Smeared Laminate Theory Of Unsymmetrical Laminated Plates: Use Of Extended Poisson Theory

Smeared Laminate Theory

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***Abstract*—Importance of transverse normal stress neglected in the classical theories is discussed for finding its proper thickness-wise distribution which is essential in the analysis of laminated plates. In the presence of vertical load along faces of the plate, it is initially shown that the use of complete set of mutually orthogonal functions like sine and cosine functions, Legendre polynomials, etc., in the thickness-wise distribution of displacements gives only the neglected transverse normal stress. The present work is based on the presumption that the author's recently proposed extended Poisson's theory is the only suitable theory along with solution of a supplementary problem to overcome lacuna in the classical theories of primary plate problems. Based on this theory, an alternate form of classical laminate plate theory (CLPT) is proposed for analysis of unsymmetrical laminated plates.**

Keywords—Elasticity; Plates; Laminates; Bending; Torsion; Extension

I. INTRODUCTION

The present work is a somewhat modest contribution to add something new based on experience gained through recent proposal of extended Poisson's theory in the analysis of bending of plates [1, 2]. In the energy methods based on stationary property of relevant total potential, equations governing 2D variables correspond to plate element equilibrium equations (PEEES). With assumed in-plane displacements and/or stresses, it becomes mandatory to use integration of equilibrium equations for obtaining converging solutions to displacements and, in particular, to *transverse* stresses. In Poisson's theory and its extension, solutions to displacements satisfy both static and integrated equilibrium equations.

In extension (stretching) problems, basic variables are u_0 and v_0 and the classical theory in which transverse stresses are zero is without any major defect as a first approximation to the 3D problem. *This is mainly due to the fact that static and z-integrated equations governing [u_0, v_0] remain unaltered.* Exception is in the error in the transverse shear stress-strain relations due to ε_z from constitutive relation. Extension of this theory in the presence of transverse stresses is discussed later.

Kirchhoff's theory [3] of plates in bending is a single variable $w_0(x, y)$ model based on plate element equilibrium equations (PEEES). Recent proposals of Poisson's theory and extended Poisson's theory [1, 2] are with one term representation of each displacement component. They are basically different from Kirchhoff's theory. In-plane displacements are determined from satisfaction of both static and z-integrated equations of the 3D infinitesimal element. Basic variable w_0 is treated as face variable in bending problem and as domain variable in the associated torsion problem. In all these theories, 2D variables in the in-plane displacements are coefficients of z. The term z is the first term of necessary odd functions of polynomials in z in a complete set of coordinate functions to express z-distributions of in-plane displacements.

In the context of present work, one should note that Touratier [4] used trigonometric function without replacing z and retained $w_0(x, y)$ as domain variable in PEEES. In view of extended Poisson's theory in which w_0 is a face variable and a theory based on one term representation of displacements, implication of replacing z with *sin (z)* function is discussed even with reference to homogeneous isotropic plates. It is primarily intended to show the necessity of retaining neglected σ_z in the constitutive relations.

II. PRELIMINARIES

The present work is initially confined to establish the relevance of extended Poisson's theory for the analysis of bending, extension and associated torsion problems. For this purpose and for simplicity in presentation [1, 5], a square plate bounded within $0 \le X, Y \le a$, $-h \le Z \le h$ with reference to Cartesian coordinate system (X, Y, Z) is considered. Material of the plate is homogeneous and isotropic with elastic constants *E* (Young's modulus), ν (Poisson's ratio) and *G* (Shear modulus) that are related to one other by $E = 2(1+\nu) G$. For convenience, coordinates X, Y, Z and displacements *(U, V, W)* in non-dimensional form x= X/a, y=Y/a, z=Z/h, *(u, v, w) = (U, V, W)/h* and half- thickness ratio α = (h/a) are used.

With the above notation, equilibrium equations in stress components are:

$$\alpha\,(\sigma_{x,x} + \tau_{xy,y}) + \tau_{xz,z} = 0 \qquad (1a)$$
$$\alpha\,(\sigma_{y,y} + \tau_{xy,x}) + \tau_{yz,z} = 0 \qquad (1b)$$
$$\alpha\,(\tau_{xz,x} + \tau_{yz,y}) + \sigma_{z,z} = 0 \qquad (2)$$

in which suffix after ',' denotes partial derivative operator. Classical theory of extension problems deals with the two in-plane equilibrium equations of infinitesimal element where as Kirchhoff theory deals mainly with the equation of transverse stresses through PEEES.

