

In the author's recent review article [10] on theories of plates in bending, emphasis is on development of new theories for proper rectification of lacuna in Kirchhoff's theory. His recently proposed Poisson's theory [11] is mainly intended for proper resolution of Poisson-Kirchhoff boundary conditions paradox and in its extended theory [12], assumed transverse shear stresses are independent of thickness co-ordinate. Equations of equilibrium are in terms of two in-plane displacements and three transverse stresses independent of vertical displacement. It is shown that the use of vertical deflection as a domain variable is not suitable for analysis of bending problems. Thickness-wise integration of transverse normal strain ϵ_z from constitutive relation is used for obtaining vertical displacement. The function $w_0(x, y)$ arising out of this integration is treated as face variable from integration of zero face shear conditions. Prescribed zero w along an edge of the plate is replaced by zero ϵ_z which is more practical at point supports.

In the case of symmetric laminates, several theories such as single layer theories, smeared laminate theories, layer-wise theories, zigzag theories, etc. in the analysis of bending, extension and torsion problems are reported in the literature. Emphasis in these theories is for accurate estimation of stresses (more so transverse stresses) along interfaces. Generally, distributions of displacements in face parallel planes are evaluated in most of these theories. Statically equivalent transverse stresses are obtained through post-processing from integration of equilibrium equations ignoring transverse shear strain-displacement relations. It is mainly due to the use of vertical displacement as a domain variable. Second order corrections in estimation of these stresses are through proper estimation of displacements. Layer-wise theories are based on ply element equilibrium equations. They are coupled with adjacent plies through the equations governing displacement and transverse stress variables along the common interface of the plies so that each ply analysis is dependent on lamination. They are solved by relating face conditions with symmetry conditions along the mid-plane in one half of the laminate through equations governing common variables between adjacent plies. It is not a convenient procedure if the stacking sequence contains large number of plies. (Moreover, there is a need to incorporate proper modifications in these theories in the analysis of unsymmetrical laminates.)

The deficiencies in the above mentioned layer-wise theories are avoided in the layer-wise theory using the author's extended Poisson's theory [12]. In this theory, ply analysis is independent of lamination and continuity of displacements and transverse stresses across interfaces is through solution of a supplementary problem in the face ply along with recurrence relations. Moreover, solutions for displacements satisfy both static and integrated equilibrium equations. It is to be noted, however, that the presentation of the theory in the above referred article gives wrong impression about the systems of equations. It consists of a fourth order system to find $[u_1, v_1]_c$ due to $[\tau_{xz0}, \tau_{yz0}, z\sigma_{z1}]$ and another one to find $[u_1, v_1]_b$ due to prescribed in-plane bending load edge conditions ignoring their influence on transverse stresses. It was intended to show through the illustrative example that one term representation of in-plane displacements gives better results in the extended Poisson's theory compared to Poisson's theory.

Here, extended Poisson's theory is presented in a more concise and precise form including proper estimation of transverse stresses due to in-plane bending loads. With priory known transverse stresses (independent of elastic deformations) from auxiliary problem, it is, indeed, shown to be a sixth order theory. Use of such a theory in the preliminary analysis of extension and associated torsion problems is explored. Further, a novel procedure based on this theory is envisaged in the analysis of unsymmetrical laminates for which no proper procedure, to the author's knowledge, exists till now in the literature.

II. PRELIMINARIES IN THE ANALYSIS OF SYMMETRIC LAMINATES

For simplicity in presentation, a symmetric laminate bounded by $0 \le X \le a$, $0 \le Y \le b$ and $-h_n \le Z \le h_n$ with interfaces $Z = h_k$ in the Cartesian coordinate system (X, Y, Z) is considered. For convenience, coordinates X, Y, and Z and displacements U, V, and W in non-dimensional form $x = X/L$, $y = Y/L$, $z = Z/h_n$, $u = U/h_n$, $v = V/h_n$, $w = W/h_n$ and half-thickness ratio $\alpha = h_n/L$ with reference to a characteristic length L [mod(x, y) ≤ 1] are utilized. The material of each ply is homogeneous and anisotropic with monoclinic symmetry. Interfaces are given by $z = \alpha_k = h_k/h_n$ (k = 1, 2…., n-1) in the upper-half of the laminate.

Equilibrium equations in terms of stress components *in each ply* (denoted with superscript 'k' but generally omitted unless it is required for clarification) are

$$\alpha\,(\sigma_{x,x} + \tau_{xy,y}) + \tau_{xz,z} = 0 \qquad (1a)$$
$$\alpha\,(\sigma_{y,y} + \tau_{xy,x}) + \tau_{yz,z} = 0 \qquad (1b)$$
$$\alpha\,(\tau_{xz,x} + \tau_{yz,y}) + \sigma_{z,z} = 0 \qquad (2)$$

in which suffix after ',' denotes partial derivative operator.

In displacement based models, stress components are expressed in terms of displacements, via, six stress-strain constitutive relations and six strain-displacement relations. In the present study, these relations are confined to the classical small deformation theory of elasticity.

Upper face values of displacements $[u, v, w]^u$ and transverse stresses $[\tau_{xz}, \tau_{yz}, \sigma_z]^u$ in a ply are related to its lower face values $[u, v, w]^b$ and $[\tau_{xz}, \tau_{yz}, \sigma_z]^b$, respectively, through the solution of equations (1, 2) together with three conditions specified later at each of constant x (and y) edges and satisfaction of continuity conditions across interfaces.

Here, it is convenient to denote displacements [u, v] as $[u_i]$, (i = 1, 2), in-plane stresses $[\sigma_x, \sigma_y, \tau_{xy}]$ and transverse stresses $[\tau_{xz}, \tau_{yz}, \sigma_z]$ as $[\sigma_i]$, $[\sigma_{3+i}]$, (i = 1, 2, 3), respectively. With the corresponding notation for strains, strain-displacement relations are

$$[\varepsilon_1, \varepsilon_2, \varepsilon_3] = \alpha\,[u_{,x}, v_{,y}, u_{,y} + v_{,x}] \qquad (3)$$
$$[\varepsilon_4, \varepsilon_5, \varepsilon_6] = [u_{,z} + \alpha\, w_{,x}, v_{,z} + \alpha\, w_{,y}, w_{,z}] \qquad (4)$$

Strain-stress and semi-inverted stress-strain relations with the usual summation convention of repeated suffix denoting summation over specified integer values:

$$\varepsilon_i = S_{ij}\,\sigma_j \quad (i, j = 1, 2, 3, 6) \qquad (5)$$
$$\varepsilon_r = S_{rs}\,\sigma_s \quad (r, s = 4, 5) \qquad (6)$$
$$\sigma_i = Q_{ij}[\varepsilon_j - S_{j6}\,\sigma_z] \quad (i, j = 1, 2. 3) \qquad (7)$$
$$\sigma_r = Q_{rs}\,\varepsilon_s \quad (r, s = 4, 5) \qquad (8)$$

With σ_i in equations (7), in-plane equilibrium equations (1) become

$$\alpha\,[Q_{1j}(\varepsilon_j - S_{j6}\,\sigma_z)_{,x} + Q_{3j}(\varepsilon_j - S_{j6}\,\sigma_z)_{,y}] + \tau_{xz,z} = 0 \quad (9a)$$
$$\alpha\,[Q_{2j}(\varepsilon_j - S_{j6}\,\sigma_z)_{,y} + Q_{3j}(\varepsilon_j - S_{j6}\,\sigma_z)_{,x}] + \tau_{yz,z} = 0 \quad (9b)$$

In expressing thickness-wise distribution of displacements, a complete set of $f_k(z)$ functions in each ply generated earlier [15] from recurrence relations with $f_0 = 1$, $f_{2k+1,z} = f_{2k}$, $f_{2k+2,z} = -f_{2k+1}$ such that $f_{2k+2}(\pm\alpha_k) = 0$ are used. They are (up to k = 5)

$$[f_1, f_2] = [z, (\alpha_k^2 - z^2)/2] \qquad (10a)$$
$$f_3 = (\alpha_k^2\, z - z^3/3)/2 \qquad (10b)$$
$$f_4 = [(5\,\alpha_k^4 - 6\,\alpha_k^2 z^2 + z^4)/24 \qquad (10c)$$
$$f_5 = z(25\,\alpha_k^4 - 10\,\alpha_k^2 z^2 + z^4)/120] \qquad (10d)$$

Displacements [u, v, w] are expressed in the form

$$[u, v, w] = f_i(z)\,[u_i, v_i, w_i],\ i = 0, 1, 2, ... \qquad (11)$$

In order to maintain continuity of a 3-D variable across interfaces and keep associated 2-D variable as a free variable, it is necessary to replace f_{2i+1} by f^*_{2i+1} given by

$$f^*_{2i+1} = f_{2i+1} - \beta_{2i-1} f_{2i-1},\ i = 1, 2, ... \quad (12)$$

in which $\beta_{2i-1}\alpha_k^2 = [f_{2i+1}(\alpha_k) / f_{2i-1}(\alpha_k)]$ so that $f^*_{2i+1}(\alpha_k) = 0$. With the above replacement of odd $f_i(z)$ functions, continuity of relevant variables across interfaces is ensured through the continuity of 2-D variables associated with f_0 and f_1 only.

Priory prescribed conditions along faces and edges of the plate are adjusted such that vertical deflection w(x, y, z) is even in z in bending and odd in extension problems. Correspondingly, displacements [u, v] are odd and even functions in z in bending and extension problems, respectively.