In displacement based models, stress components are expressed in terms of displacements, via, six stress-strain constitutive relations and six strain-displacement relations. These relations within the classical small deformation theory of elasticity are:

Strain-stress and semi-inverted stress-strain relations:

$$E\,\varepsilon_x = \sigma_x - \nu\,(\sigma_y + \sigma_z) \qquad (3a)$$
$$E\,\varepsilon_y = \sigma_y - \nu\,(\sigma_x + \sigma_z) \qquad (3b)$$
$$E\varepsilon_z = \sigma_z - \nu\,(\sigma_x + \sigma_y) \qquad (3c)$$
$$G\,[\gamma_{xy}, \gamma_{xz}, \gamma_{yz}] = [\tau_{xy}, \tau_{xz}, \tau_{yz}] \qquad (4)$$
$$\sigma_x = E'(\varepsilon_x + \nu\,\varepsilon_y) + \mu\,\sigma_z \qquad (5a)$$
$$\sigma_y = E'(\varepsilon_y + \nu\,\varepsilon_x) + \mu\,\sigma_z \qquad (5b)$$
$$\varepsilon_z = -\mu\,e + (1 - 2\,\nu\,\mu)\,\sigma_z/E \qquad (6)$$

in which $e = (\varepsilon_x + \varepsilon_y)$, $E' = E/(1-\nu^2)$ and $\mu = \nu/(1-\nu)$.

Strain-displacement relations:

$$[\varepsilon_x, \varepsilon_y, \varepsilon_z] = [\alpha u_{,x}, \alpha v_{,y}, w_{,z}] \qquad (7)$$
$$\gamma_{xy} = \alpha u_{,y} + \alpha v_{,x} \qquad (8a)$$
$$[\gamma_{xz}, \gamma_{yz}] = [u_{,z} + \alpha w_{,x}, v_{,z} + \alpha w_{,y}] \qquad (8b)$$

In-plane equilibrium equations in terms of displacements are

$$E'[\alpha^2\Delta u - ½(1+\nu)\,\alpha^2(v_{,x} - u_{,y})_{,y}] + \mu\,\alpha\sigma_{z,x} + \tau_{xz,z} = 0 \qquad (9a)$$
$$E'[\alpha^2\Delta v + ½(1+\nu)\,\alpha^2(v_{,x} - u_{,y})_{,x}] + \mu\,\alpha\sigma_{z,y} + \tau_{yz,z} = 0 \qquad (9b)$$

Polynomials in z to express thickness-wise distributions of displacements are generated earlier [5], for convenience, from recurrence relations with $f_0 = 1$, $f_{2k+1,z} = f_{2k}$, $f_{2k+2,z} = -f_{2k+1}$ such that $f_{2k+2}(\pm 1) = 0$. They are, (up to k = 5),

$$[f_1, f_2, f_3] = [z, ½(1 - z^2), ½(z - z^3/3)] \qquad (10a)$$
$$f_4 = (5 - 6z^2 + z^4)/24 \qquad (10b)$$
$$f_5 = z\,(25 - 10z^2 + z^4)/120 \qquad (10c)$$

In order to keep associated 2-D variables of transverse stresses as free variables, f_{2k+1} are replaced with f^*_{2k+1} given by

$$f^*_{2k+1} = f_{2k+1} - \beta_{2k-1}\,f_{2k-1},\ k = 1, 2, \ldots.. \qquad (11)$$

in which $\beta_{2k-1} = [f_{2k+1}(1)/f_{2k-1}(1)]$ so that $f^*_{2k+1}(1) = 0$.

A. Some observations on the classical theories of extension and bending problems

In the classical theory of extension problems, the plate with its faces free of transverse stresses is in a state of plane stress. In-plane equilibrium equations become

$$\alpha\,(\sigma_{x,x} + \tau_{xy,y}) = 0\ ,\ \alpha\,(\sigma_{y,y} + \tau_{xy,x}) = 0 \qquad (12)$$

to be solved with two conditions at each of x = constant edge (with analogue conditions along y = constant edges) in the primary problem

$$\text{(i)}\ u = \tilde{u}_0(y)\ \text{or}\ \sigma_{x0}(y) = T_{x0}(y) \qquad (13a)$$
$$\text{(ii)}\ v = \tilde{v}_0(y)\ \text{or}\ \tau_{xy0}(y) = T_{xy0}(y) \qquad (13b)$$

If the edge conditions in (13) and similar conditions at y = constant edges correspond to stress components, it is convenient to express them in terms of second order derivatives of well-known and widely used Airy's stress function Φ(x, y) such that equations (1, 2) are identically satisfied. In such a case, stress components are independent of material constants, thereby, elastic deformations. Stress components are

$$\sigma_x = \Phi_{,yy}\ ,\ \tau_{xy} = -\,\Phi_{,xy}\ ,\ \sigma_y = \Phi_{,xx} \qquad (14)$$

Airy's function Φ is governed by the bi-harmonic equation $\Delta\Delta\Phi = 0$ $(\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2)$ from compatibility condition and the strains are compatible if $\epsilon_{x,yy} - 2\gamma_{xy,xy} + \epsilon_{y,xx} = 0 \Longrightarrow \Delta\Delta\,\Phi = 0$.