A. Edge conditions in each ply

Equilibrium equations (1, 2) in each ply have to be solved along with three edge conditions and ensure continuity conditions of three displacements and three transverse stresses across interfaces. **In the primary bending** and associated torsion problems, prescribed conditions at each of x = constant edge (with analogous conditions along y = constant edge) in the primary problems are

$$\text{(i)}\ u_1(y) = 0 \text{ or } \sigma_{x1}(y) = T_{x1}(y) \quad (13a)$$
$$\text{(ii)}\ v_1(y) = 0 \text{ or } \tau_{xy1}(y) = T_{xy1}(y) \quad (13b)$$
$$\text{(iii)}\ w_0(y) = 0 \text{ or } \tau_{xz0}(y) = T_{xz0}(y) \quad (14)$$

Contradiction between zero face shear conditions and prescribed transverse shears along wall of the plate in the bending problem is resolved earlier [12]. In this earlier analysis, derived 2-D problems are independent of lamination. Interface continuity of displacements and transverse stresses is through solution of a supplementary problem defined from Levy's work [13] in the face ply and recurrence relations across interfaces. It is to be noted that vertical displacement is a face (and interface) variable in bending problems and domain variable in the associated torsion problems.

Corresponding edge conditions in primary extension problems are

$$\text{(i)}\ u = \tilde{u}_0(y) \text{ or } \sigma_{x0}(y) = T_{x0}(y) \quad (15a)$$
$$\text{(ii)}\ v = \tilde{v}_0(y) \text{ or } \tau_{xy0}(y) = T_{xy0}(y) \quad (15b)$$
$$\text{(iii)}\ w_1(y) = 0 \text{ or } \tau_{xz1}(y) = T_{xz1}(y) \quad (16)$$

(It would be shown later that the condition $w_1(y) = 0$ cannot be imposed)

III ANALYSIS OF PRIMARY BENDING PROBLEM

In a primary bending problem, the plate is subjected to zero transverse shear stresses and asymmetric normal stress $\sigma_z = \pm\, q_0(x, y)/2$ along $z = \pm 1$ faces.

Primary transverse stresses in the initial analysis are

$$[\tau_{xz}, \tau_{yz}] = [\tau_{xz0}, \tau_{yz0}] + [f_2\, \tau_{xz2}, f_2\, \tau_{yz2}]^k \quad (17a)$$
$$\sigma_z = z\sigma_{z1} + [f_3\, \sigma_{z3}]^k \quad (17b)$$

It is to be noted that $[\tau_{xz0}, \tau_{yz0}, z\sigma_{z1}]$ are independent of lamination and material constants. Second expression consists of reactive stresses in the ply which are also independent of lamination due to the chosen $f_k(z)$ functions. In most of the reported layer-wise theories, $[\tau_{xz0}, \tau_{yz0}, z\sigma_{z1}]$ are dependent on material constants due to lamination dependent analysis.

In view of eq. (2), $[\tau_{xz0}, \tau_{yz0}]$ are expressed in the form

$$[\tau_{xz0}, \tau_{yz0}] = \alpha\, [\psi_{0,x}, \psi_{0,y}] \quad (18)$$

so that ψ_0 (x, y), with plane Laplace operator $\Delta = (\partial^2/\partial x^2 + \partial^2/\partial y^2)$, is governed by

$$\alpha^2\, \Delta\psi_0 + \sigma_{z1} = 0 \ \ (\sigma_{z1} = q/2) \quad (19)$$

with condition along x = constant edge (with analogous conditions along y = constant edge)

$$\psi_0 = 0 \quad \text{or} \quad \alpha\, \psi_{0,x} = T_{xzo}(y) \qquad (20)$$

Solution for ψ_0 from the above auxiliary problem is designated as *universal solution* being independent of elastic deformations and material constants.

With displacements $[w_0, z\, u_1, z\, v_1]$ and strain-displacement relations, w_{0f} is given by

$$w_{0f}(x, y) = \int [\gamma_{xz0}\, dx + \gamma_{yz0}\, dy] - \int [u_1\, dx + v_1\, dy] \qquad (21)$$

Vertical deflection $w_0(x, y)$ from integration of ε_z is a known variable only with $w_{0f}(x, y)$. Otherwise, it remains as an unknown domain variable which is necessary in the associated torsion problem due to its participation in the static equilibrium equations. Static equilibrium equations have to be independent of $w_0(x, y)$ in bending problem so as to decouple from torsion problem. For this purpose, $[u_1, v_1]$ are modified as

$$u_{*1} = (u_1 + \gamma_{xz0} - \alpha\, w_{0,x}) \qquad (22a)$$
$$v_{*1} = (v_1 + \gamma_{yz0} - \alpha\, w_{0,y}) \qquad (22b)$$

(Inclusion of transverse shear strains is necessary for finding face deflection w_{0f})

With displacements $[u, v] = z\, [u_1, v_1]$, determination of $[u, v]$ satisfying equilibrium equations requires $f_2[\tau_{xz2}, \tau_{yz2}]$ along with $\sigma_z = f_3\sigma_{z3}$. Since $f_3(\alpha_k) \neq 0$, σ_{z3} becomes a free variable by replacing $f_3(z)$ with $f^*_3(z)$. Normal stress σ_z along with $(z\, q_1/2)$ in the extended Poisson's theory takes the form

$$\sigma_z = z\, [(1/2)q_1 - \beta_1\, \sigma_{z3}] + f_3\, \sigma_{z3} \qquad (23)$$

Transverse shear stresses along with those in the auxiliary problem are

$$[\tau_{xz}, \tau_{yz}] = [\tau_{xz}, \tau_{yz}]_0 + f_2(z)\, [\tau_{xz}, \tau_{yz}]_2 \qquad (24)$$

in which

$$\tau_{xz2} = Q_{44}u_1 + Q_{45}\, v_1 + \tau_{xz0} \qquad (25a)$$
$$\tau_{yz2} = Q_{55}v_1 + Q_{45}\, u_1 + \tau_{yz0} \qquad (25b)$$

Note that $[\tau_{xz0}, \tau_{yz0}]$ in eq. (24) are included due to participation of σ_{z1} in Eq. (1). From equations (1, 2, 23, 25), one obtains

$$\alpha\, [(Q_{44}\, u_1 + Q_{45}\, v_1)_{,x} + (Q_{54}\, u_1 + Q_{55}\, v_1)_{,y} = \beta_1\, \sigma_{z3} \qquad (26)$$

(σ_{z3} is from coefficient of f_1. Note that one cannot prescribe zero τ_{xz2} (and τ_{yz2}) at x (and y) constant edges since τ_{xz0} (and τ_{yz0}) are independent of elastic deformations)

For the use of $[u^*, v^*]_1$ in the integration of equilibrium equations (1), it is convenient to express $[u_1, v_1]$ in terms of gradients of two functions $[\psi, \varphi]$ in the form

$$[u_1, v_1] = -\alpha\, [\psi_{1,x} + \varphi_{1,y},\ \psi_{1,y} - \varphi_{1,x}] \qquad (27)$$

In the case of isotropic face ply, ψ_1 is governed by the bi-harmonic operator and φ_1 by the plane Laplace equation (φ_1 was originally introduced as 'stress function' in [2]). We note here that φ_1 and harmonic part of bi-harmonic ψ_1 are not conjugate to each other. Same set of harmonic functions in ψ_1 has to be used in φ_1 (with different multiplying algebraic coefficients) so as to replace tangential edge displacement from ψ_1 with normal gradient of φ_1. As such, in-plane displacements are in terms of gradients of $[\psi_1, \varphi_1]$ but without the need to use gradients of φ_1 in transverse shear stresses.

Contributions of ψ_1 and w_0 in $[u^*, v^*]_1$ are one and the same in giving corrections to $w(x, y, z)$ and transverse stresses. Hence, w_0 in $[u^*, v^*]_1$ is replaced by ψ_1 so that $[u^*, v^*]_1$ are

$$u^*_1 = -\alpha\, (2\psi_{1,x} + \varphi_{1,y}) + \gamma_{xz0} \qquad (28a)$$
$$v^*_1 = -\alpha\, (2\psi_{1,y} - \varphi_{1,x}) + \gamma_{yz0} \qquad (28b)$$

Correspondingly, in-plane strains $[\varepsilon_x^*, \varepsilon_y^*, \gamma_{xy}^*]_1$ with $[\tilde{\varepsilon}_{x1}, \tilde{\varepsilon}_{y1}, \tilde{\gamma}_{xy1}] = -\alpha^2\, [(2\psi_{1,xx} + \varphi_{1,xy}), (2\psi_{1,yy} - \varphi_{1,xy}), (4\psi_{1,xy} + \varphi_{1,yy} - \varphi_{1,xx})]$ are given by

$$[\varepsilon^*_x, \varepsilon^*_y]_1 = [(\tilde{\varepsilon}_{x1} + \alpha\, \gamma_{xz0,x}), (\tilde{\varepsilon}_{y1} + \alpha\, \gamma_{xz0,y})] \qquad (29a)$$
$$\gamma^*_{xy1} = [\tilde{\gamma}_{xy1} + \alpha\, (\gamma_{xz0,y} + \gamma_{yz0,x})] \qquad (29b)$$

From integration of eq. (2) and equilibrium equations (9) using the strains in equations (29), reactive transverse stresses are (with sum j = 1, 2. 3)

$$\tau^*_{xz2} = \alpha\, [Q_{1j}(\tilde{\varepsilon}_j - S_{j6}\, \sigma_{z1})_{,x} + Q_{3j}(\tilde{\varepsilon}_j - S_{j6}\, \sigma_{z1})_{,y}] \quad (30a)$$
$$\tau^*_{yz2} = \alpha\, [Q_{2j}(\tilde{\varepsilon}_j - S_{j6}\, \sigma_{z1})_{,y} + Q_{3j}(\tilde{\varepsilon}_j - S_{j6}\, \sigma_{z1})_{,x}] \quad (30b)$$
$$\sigma_{z3} = -\alpha\, (\tau^*_{xz2,x} + \tau^*_{yz2,y}) \quad (31)$$

(Here, σ_{z3} is coefficient of f_3)

Noting that σ_{z3} from eq. (26) is negative of the one from eq. (31) due to $(f_{3,zz} + f_1) = 0$, the equation governing in-plane displacements is given by

$$\alpha\, \beta_1\, (\tau^*_{xz2,x} + \tau^*_{yz2,y}) = \alpha\, [(Q_{44}\, u_1 + Q_{45}\, v_1)_{,x} + (Q_{54}\, u_1 + Q_{55}\, v_1)_{,y}] \quad (32)$$

With the condition zero ω_z (i.e., $v_{,x} = u_{,y}$) required to decouple bending and torsion, eq. (32) consists of Laplace equation $\Delta\varphi_1 = 0$ and a fourth order equation in ψ_1 to be solved with the following three conditions at x = constant edges (with analogous conditions at y = constant edges)

$$\text{(i) } (u^* \text{ or } \sigma^*)_1 = 0, \text{ (ii) } (v^* \text{ or } \tau^*_{xy})_1 = 0 \quad (33a)$$
$$\text{(iii) } \psi_1 \text{ or } \tau^*_{xz2} = 0 \quad (33b)$$

A Supplementary problem in the face ply

Transverse stresses in the face ply are

$$[\tau_{xz}, \tau_{yz}] = [\tau_{xz0}, \tau_{yz0}] + f_2\, [\tau_{xz2}, \tau_{yz2}] \quad (34)$$
$$\sigma_z = z\, \sigma_{z1} + f_3\, \sigma_{z3} \quad (35)$$

Corrective in-plane displacements in the supplementary problem are assumed as

$$[u, v]_s = [u_1, v_1]_s \sin(\pi z/2) \quad (36)$$

In the case of homogeneous isotropic plate, single term sin (π z/2) implied in Levy's work [13] is shown to be adequate for proper estimation of w(x, y, 0) (see Appendix). All shear deformation theories with two term representation of [u, v] in the illustrative example [15, 16] give more or less same estimates around 3.48 to the maximum neutral plane deflection w_n.

In-plane stresses are

$$\sigma_{is} = Q_{ij}\, \varepsilon_{1sj} \sin(\pi z/2) \quad (i, j = 1, 2, 3) \quad (37)$$

From integration of equilibrium equations using in-plane stresses in Eq. (37) along with $[\tau_{xz}, \tau_{yz}] = [\tau_{xz2}, \tau_{yz2}]_s \cos(\pi z/2)$ and $\sigma_{z3} = \sigma_{z3s} \sin(\pi z/2)$, transverse stresses in the supplementary problem are given by

$$\tau_{xz2s} = -(2/\pi)\, \alpha\, [Q_{1j}\, \varepsilon_{1sj,x} + Q_{3j}\, \varepsilon_{1sj,y}] \quad (38a)$$
$$\tau_{yz2s} = -(2/\pi)\, \alpha\, [Q_{2j}\, \varepsilon_{1sj,y} + Q_{3j}\, \varepsilon_{1sj,x}] \quad (38b)$$
$$\sigma_{z3s} = (2/\pi)^2 \alpha^2\, [Q_{1j}\, \varepsilon_{1sj,xx} + 2Q_{3j}\, \varepsilon_{1sj,xy} + Q_{2j}\, \varepsilon_{1sj,yy}] \quad (39)$$

In-plane distributions u_{1s} and v_{1s} are added as corrections to the known in-plane displacements $[u_1, v_1]$ so that [u, v] in the supplementary problem are

$$[u, v] = [(u_1 + u_{1s}), (v_1 + v_{1s})] \sin(\pi z/2) \quad (40)$$

From equations (32, 38.39), one gets

$$\beta_1 \sigma_{z3} = (2/\pi)^2 \alpha^2\, [Q_{1j}\, \varepsilon_{1sj,xx} + 2\, Q_{3j}\, \varepsilon_{1sj,xy} + Q_{2j}\, \varepsilon_{1sj,yy}] \quad (41)$$

By expressing $[u_{1s}, v_{1s}] = -\alpha\, [\psi_{1s,x}, \psi_{1s,y}]$, eq. (41) becomes a fourth order equation in ψ_{1s} to be solved with two in-plane conditions at x = constant edges (with analogous conditions along y = constant edges)

$$\text{(i) } (u_{1s} \text{ or } \sigma_{xs1}) = 0, \text{ (ii) } (v_{1s} \text{ or } \tau_{xys1}) = 0 \quad (42)$$

B. Continuity of displacements and transverse stresses across interfaces

In-plane displacements and transverse stresses in the face ply, now, become

$$u = z\, u_1 + (u_1 + u_{1s}) \sin(\pi z/2) \quad (43a)$$
$$v = z\, v_1 + (y_1 + v_{1s}) \sin(\pi z/2) \quad (43b)$$
$$\tau_{xz} = \tau_{xz0} + f_2\, \tau_{xz2} + (\tau_{xz2} + \tau_{xz2s})\, (\pi/2) \cos(\pi z/2) \quad (44a)$$
$$\tau_{yz} = \tau_{yz0} + f_2\, \tau_{yz2} + (\tau_{yz2} + \tau_{yz2s})\, (\pi/2) \cos(\pi z/2) \quad (44b)$$
$$\sigma_z = z\, (q/2) + [f_3 - \sin(\pi z/2)]\, \beta_1 \sigma_{z3} - \sigma_{z3s} \sin(\pi z/2) \quad (45)$$

In the interior plies, displacements $[u_1, v_1]$, thereby, in the neighboring plies are obtained from solution of sixth order system of equations (1) governing $[u_1, v_1]$. They are dependent on material constants but independent of lamination. Displacements $[u_1, v_1]_s$ and transverse stresses $[\tau_{xz2}, \tau_{yz2}, \sigma_{z3}]_s$ are obtained from continuity conditions across interfaces.

Continuity of u (with similar expressions for v) across interfaces is simply assured through the following recurrence relations

$$[u_{1s}^{(k)} - u_{1s}^{(k+1)}]\sin \tfrac{\pi}{2}\alpha_k = \alpha_k[u_1^{(k+1)} - u_1^{(k)}] + [\alpha_k + \sin \tfrac{\pi}{2}\alpha_k]\,[u_1^{(k+1)} - u_1^{(k)}] \quad (46)$$

Since $[\tau_{xz0}, \tau_{yz0}]$ and $\sigma_{z1} = z\ q/2$ are same throughout the laminate, recurrence relations for τ_{xz2} (with similar expressions for τ_{yz2}) and σ_{z3} are

$$f_2^{(k+1)}(\alpha_k)\ \tau_{xz2}^{(k+1)} = \{[\tau_{xz2s}^{(k)} - \tau_{xz2s}^{(k+1)}] + [\tau_{xz2}^{(k)} - \tau_{xz2}^{(k+1)}]\}\tfrac{\pi}{2} \cos \tfrac{\pi}{2}\alpha_k \quad (47)$$

$$\{[\sigma_{z3s}^{(k)} - \sigma_{z3s}^{(k+1)}] + \beta_1[\sigma_{z3}^{(k)} - \sigma_{z3}^{(k+1)}]\} \sin \tfrac{\pi}{2}\alpha_k = \beta_1[f_3^{(k+1)}(\alpha_k)\ \sigma_{z3}^{(k+1)} - f_3^{(k)}(\alpha_k)\ \sigma_{z3}^{(k)}] \quad (48)$$

With ε_{z1} from constitutive relation, vertical deflection w(x, y, z) is given by

$$w = w_0 - f_2\ \varepsilon_{z1} + (\pi/2)\ w_{os} \cos(\pi z/2) \quad (49)$$

Note that vertical deflections w_0 and w_{0s} are obtained from integration of shear strain-displacement relations (whereas ε_{z1} which does not participate in determination of $[u_1, v_1]_b$ is obtained from ε_6 in the constitutive relations (5) in the interior of each ply). They are

$$\alpha\ w_0 = \int [(\varepsilon_{40} - u_1)\ dx + (\varepsilon_{50} - v_1)\ dy] \quad (50)$$

$$\alpha\ w_{0s} = \int [(\varepsilon_{40} - u_{1s})\ dx + (\varepsilon_{50} - v_{1s})\ dy] \quad (51)$$

Due to zero face shear conditions, ε_{40} and ε_{50} are zero in the face ply and w_0 corresponds to face deflection. To satisfy edge support condition, one needs only a support so as to prevent vertical movement of intersections of supported segments of the faces and wall of the plate.