In-plane strains from constitutive relations are dependent on material constants and displacements $[u_0, v_0]$ are determined from strain-displacement relations. (Such a simplistic procedure is not convenient in the analysis of laminated plates and if the material is orthotropic or anisotropic even in the case of homogeneous plates.)

Equations corresponding to (12) are

$$(E'/3)\,[\alpha^2\Delta u_0 - ½(1+\nu)\,\alpha^2\,(v_{0,x} - u_{0,y})_{,y}] = 0 \qquad (15a)$$
$$(E'/3)\,[\alpha^2\Delta v_0 + ½(1+\nu)\,\alpha^2\,(v_{0,x} - u_{0,y})_{,y}] = 0 \qquad (15b)$$

to be solved with edge conditions (13).

In the corresponding bending problem, in-plane displacements $[zu_1, zv_1]$ are governed by the same set of above equations with [u_1, v_1] replacing [u_0, v_0]. In Kirchhoff's theory, [u_1, v_1] are expressed as gradients of a single function $w_0(x, y)$ from zero transverse shear strains. As such, static equations (1) have to be ignored due to two equations governing a single variable w_0 and the required single equation through PEEES is given by the bi-harmonic equation $\Delta\Delta w_0 = 0$. However, tangential edge condition is replaced by an artificial condition. This problem does not arise in the extension problem due to the absence of w_0. However, it is possible to express [u_0, v_0] in the form of gradients of a single function.

Like in the Poisson's theory and its extension [1, 2], [u_0, v_0] are expressed in terms of gradients of two functions $\psi_0(x, y)$ and $\varphi_0(x, y)$ in the form [u_0, v_0] $=\alpha$ [$\psi_{0,x} + \varphi_{0,y}$, $\psi_{0,y} - \varphi_{0,x}$]. By simple manipulations of equations (15) with derivatives of these equations, $\psi_0(x, y)$ and $\varphi_0(x, y)$ are governed by two uncoupled bi-harmonic equations. Since only two edge conditions are prescribed, one obtains solution for ψ_0 with $\varphi_0 \equiv 0$ or vice versa. Solutions thus obtained are one and the same for [u_0, v_0] so that [u_0, v_0] = [$\psi_{0,x}$, $\psi_{0,y}$] or [$\varphi_{0,y}$, $-\varphi_{0,x}$]. (Note that the solution is a combination of that of Laplace equation and the corresponding Poisson equation. *Due to the later equation, harmonic parts of ψ_0 and φ_0 need not be conjugate to each other*)

Replacement of w_0 in Kirchhoff's theory with $\boldsymbol{\psi}_1$

Similar analysis with suffix '$_0$' replaced with suffix '$_1$' is missing, rather surprisingly, in the bending problem in the reported literature (e.g., [6]). Kirchhoff's theory [3] is used even in the absence of transverse stresses whereas replacing $w_0(x, y)$ with ψ_1 is proper and there was no need to use artificial transverse shear resultants along the edges. Use of ψ_1 instead of φ_1 is proper since φ_1 does not participate in (2) in the presence of transverse stresses. In this case, however, $\varphi_1 \not\equiv 0$ since it exists as a harmonic function in the in-plane displacements necessary to satisfy three edge conditions in the 3D problem like in the proposed Poison's theories (this function φ_1 was originally introduced as 'stress function' by Reissner [7]). It is now clear that if one considers harmonic part of bi-harmonic ψ_1, same set of harmonic functions has to be used in φ_1 so as to replace tangential edge displacement from ψ_1 with normal gradient of φ_1. As such, in-plane displacements are in terms of gradients of [ψ_1, φ_1] and there is no need to use gradients of φ_1 in transverse shear stresses. It justifies the extended Poison's theory for proper initial solutions in the analysis of bending problems.