Continuity across interfaces gives the recurrence relation

$$\alpha[w_{0s}^{(k)} - w_{0s}^{(k+1)}]\tfrac{\pi}{2} \cos \tfrac{\pi}{2}\alpha_k = \alpha[w_0^{(k+1)} - w_0^{(k)}] - [f_2(\alpha_k)\ \varepsilon_{z1}]^{(k+1)} \quad (52)$$

C Higher order corrections in the ply from iterative procedure

Above solutions for displacements and transverse stresses are initial solutions in the iterative procedure in solving 3-D problems to generate a proper sequence of sets of 2-D equations. The only error with reference to 3-D problems is in the transverse shear strain-displacement relations. Displacements $f_3\ [u_3, v_3]$ (thereby, ε_{z3}) consistent with $f_2\ [\tau_{xz2}, \tau_{yz2}]$ and reactive transverse stresses $(\tau_{xz4}, \tau_{yz4}, \sigma_{z5})$ have to be obtained from the first stage of iterative procedure.

Initial transverse shear strains from constitutive relations are

$$\gamma_{xz2} = S_{44}\ \tau_{xz2} + S_{45}\ \tau_{yz2} \quad (53a)$$

$$\gamma_{yz2} = S_{45}\ \tau_{xz2} + S_{55}\ \tau_{yz2} \quad (53b)$$

Displacements $[u_3, v_3]$ are modified such that they are corrections to face parallel plane distributions of the preliminary solution so that

$$w = f_2(z)(w_2 - \varepsilon_{z1}) \quad (54)$$

$$u_3^* = u_3 + \gamma_{xz2} - \alpha\ (w_2 - \varepsilon_{z1})_{,x} \quad (55a)$$

$$v_3^* = v_3 + \gamma_{yz2} - \alpha\ (w_2 - \varepsilon_{z1})_{,y} \quad (55b)$$

in which $[u, v]_3 = [u, v]_1 + [u, v]_{3c}$ with $[u,v]_{3c}$ denoting corrections due to transverse shear strain-displacement relations.

Corresponding correction to vertical displacement is w_2 which is treated as virtual variable in strain-displacement relations due to its real contribution in the integrated equilibrium equations.

Due to replacement of f_5 by f^*_5, one gets after some algebra

$$\alpha\beta_3\,(\tau_{xz4,x}+\tau_{yz4,y}) = \alpha^2[(Q_{44}u_3+Q_{45}v_3)_{,xx} + (Q_{54}\,u_3 + Q_{55}\,v_3)_{,yy}\,]+ \alpha^2\Delta\sigma_{z1} \qquad (56)$$

With the condition zero ω_z ($v_{,x} = u_{,y}$) necessary for analysis of bending problem, eq. (56) consists of Laplace equation $\Delta\varphi_3 = 0$ and a fourth order equation in ψ_3 to be solved with conditions at x = constant edges (with analogous conditions along y = constant edges)

$$\text{(i) } u^*_3 \text{ or } \sigma^*_3 = 0, \text{ (ii) } v_3 \text{ or } \tau^*_{xy3} = 0, \text{ (iii) } \psi_3 \text{ or } \tau^*_{xz4} = 0 \qquad (57)$$

C Supplementary problem in the face ply

Corrective displacements in the supplementary problems are assumed in the form:

$$w = w_{2s}\,(\pi/2)\cos(\pi z/2) \qquad (58)$$
$$[u_s, v_s] = [u_{3s}, v_{3s}]\sin(\pi z/2) \qquad (59)$$
$$\sigma_{3si} = Q_{ij}\,\varepsilon_{3sj} \qquad (i, j = 1, 2, 3) \qquad (60)$$

Analysis here is a repetition of the corresponding analysis in the supplementary problem in the section III *A*. Necessary equations are given below:

$$\tau_{xz2s} = -\,(2/\pi)\,\alpha\,[Q_{1j}\,\varepsilon_{3sj,x} + Q_{3j}\,\varepsilon_{3sj,y}] \qquad (61a)$$
$$\tau_{yz2s} = -\,(2/\pi)\,\alpha\,[Q_{2j}\,\varepsilon_{3sj,y} + Q_{3j}\,\varepsilon_{3sj,x}] \qquad (61b)$$
$$[u, v] = [(u^*_3 + u_{3s}), (v^*_3 + v_{3s})]\sin(\pi z/2) \qquad (62)$$
$$\beta_3\,\sigma_{z5} = (2/\pi)^2\alpha^2\,[Q_{1j}\,\varepsilon^*_{sj,xx} + 2Q_{3j}\,\varepsilon^*_{sj,xy} + Q_{2j}\,\varepsilon^*_{sj,yy}]_{(3)} \qquad (63)$$

Corrective displacements are

$$w = w_{04}(x, y) - f_4\,\varepsilon_{z3} + w_{04s}\,(\pi/2)\cos(\pi z/2) \qquad (64)$$
$$u = f_3\,u^*_3 + (u^*_3 + u_{3s})\sin(\pi z/2) \qquad (65a)$$
$$v = f_3\,v^*_3 + (v^*_3 + v_{3s})\sin(\pi z/2) \qquad (65b)$$
$$\alpha\;w_{04} = \int[(\gamma_{xz2} - u_3)\,dx + (\gamma_{yz2} - v_3)\,dy] \qquad (66)$$
$$\alpha\;w_{04s} = \int[(\gamma_{xz2s} - u_{3s})\,dx + (\gamma_{yz2s} - v_{3s})\,dy] \qquad (67)$$

D Continuity of displacements across interfaces

Continuity of $[\tau_{xz4}, \tau_{yz4}, \sigma_{z5}]$ across interfaces is not presented here since these stresses require further correction due to the next higher order terms. Recurrence relations for continuity of displacement u (with similar expressions for v and analogues recurrence relations for w_{04} and w_{04s}) are

$$[u_{3s}{}^{(k)} - u_{3s}{}^{(k+1)}]\sin(\pi\alpha_k/2) = \alpha_k[u_1{}^{(k+1)} - u_1{}^{(k)}] + [f_3{}^{(k+1)}(\alpha_k) + \sin(\pi\alpha_k/2)][u^*_3{}^{(k+1)} - u^*_3{}^{(k)}] \qquad (68)$$

Successive use of eq. (68) starting from top ply ensures continuity of displacements across the interfaces.

In the face ply, the equations governing $[\psi_3, \varphi_3]$ and ψ_{3s} consists of sixth and fourth order system of equations, respectively. Obviously, similar sets of equations govern $[\psi, \varphi]_{2n+3}$ and ψ_{2n+3s}, (n = 1, 2, 3 …), corresponding to higher order displacement terms. In the interior plies, only a sixth order system of equation govern $[\psi, \varphi]_{2n+3}$.

IV ASSOCIATED TORSION PROBLEMS

Kirchhoff's theory and FSDT due to their inherent deficiencies are not suitable in the analysis of face ply but $w_0(x, y)$ as domain variable is required in the analysis of associated torsion problem. In the analysis, it is convenient and proper to include transverse shear stresses $[\tau_{xz0}, \tau_{yz0}]$ and $\sigma_z = z\,q/2$ used in the extended Poisson's theory.

Here, the analysis requires only modification of the analysis of bending problem replacing ψ_1 with vertical displacement $w_0(x, y)$ so that

$$w = w_0(x, y) \qquad (69)$$
$$[u, v] = -\,\alpha\,z\,[(w_{0,x}+\varphi_{1,y}), (w_{0,y}-\varphi_{1,x})] \qquad (70)$$
$$\tau_{xz} = \tau_{xz0} + f_2\,(\tau_{xz0} + \tau_{xz2}) \qquad (71a)$$
$$\tau_{yz} = \tau_{yz0} + f_2\,(\tau_{yz0} + \tau_{yz2}) \qquad (71b)$$
$$\sigma_{z3} = z\,q/2 + f_3\,\sigma_{z3} \qquad (72)$$

Note that transverse shear strains from strain-displacement relations are $[-\alpha\;\varphi_{1,y},\;\alpha\;\varphi_{1,x}]$ which correspond to self-equating stresses. From zero face shear conditions, one gets an additional face deflection

$\tilde{w}_{0c}(x, y)$ given by

$$\tilde{w}_{0c} = \int [\varphi_{1,y}\, dx - \varphi_{1,x}\, dy] \qquad (73)$$

Above $\tilde{w}_{0c}$ is a face variable which becomes zero with prescribed zero w_0 at the edge of the plate by preventing vertical movement of intersection of the face with cylindrical surface of the side wall of the plate.

For obtaining second order corrections, ψ_3 is replaced by w_2 which is treated as virtual variable in strain-displacement relations with real contribution in the integrated equilibrium equations.

V EXTENSION PROBLEMS

In a primary extension problem, the plate is subjected to symmetric normal stress $\sigma_{zo} = q_0(x, y)/2$, asymmetric shear stresses $[\tau_{xz}, \tau_{yz}] = \pm [T_{xz1}(x, y), T_{yz1}(x, y)]$ along top and bottom faces of the plate.

Due to out-off plane equilibrium equation, applied face shears $[T_{xz}, T_{yz}]$ have to be gradients of a harmonic function ψ_1 so that $[T_{xz}, T_{yz}] = -\alpha\, [\psi_{1,x}, \psi_{1,y}]$. Transverse shear stresses and normal stress satisfying face conditions are

$$[\tau_{xz}, \tau_{yz}] = -\alpha z\, [\psi_{1,x}, \psi_{1,y}],\ \sigma_{z0} = q_0(x, y)/2 \qquad (74)$$

Above transverse stresses which are independent of material constants remain same in the *interior* of the laminate.

One should note here that σ_{z0} does not participate in the equilibrium equations but contributes in the in-plane constitutive relations. $[\tau_{xz}, \tau_{yz}]$ in the above equation are related to in-plane displacements $[u_0, v_0]$ through equilibrium equations (1). From constitutive relation,

$$\varepsilon_{z0} = S_{6j}\, \sigma_{j0} + S_{66}\, q_0/2 \quad (j = 1, 2, 3) \qquad (75)$$

Correspondingly, vertical deflection w is linear in z and cannot be prescribed to be zero at the edge of the plate due to $S_{6j}\, \sigma_{j0}$ even if the faces are free of transverse stresses. It is complementary to the fact that transverse shear strains in the bending problem from strain-displacement relations, thereby, transverse shear stresses cannot be prescribed to be zero along the wall of the plate even if the faces are free of transverse stresses. It is, in a way, justifies the extended Poisson's theory of bending of plates.

A Preliminary analysis of the ply

In-plane equilibrium equations in the preliminary analysis with $[u, v] = [u_0(x, y), v_0(x, y)]$ are

$$[Q_{1j}\, (\varepsilon_{j0} - S_{j6}\, \sigma_{z0})_{,x} + Q_{3j}\, (\varepsilon_{j0} - S_{j6}\, \sigma_{z0})_{,y}] = \alpha\psi_{1,x} \qquad (76a)$$
$$[Q_{2j}\, (\varepsilon_{j0} - S_{j6}\, \sigma_{z0})_{,y} + Q_{3j}\, (\varepsilon_{j0} - S_{j6}\, \sigma_{z0})_{,x}] = \alpha\psi_{1,y} \qquad (76b)$$

subjected to the edge conditions (15) at x (and y) constant edges.

Solutions of $[u_0, v_0]$ from the above equations with reference to 3-D problem are in error in transverse shear strain-displacement relations due to $w_1 = \varepsilon_{z0}$ (i.e., $w = z\, \varepsilon_{z0}$) from constitutive relation. In the analysis of laminated plates, transverse stresses along interfaces are important even in the case of applied face stresses and edge transverse shear stresses are zero. Moreover, analysis of the face and interior plies in the layer-wise theory adapted in the present work is independent of lamination. As such, one requires higher order approximate solutions even in the homogeneous plate problem.

B Higher order corrections in the ply

One should note that transverse shear stresses are zero through thickness at locations of zeros of gradients of ψ. To overcome this limitation, transverse stress components are assumed in the form

$$\tau^*_{xz2n+1} = (\tau_{xz2n+1} - \beta_{2n-1}\tau_{xz2n-1}) \qquad (77a)$$
$$\tau^*_{yz2n+1} = (\tau_{yz2n+1} - \beta_{2n-1}\tau_{yz2n-1}) \qquad (77b)$$
$$\sigma^*_{z2n+2} = \sigma_{z2n+2} - \beta_{2n-1}\, \sigma_{z2n} \qquad (78)$$

At the n^{th} stage of iteration ($n \geq 1$), transverse stresses $[\tau_{xz}, \tau_{yz}]_{2n-1}$ and w_{2n-1} are known in the preceding stage. With regard to in-plane displacements, one should include additional terms such that they are consistent with known stresses $[\tau_{xz}, \tau_{yz}]_{2n-1}$ and are free to obtain stresses $[\tau_{xz}, \tau_{yz}]_{2n+1}$, σ_{z2n+2} and w_{2n+1}.

We have from constitutive relations,

$$\gamma_{xz2n-1} = S_{44}\, \tau_{xz2n-1} + S_{45}\, \tau_{yz2n-1} \quad (79a)$$
$$\gamma_{yz2n-1} = S_{45}\, \tau_{xz2n-1} + S_{55}\, \tau_{yz2n-1} \quad (79b)$$

Modified displacements and the corresponding derived quantities denoted with * are with w_{2n-1} as correction to ε_{z2n-2} due to $[u, v]_{2n}$

$$u^*_{2n} = u_{2n} - \alpha\, (\varepsilon_{z2n-2} + w_{2n-1})_{,x} + \gamma_{xz2n-1} \quad (80a)$$
$$v^*_{2n} = v_{2n} - \alpha\, (\varepsilon_{z2n-2} + w_{2n-1})_{,y} + \gamma_{yz2n-1} \quad (80b)$$

Strain-displacement relations give

$$\varepsilon^*_{x2n} = \varepsilon_{x2n} - \alpha^2(\varepsilon_{z2n-2} + w_{2n-1})_{,,xx} + \alpha\, \gamma_{xz2n-1,x} \quad (81a)$$
$$\varepsilon^*_{y2n} = \varepsilon_{y2n} - \alpha^2(\varepsilon_{z2n-2} + w_{2n-1})_{,,yy} + \alpha\, \gamma_{yz2n-1,y} \quad (81b)$$
$$\gamma^*_{xy2n} = \gamma_{xy2n} - 2\,\alpha^2(\varepsilon_{z2n-2} + w_{2n-1})_{,xy} + $$
$$+ \alpha\, (\gamma_{xz,y} + \gamma_{yz,x})_{2n-1} \quad (81c)$$
$$\gamma^*_{xz2n-1} = \gamma_{xz2n-1} - (u_{2n-2} + u_{2n}) \quad (82a)$$
$$\gamma^*_{yz2n-1} = \gamma_{yz2n-1} - (v_{2n-2} + v_{2n}) \quad (82b)$$

In-plane stresses and transverse shear stresses from constitutive relations are

$$[\sigma^*_i]_{2n} = [Q_{ij}\, \varepsilon^*_j]_{2n} \quad (i, j = 1, 2, 3) \quad (83)$$
$$\tau^*_{xz2n-1} = \tau_{xz2n-1} - (Q_{44}\, u + Q_{45}\, v)_{2n} \quad (84a)$$
$$\tau^*_{yz2n-1} = \tau_{yz2n-1} - (Q_{54}\, u + Q_{55}\, v)_{2n} \quad (84b)$$

One gets from equations (1, 2, 77, 78) noting that $\sigma_{z2n}{}^* = \sigma_{z2n} - \beta_{2n-1}\, \sigma_{z2n+2}$

$$\alpha\, [(Q_{44}\, u_{2n} + Q_{45}\, v_{2n})_{,x} + (Q_{54}\, u_{2n} + Q_{55}\, v_{2n})_{,y}] + $$
$$+ \beta_{2n-1}\, \sigma_{z2n+2} = 0 \quad (85)$$

(Note that w_{2n-1} is not present in the above equation)

For the use of $[u^*, v^*]_{2n}$ in the integration of equilibrium equations, displacements $[u, v]_{2n}$ are expressed in the form

$$[u, v]_{2n} = -\alpha\, [\psi_{2n,x}, \psi_{2n,y}] \quad (86)$$

Contributions of ψ_{2n} and w_{2n-1} in $[u^*, v^*]_{2n}$ are one and the same in giving corrections to w(x, y, z) and transverse stresses (in fact, contribution of w_{2n-1} is through strain-displacement relations in static equilibrium equations, and through constitutive relations in through-thickness integration of equilibrium equations). Hence, w_{2n-1} in $[u^*, v^*]_{2n}$ is replaced by ψ_{2n} (so as to be independent of w_{2n-1} used in strain-displacement relations) so that $[u^*, v^*, \varepsilon^*_x, \varepsilon^*_y, \gamma^*_{xy}]_{2n}$ are

$$u^*_{2n} = (2\, u_{2n} + \gamma_{xz2n-1} - \alpha\, \varepsilon_{z2n-2,x}) \quad (87a)$$
$$v^*_{2n} = (2\, v_{2n} + \gamma_{yz2n-1} - \alpha\, \varepsilon_{z2n-2,y}) \quad (87b)$$
$$\varepsilon^*_{x2n} = (2\, \varepsilon_{x2n} + \alpha\, \gamma_{xz2n-1,x} - \alpha^2 \varepsilon_{z2n-2,xx}) \quad (88a)$$
$$\varepsilon^*_{y2n} = (2\, \varepsilon_{y2n} + \alpha\, \gamma_{yz2n-1,y} - \alpha^2 \varepsilon_{z2n-2,yy}) \quad (88b)$$
$$\gamma^*_{xy2n} = [2\, \gamma_{xy2n} + \alpha\, (\gamma_{xz2n-1,y} + \gamma_{yz2n-1,x}) -$$
$$-2\alpha^2 \varepsilon_{z2n-2,xy}] \quad (88c)$$

Note that the role of w_{2n-1} is in its contribution to the integrated equilibrium equations where as it is a virtual quantity in transverse strain-displacement relations.

From integration of equilibrium equations using the strains in equations (90), reactive transverse stresses are

$$\tau^*_{xz2n+1} = \alpha\, [Q_{1j}(\varepsilon^*_j - S_{j6}\, \sigma_z)_{,x} + $$
$$+ Q_{3j}(\varepsilon^*_j - S_{j6}\, \sigma_z)_{,y}]_{2n}, \quad (j = 1, 2, 3) \quad (89a)$$
$$\tau^*_{yz2n+1} = \alpha\, [Q_{2j}(\varepsilon^*_j - S_{j6}\, \sigma_z)_{,y} + $$
$$+ Q_{3j}(\varepsilon^*_j - S_{j6}\, \sigma_z)_{,x}]_{2n}, \quad (j = 1, 2, 3) \quad (89b)$$
$$\sigma_{z2n+2} = -\alpha\, (\tau^*_{xz,x} + \tau^*_{yz,y})_{2n+1} \quad (90)$$

One equation governing in-plane displacements $(u, v)_{2n}$, noting that σ_{z2n+2} from eq. (87) is negative of the one from eq. (92) due to $(f_{2n+1,zz} + f_{2n-1}) = 0$, is given by

$$\alpha\, \beta_{2n-1}(\tau^*_{xz,x} + \tau^*_{yz,y})_{2n+1} = \alpha\, [(Q_{44}\, u + Q_{45}\, v)_{,x} + $$
$$+ (Q_{54}\, u + Q_{55}\, v)_{,y}]_{2n} \quad (91)$$

With the second equation $v_{2n,x} = u_{2n,y}$, the above equation becomes a fourth order equation in ψ_{2n} to be solved along with harmonic function φ_{2n} with three conditions along constant x = constant edges (with analogous conditions along y = constant edge)

$$\text{(i) } (u_{2n} \text{ or } \sigma_{2n})^* = 0, \text{ (ii) } (v_{2n} \text{ or } \tau_{xy2n})^* = 0, \quad \text{(iii) } \tau_{xz2n-1}^* = 0 \qquad (92)$$

C Continuity of displacements across interfaces

Since $[u, v, \sigma_z]$ are in terms of even functions f_{2n} (z), continuity of u, v and σ_z across interface z = α_k, (k = 1, 2, n-1), is ensured except u_0 and v_0. Note that the analysis of face ply is independent of lamination. In the bending problem, solution of a supplementary problem in the face ply is used to ensure continuity of displacements across interfaces. Here also, one should adapt the same procedure with interchange of sine and cosine trigonometric terms used in the bending problem. Continuity of $[u_0, v_0]$ is achieved through solution of the following supplementary problem.

D Supplementary problem

Corrective in-plane displacements in the face ply in the supplementary problems are assumed in the form:

$$[u, v]_s = \frac{\pi}{2}\,[u, v]_s \cos(\pi z/2) \qquad (93)$$

$$\sigma_{0si} = Q_{ij}\, \varepsilon_{0sj} \qquad (i, j = 1, 2, 3) \qquad (94)$$

If transverse shear stresses are expressed in terms of f_{2n+1}^* (n ≥ 1), their continuity across interface is simply given by continuity of ψ_1 which is independent of material constants and lamination.

With the in-plane stresses σ_{0is} along with $[\tau_{xz}, \tau_{yz}] = [\tau_{xz1}, \tau_{yz1}]_s \sin(\pi\, z/2)$ and $\sigma_{z2} = \sigma_{z2s} \cos(\pi\, z/2)$, integration of equilibrium equations give

$$\tau_{xz1s} = -(2/\pi)\,\alpha\,(Q_{1j}\, \varepsilon_{0j,x} + Q_{3j}\, \varepsilon_{0j,y})_s \qquad (95a)$$
$$\tau_{yz1s} = -(2/\pi)\,\alpha\,(Q_{2j}\, \varepsilon_{0j,y} + Q_{3j}\, \varepsilon_{0j,x})_s \qquad (95b)$$
$$\sigma_{z2s} = (2/\pi)^2 \alpha^2\, [Q_{1j}\, \varepsilon_{0j,xx} + 2Q_{3j}\, \varepsilon_{0j,xy} + Q_{2j}\, \varepsilon_{0j,yy}]_s \qquad (96)$$

In-plane distributions of $[u_0, v_0]_s$ are added as corrections to the known in-plane displacements $[u_0, v_0]$ so that [u, v] in the supplementary problem are

$$[u, v] = [(u_0 + u_{0s}), (v_0 + v_{0s})] \cos(\pi z/2) \qquad (97)$$
$$\beta_0\, \sigma_{z2c} = (2/\pi)^2 \alpha^2\, [Q_{1j}\, \varepsilon_{0j,xx} + 2Q_{3j}\, \varepsilon_{0j,xy} + Q_{2j}\, \varepsilon_{0j,yy}]_s \qquad (98)$$

By expressing $[u_{0s}, v_{0s}] = -\alpha\, [\psi_{0s,x}, \psi_{0s,y}]$, eq. (98) becomes a fourth order equation in ψ_{0s} to be solved with two conditions along x = constant edges (with analogous conditions along y = constant edge)

$$\text{(i) } (u_{0s} \text{ or } \sigma_{xs0}) = 0, \text{ (ii) } (v_{0s} \text{ or } \tau_{xys0}) = 0 \qquad (99)$$

E Continuity of displacements and transverse stresses across interfaces

In-plane displacements and transverse stresses in the face ply from above analysis are

$$[u, v] = [(u_0 + u_{0s}), (v_0 + v_{0s})] \cos(\pi z/2) \qquad (100)$$
$$\tau_{xz} = f_1\, \tau_{xz1} + (\tau_{xz1} + \tau_{xz1s})(\pi/2) \sin(\pi z/2) \qquad (101a)$$
$$\tau_{yz} = f_1\, \tau_{yz1} + (\tau_{yz1} + \tau_{yz1s})(\pi/2) \sin(\pi z/2) \qquad (101b)$$
$$\sigma_z = (q/2) + [f_2 - \cos(\pi z/2)]\, \beta_0\, \sigma_{z2} - \sigma_{z2s} \cos(\pi z/2) \qquad (102)$$

In the interior plies, displacements $[u_0, v_0]$, thereby, in the neighboring plies are obtained from solution of sixth order system of equations (1, 2) governing $[u_0, v_0]$. They are dependent on material constants but independent of lamination. Displacements $[u_0, v_0]_s$ and transverse stresses $[\tau_{xz1}, \tau_{yz1}, \sigma_{z2}]_s$ are obtained from continuity conditions across interfaces.

Continuity of u (and v) across interfaces is simply assured through the recurrence relations (similar relations for v)

$$[u_{0s}^{(k)} - u_{0s}^{(k+1)}] \cos(\pi\alpha_k/2) = \alpha_k [u_0^{(k+1)} - u_0^{(k)}] + [\alpha_k + \cos(\pi\alpha_k/2)]\, [u_0^{(k+1)} - u_0^{(k)}] \qquad (103)$$

Above analysis gives displacements and transverse stresses (with sum n = 1, 2, …, , n-1)

$$w = f_1 w_1 + (\pi/2)\, w_{1s} \sin(\pi z/2) + f^*_{2n+1}\, w_{2n+1} \qquad (104)$$
$$u = u_0 + u_{2s} \cos(\pi z/2) + f_{2n}\, u_{2n} \qquad (105a)$$
$$v = v_0 + v_{2s} \cos(\pi z/2) + f_{2n}\, v_{2n} \qquad (105b)$$
$$[\tau_{xz}, \tau_{yz}] = f_1 [\tau_{xz}, \tau_{yz}]_1 + f^*_{2n+1}\, [\tau_{xz}, \tau_{yz}]_{2n+1} \qquad (106)$$
$$\sigma_z = q_0(x, y)/2 + f_2\, \sigma^*_{z2} + \sigma_{z2s} \cos(\pi z/2) +$$

$+ f_{2n}\, \sigma_{z2n}$ (107)

VI. ASSOCIATED TORSION TYPE PROBLEMS IN STRETCHING OF PLATES

In-plane displacements $[u_0, v_0]$ in extension problem are obtained from solution of 2-D equations in the classical theory. Even in the case of faces of the plate free of applied loads, constitutive relation gives normal strain ε_{z0} from which one gets $w = z\, \varepsilon_{z0}$. Due to this vertical displacement, transverse strain-displacement relations in 3-D problem are not satisfied (but generally ignored). One needs higher order corrections in transverse stresses to satisfy these relations, particularly, in the analysis of laminates. If one uses shear deformation theory like in the bending problem, one has to replace ε_{z0} by a domain variable w_1 and introduce shear correction factor denoted by k_e^2 in transverse shear energy terms. k_e^2 is evaluated from consideration of distribution correction factor β_e = 17/42 from

$$\beta_e = [\int[f_3^2\, dz]/[\int[z\, f_3(z)dz] \qquad (108)$$

so that k_e^2 = 52/105 from integration of $(z + \beta_e\, f_3)^2$. Note here that transverse stresses are zero along faces of the plate.

In the associated torsion problem in bending of the plate, normal strains $[\varepsilon_x, \varepsilon_y, \varepsilon_z]$ are zero along mid-plane in pure torsion problem. Similarly, they are zero along faces of the plate with reference to second order corrections in extension problem and shear stresses correspond to non-torsion problem. In the isotropic rectangular plate, warping function u(y, z) is also harmonic function in y-z plane and analysis is same as pure torsion problem in bending except that $g_{un} = \cos \lambda_n z$ and $\lambda_n = n\pi$ satisfying zero face shear conditions. Here also, τ_{xy} distribution like in bending is nullified in shear deformation theories due to non-torsion problem in the limit.

VII. UNSYMMETRICAL LAMINATES

Here, suffix n in $(-\, h_n)$ of the bottom face is replaced by m with number of layers 'm' in the bottom-half $z \le 0$ need not be equal to number of layers 'n' in the upper-half $z \ge 0$. Initial set of solutions in the upper-half of the laminate in all the problems presented above are unaltered up to the reference plane z = 0. One has to consider continuity of non-zero displacements and transverse stresses across reference plane. They will be different due to asymmetry from similar analysis in the bottom-half of the laminate. A novel procedure is proposed here to maintain necessary continuity across z = 0 plane in bending problems and a similar procedure in extension problems. Corresponding procedures in torsion problems which involve simple modifications are not presented.

A Bending problem

From analysis of the upper-half the laminate, σ_z = 0 and transverse shear stresses including first order corrections due to σ_{z1} in the in-plane constitutive relations along the reference plane z = 0 are

$$(\tau_{xz})_{z=0} = \tau_{xz0} + f_2(0)\, \tau_{xz2} + (\pi/2)\, [\tau^*_{xz2} + \tau_{xz2s}] \qquad (109a)$$
$$(\tau_{yz})_{z=0} = \tau_{yz0} + f_2(0)\, \tau_{yz2} + (\pi/2)\, [\tau^*_{yz2} + \tau_{yz2s}] \qquad (109b)$$

in which

$$\tau^*_{xz2} = \tau_{xz0} + (Q_{44}\, u_1 + Q_{45}\, v_1)_c \qquad (110a)$$
$$\tau^*_{yz2} = \tau_{yz0} + (Q_{54}\, u_1 + Q_{55}\, v_1)]_c \qquad (110b)$$

Continuation of the same analysis with $\bar{z}$ (= − z) ≥ 0, one obtains along $\bar{z}$ = 0 plane that normal stress $\bar{\sigma}_z$ = 0 and

$$\bar{\tau}_{xz} = \bar{\tau}_{xz0} + \bar{f}_2(0)\, \bar{\tau}_{xz2} + (\pi/2)\, [\, \bar{\tau}^*_{xz2} + \bar{\tau}_{xz2s}] \qquad (111a)$$
$$\bar{\tau}_{yz} = \bar{\tau}_{yz0} + \bar{f}_2(0)\, \bar{\tau}_{yz2} + (\pi/2)\, [\, \bar{\tau}^*_{yz2} + \bar{\tau}_{yz2s}] \qquad (111b)$$

in which

$$\bar{\tau}^*_{xz2} = \bar{\tau}^*_{xz0} + (\bar{Q}_{44}\bar{u}_1 + \bar{Q}_{45}\bar{v}_1)_c \qquad (112a)$$
$$\bar{\tau}^*_{yz2} = \bar{\tau}^*_{yz0} + (\bar{Q}_{45}\bar{u}_1 + \bar{Q}_{55}\bar{v}_1)_c \qquad (112b)$$

B Associated extension problem in bending

In the initial set of solutions, transverse shear stresses obtained along z = 0 plane are sum of the stresses in equations (110, 112). For continuity of these stresses across z = 0 interface, one has to consider the adjacent plies above and below the interface subjected to shear stresses

$$\tau_{xz}' = \pm [\bar{\tau}_{xz} - \tau_{xz}]_{z=0};\ \tau_{yz}' = \pm [\bar{\tau}_{yz} - \tau_{yz}]_{z=0} \quad (113)$$

Continuity of these stresses is ensured by adding solutions of the laminate with free top and bottom faces along with above stresses in the adjacent plies of the interface z = 0 to the solutions of problems in the initial set. Continuity of these stresses ensures also continuity of vertical displacement across z = 0 plane.

It is convenient to introduce the coordinate z' = (1− z) for (z ≥ 0) so that the reference plane z = 0 corresponds to z' = 1. $h_k' = 1- h_k$, interfaces $\alpha_k' = (1-\alpha_k)$. Here, q = 0 along z' = 1 and the faces z'=0 are free of transverse stresses. It is inconvenient to use linear $z'\tau_{xz}'$ along edges satisfying above face conditionssince the corresponding solutions for in-plane displacements $[u_0', v_0']$ from in-plane equilibrium equations are lamination independent thereby not satisfying continuity across interfaces. As such, edge conditions at x = constant edges (and analogous conditions along y = constant edges) are assumed in the form

$$\tau_{xz} = \tau_{xz}'(y) \sin (\pi z'/2) \quad (114)$$

Since $[\tau_{xz}', \tau_{yz}']$ are gradients $\alpha'[\psi_{1,x}, \psi_{1,y}]$ of a harmonic function ψ_1 and $\partial/\partial z = -\partial/\partial z'$, in-plane static equilibrium equations governing $[u_0', v_0'] \cos(\pi z'/2)$ are

$$\alpha'[Q_{1j}\,\varepsilon'_{j,x} + Q_{3j}\,\varepsilon'_{j,y}] = \alpha' (\pi/2)\,\psi_{1,x} \quad (115a)$$
$$\alpha'[Q_{2j}\,\varepsilon'_{j,y} + Q_{3j}\,\varepsilon'_{j,x}] = \alpha' (\pi/2)\,\psi_{1,y} \quad (115b)$$

Solutions of the above equations with zero bending and twisting stresses along edges give $[u_0', v_0']$ in each ply independent of lamination. Continuity of these displacements across interfaces is through recurrence relations

$$[u'_0{}^{(k)} - u'_0{}^{(k+1)}] \cos (\pi\alpha'_k/2) = [u'_0{}^{(k+1)} - u'_0{}^{(k)}] \quad (116)$$

Normal strain $\varepsilon'_z = \varepsilon'_{z0} \cos (\pi z'/2)$ from constitutive relation in which ε'_{z0} is given by

$$\varepsilon'_{z0} = S_{ij}\,\sigma'_j \quad (i, j = 1, 2, 3) \quad (117)$$

Vertical deflection $w' = -\varepsilon'_{z0}\,(2/\pi) \sin (\pi z'/2)$ from integration of ε'_z in the interior of the ply. This deflection along the interface is obtained from shear stress-strain and strain displacement relations in the form

$$\alpha\ w' = (\pi/2) \cos (\pi\alpha'_k/2) \int [(\varepsilon'_{40} - u'_0)\, dx + + (\varepsilon'_{50} - v'_0)\, dy] \quad (118)$$

Its continuity across interfaces is through recurrence relations

$$\alpha(\varepsilon'^{(k)} - \varepsilon'^{(k-1)})_{z0}\,(2/\pi) \sin (\pi\alpha'_k/2) = (\pi/2) \int [(\varepsilon'_{40} - u'_0)\, dx + (\varepsilon'_{50} - v'_0)\, dy]^{(k)} \cos (\pi\alpha'_k/2) \quad (119)$$

Similar analysis is to be carried out with $\bar{z}$ (= − z) ≥ 0 and $\bar{z}' = (1 - \bar{z})$ so that $\bar{z} = 0$ plane is $\bar{z}' = 1$ (this part of the analysis is omitted). By adding the above vertical displacements to the corresponding displacements obtained in the upper-half and bottom-half of the relevant symmetric laminate ensure continuity across reference plane. One can choose, in principle, any one interface (excluding faces of the laminate) as reference plane but from consideration of limitations of small deformation theory, it is better to choose either mid-plane or its adjacent interface as reference plane.

B Extension problem

Along the reference plane z = 0, w and transverse shear stresses are zero and in-plane displacements and σ_z are (sum n ≥ 1)

$$u = [u_0 + (u^*_2 + u_{2s}) + f_{2n}(0)\, u_{2n} \quad (120a)$$
$$v = [v_0 + [v^*_2 + v_{2s}] + f_{2n}(0)\, v_{2n} \quad (120b)$$
$$\sigma_z = q_0(x, y)/2 + [\sigma^*_{z2} + \sigma_{z2s}] + f_{2n}(0)\sigma_{z2n} \quad (121)$$

Continuation of the same analysis with $\bar{z}$ (= − z) ≥ 0, $\bar{u}$, $\bar{v}$ and $\bar{\sigma}_z$ along $\bar{z} = 0$ plane are

$$\bar{u} = \bar{u}_0 + [\bar{u}_2^* + \bar{u}_{2s}] + \bar{f}_{2n}(0)\bar{u}_{2n} \quad (122a)$$
$$\bar{v} = \bar{v}_0 + [\bar{v}_2^* + \bar{v}_{2s}] + \bar{f}_{2n}(0)\bar{v}_{2n} \quad (122b)$$
$$\bar{\sigma}_z = q_0(x, y)/2 + [\bar{\sigma}^*_{z2} + \bar{\sigma}_{z2s}] + \bar{f}_{2n}(0)\bar{\sigma}_{z2n} \quad (123)$$

Continuity of σ_z and in-plane displacements across z = 0 plane requires them to be same in the adjacent ply on each side of the z = 0 plane. For this purpose, one needs solutions of associated bending problems.

C Associated bending problem in Extension

In the initial set of solutions, in-plane displacements obtained along z = 0 plane are sum of the displacements in equations (122, 124). For continuity of these displacements across z = 0 interface, one has to consider the adjacent plies above and below the interface z = 0 with

$$u' = \pm [\bar{u} - u];\ v' = \pm [\bar{v} - v];\ \sigma_z' = \pm [\bar{\sigma}_z - \sigma_z] \quad (124)$$

Continuity of u, v and σ_z is ensured by adding solutions of the laminate with free top and bottom faces along with above displacements and σ_z' in the adjacent plies of the interface z = 0 to the solutions of problems in the initial set.

It is convenient to introduce the coordinate z' = (1− z) for (z ≥ 0) so that z'=1 is reference plane z = 0, $h_k' = 1 - h_k$ and $\alpha_k' = (1 - \alpha_k)$ are interfaces. Here, $\sigma_z' = [\bar{\sigma}_z - \sigma_z]$ and [u', v'] = $[(\bar{u} - u), (\bar{v} - v)]$ along z' = 1 plane and the faces z'=0 are free of transverse stresses. It is convenient to assume $\sigma_z' = [\bar{\sigma}_z - \sigma_z] \sin(\pi z'/2)$. Then, equation governing ψ' is $\alpha^2 \Delta \psi'_1 = \sigma_z'$ with $\partial/\partial z = -\partial/\partial z'$ and

$$\tau_{xz}'(\pi/2)\cos(\pi z'/2) = \alpha'\psi'_{1,x}(\pi/2)\cos(\pi z'/2) \quad (125a)$$
$$\tau_{yz}'(\pi/2)\cos(\pi z'/2) = \alpha'\psi'_{1,y}(\pi/2)\cos(\pi z'/2) \quad (125b)$$

Above equation is to be solved with zero normal gradient $(\psi'_1)_n$ along the edge of the plate.

In-plane static equilibrium equations governing $[u_0', v_0'] \sin(\pi z'/2)$ are

$$\alpha'[Q_{1j}\,\varepsilon'_{j,x} + Q_{3j}\,\varepsilon'_{j,y}] + \alpha'(\pi/2)^2\psi'_{1,x} = 0 \quad (126a)$$
$$\alpha'[Q_{2j}\,\varepsilon'_{j,y} + Q_{3j}\,\varepsilon'_{j,x}] + \alpha'(\pi/2)^2\psi'_{1,y} = 0 \quad (126b)$$

Solutions of the above equations with zero bending and twisting stresses along edges give $[u_0', v_0']$ in each ply independent of lamination. Continuity of these displacements and σ_z' across interfaces is through recurrence relations

$$[u'^{(k)}_0 - u'^{(k+1)}_0]\sin(\pi\alpha'_k/2) = [u'^{(k+1)}_0 - u'^{(k)}_0] \quad (127a)$$
$$[v'^{(k)}_0 - v'^{(k+1)}_0]\sin(\pi\alpha'_k/2) = [v'^{(k+1)}_0 - v'^{(k)}_0] \quad (127b)$$
$$[\sigma_z'^{(k)} - \sigma_z'^{(k+1)}]\sin(\pi\alpha'_k/2) = [\sigma_z'^{(k+1)} - \sigma_z'^{(k)}] \quad (128)$$

Similar analysis is to be carried out with $\bar{z}$ (= −z) ≥ 0 and $\bar{z}' = (1 - \bar{z})$ so that $\bar{z}$ = 0 plane is $\bar{z}'$ = 1 (this part of the analysis is omitted). By adding in-plane displacements and normal stress σ_z' with those in the analysis of the upper-half and bottom-half of relevant symmetric laminates ensure continuity across reference plane.

VIII. CONCLUDING REMARKS

An attempt is made here to present a proper sequence of sets of 2-D problems necessary for analysis of laminated plates within small deformation theory. Emphasis is on the usage of vertical displacement variable. If it is used as a domain variable, analysis corresponds to the solution of associated torsion problem in which normal strains are not zero unlike in the St. Venant's torsion problem. In bending and extension problems, it cannot be used as a domain variable. In the interior of the domain, it is from thickness-wise integration of normal strain ϵ_z. Zero vertical displacement along the edge of the plate is to be replaced by zero ϵ_z. Displacement w(x, y) arising out of integration of ϵ_z is to be obtained as a face variable from integration of transverse strain-displacement relations. Zero w(x, y) along the prescribed edge condition requires only the prevention of vertical movement of line segment of intersection of face and wall of the plate. 2-D variables associated in assumed polynomials in z of vertical displacement are virtual quantities in transverse shear strain-displacement relations but contribute to higher order corrections in stress components from integration of equilibrium equations.

Set of polynomials generated in z is necessary in satisfying both static and integrated equilibrium equations. (It is, however, not simple to develop software for generation of $f_k(z)$ functions and β_{2k+1}, necessary for application of the theory with thickness ratio varying up to unit value. This problem is avoided recently in [14]. *One significant feature of the present work is that the ply analysis is independent of lamination. Moreover, Corrections to the solutions at each stage of the adapted iterative procedure are determined without disturbing solutions in the preceding stages of iterations. This facility is not available with solutions from PEEES.*

The present theory needs exploitation in investigations on optimum ply lay-up, its utility in the analysis of associated eigen-value problems of free vibration and buckling of plates, and even in the area of fracture mechanics. However, polynomials in z are not adequate for proper solutions of 3-D problems. Solution of a supplementary problem based on appropriate trigonometric function in z representing each of displacement and stress components is required. Solution of additional similar problem is required in the analysis of unsymmetrical laminates.

Highlights of the present work

- 3-D equations in displacements and sequence of 2-D problems with vertical displacement as domain variable correspond to associated torsion problems.
- Kirchhoff's theory is a 0^{th} order shear deformation theory.
- FSDT and higher order shear deformation theories with shear correction factors deal with artificial torsion problems
- Extended Poisson's theory is necessary to rectify lacuna in Kirchhoff's theory.
- Extended Poisson's theory is based on satisfaction of both static and integrated equilibrium equations.
- Solutions of auxiliary and supplementary problems are necessary to rectify lacuna in Kirchhoff's theory.
- $f_k(z)$ functions are chosen such that ply analysis is independent of lamination
- A novel procedure is proposed for analysis of unsymmetrical laminates
- More or less uniform accuracy of face parallel plane deflections along each normal to the plane is achieved through solution of this secondary problem. *Such approximation to deformations of parallel planes is useful in the analysis of laminates embedded with piezoelectric actuators.*

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Nomenclature

$f_k(z)$ = functions in z

h_k, = interfaces, k = 1, 2…

$h_m = h_n$ = plate half-thickness.

L = Characteristic length in X-Y plane

m = number of plies below mid-plane z = 0 of the plate

n = number of plies above mid-plane z = 0 of the plate

$q_0(x, y)/2$ = prescribed $\sigma_z(x, y)$ along a face of the plate

Q_{ij} = Stiffness coefficients, (i, j = 1, 2. 3)

Q_{rs} = Stiffness coefficients, (r, s = 4, 5

S_{ij} = elastic compliances, (i, j = 1, 2, 3, 6)

S_{rs} = elastic compliances, (r, s = 4, 5)

$T_{x1}(y)$, $T_{xy1}(y)$ = prescribed stresses at an x = constant edge

U, V, W = displacements in X, Y, Z directions, respectively

u, v, w = U/h_n, V/h_n, W/h_n, respectively

$[u, v, w]^u$ = displacements in z = h_k plane

$[u, v, w]^b$ = displacements in z = h_{k-1} plane

O-X Y Z = Cartesian coordinate system

x, y, z = X/L, Y/L, Z/L, respectively

$\bar{z}$ = − z, below mid-plane z = 0 of the plate

α = h_n/L

α_k = h_k/h_n

β_e = z-distribution correction factor in extension problem

β_{2k-1} = $f_{2k+1}(\alpha_k)/[\alpha_k^2\ f_{2k-1}(\alpha_k)]$, k = 1, 2, ...

Δ = plane Laplace operator $(\partial^2/\partial x^2 + \partial^2/\partial y^2)$

$\varepsilon_x, \varepsilon_y, \varepsilon_z$ = ε_i, (i = 1, 2, 3)

$\gamma_{xy}\ \gamma_{xz}\ \gamma_{yz}$ = ε_{3+i}, (i = 1, 2, 3)

$\sigma_x, \sigma_y, \tau_{xy}$ = σ_i, (i = 1, 2, 3)

$\tau_{xz}, \tau_{yz}, \sigma_z$ = σ_{3+i}, (i = 1, 2, 3)

$\tau_{xzs}, \tau_{yzs}, \sigma_{zs}$ = transverse stresses in supplementary problems

$[\tau_{xz0}, \tau_{yz0}]$ = $\alpha\ [\psi_{0,x}, \psi_{0,y}]$

ψ, φ = variables in face parallel x-y planes

τ_{xz}', τ_{yz}' = shear stresses in mid- plane of unsymmetrical laminates

ω_z = $\alpha(v_{,x} - u_{,y})$, rotation of inclined line about z-axis

Appendix: Homogeneous isotropic plate

Table 1: Face and Neutral Plane Displacements

$w_f = (E/2q_0)\ w\ (1/2, 1/2, 1)$, $w_n = (E/2q_0)\ w\ (1/2, 1/2, 0)$; $\alpha = 1/6$, $\nu = 0.3$.

	1	2	3	4	5	6	7	8
w_f	4.12	4.17	2.27	2.54	3.72	3.69	4.07	4.07
w_n	3.49	4.49	2.27	2.54	3.72	3.41[a]	4.36	4.46

First row: 1. *Exact: w(x, y, z) as domain variable*, 2.*Exact: w as face variable*, 3. *Kirchhoff's theory*, 4. *Poisson's theory*, 5. *Poisson's theory with one iteration*, 6. *FSDT ([a] without k^2)*, 7. *Extended Poisson's theory (without ε_{z1})* 8.*Extended Poisson's theory (with ε_{z1}). (Results in Poison's theories are based with $\varphi_1 \equiv 0$.*