On transverse stresses in extension problem

If the prescribed transverse shear stresses along faces of the plate in the extension problem are gradients of a given potential function $\tilde{\psi}_1(x, y)$ other than ϵ_{z0} from constitutive relation, they can be absorbed in the normal stress components in the above analysis. If one uses gradients of ϵ_{z0} consistent with $zw_1(x, y)$, then ϵ_{z0} becomes zero from the analysis implying that w_1 cannot be used as a domain variable. Moreover, w_1 cannot be a priory prescribed condition along an edge of the plate. Zero ϵ_{z0} implies that the problem is equivalent to the plate subjected to pure in-plane shear and the rotation (of infinitesimal element about the normal to the plate) $\omega_z = \alpha[v_{0,x} - u_{0,y}] = \alpha^2\Delta\varphi_0 \equiv 0$ so that φ_0 is a solution of Laplace equation in place of bi-harmonic equation.

III. SEQUENCE OF UNCOUPLED 2-D PROBLEMS

Here, relevance of sequence of uncoupled 2D static equations from expressing *f(z)* in Fourier series of trigonometric functions is examined. In the previous section, it is mentioned that the analysis is similar in extension and bending problems if the faces of the plate are free of transverse stresses. Such a similarity exists even in the case of prescribed stresses along the faces of the plate but slightly different in determining basic variables [u_0, v_0, w_0].

It is known to be mandatory to satisfy thickness-wise integrated equilibrium equations with assumed in-plane displacements [u, v] in bending problems. This procedure can be reversed by the z-distribution of [u, v] obtained from integration of *f(z)* distribution of transverse shear stresses so that [u, v] are determined from static equilibrium equations avoiding PEEES. For this purpose, possible expansion of σ_z considered here in series in terms of cosine and sine functions in the extension and bending problems, respectively, with $\lambda_n = 2/[(2n\text{-}1)\pi]$, (sum n = 1, 2, …..) are

$$\tilde{\sigma}_{ze} = \sigma_{z0e} + \cos(z/\lambda_n)\, \sigma_{z2n} \text{ (extension problem)} \tag{16a}$$

$$\tilde{\sigma}_{zb} = \sin(z/\lambda_n)\, \sigma_{z2n\text{-}1} \text{ (bending problem)} \tag{16b}$$

In the extension problem, face condition gives $\sigma_{z0e} = q_0/2$ and it is connected with other 2D variables through constitutive relations only and σ_{z2n} need not be zero. Since σ_{z0e} by itself does not participate in (2), it is omitted in the subsequent integration process.

In the bending problem, face condition can be satisfied by any one 2D variable $\sigma_{z2n\text{-}1}$. Choice of $\sigma_{z2n\text{-}1}$ is dependent on the appropriate z-distribution of edge conditions. Like in the primary extension problem, relevant 2D variables with $w_0(x, y)$ as face variable satisfy both static and integrated equilibrium equations, thereby, in complete conformity with 3D equations.

Extended Poisson's theory is, however, based on $\sigma_z = z\, q_1/2$ associated with more practical

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reactive or prescribed transverse shear, in place of parabolic or cosine distribution, along edges of the plate. In such a case, the above sinusoidal series becomes the expansion of z q_1/2. Note that σ_{z0} and σ_{z1} participate in the in-plane equilibrium equations through semi-inverted constitutive relations.

For convenience, transverse shear stresses by z-integration of (16) are expressed as

$$[\tilde{\tau}_{xz}, \tilde{\tau}_{yz}]_e = \sum \lambda_n \sin(z/\lambda_n)\, [\tilde{\tau}_{xz}, \tilde{\tau}_{yz}]_{2n} \quad (17a)$$
$$[\tilde{\tau}_{xz}, \tilde{\tau}_{yz}]_b = [\tilde{\tau}_{xz}, \tilde{\tau}_{yz}]_{0b} + \sum \lambda_n \cos(z/\lambda_n)\, [\tilde{\tau}_{xz}, \tilde{\tau}_{yz}]_{b2n-1} \quad (17b)$$

It is obvious that [τ_{xz0b}, τ_{yz0b}] are from solutions of auxiliary problem used in the extended Poison's theory. By z-integration of [$u_{,z}$, $v_{,z}$] in [$\tilde{\tau}_{xz}$, $\tilde{\tau}_{yz}$], [u, v] apart from [u_0, v_0] in the extension problems are assumed for convenience in the form

$$[u, v]_e = \sum \lambda_n^2 [u, v]_{2n} \cos(z/\lambda_n) \quad (18a)$$
$$[u, v]_b = \sum \lambda_n^2 [u, v]_{2n-1} \sin(z/\lambda_n) \quad (18b)$$

(Note that term by term differentiation of u_b (or v_b) is valid whereas it is not valid in the expansion of zu_b (or z v_b) in the Fourier expansion in terms of the corresponding sine functions.)

The components [$\tilde{\tau}_{xz}$,$\tilde{\tau}_{yz}$]$_{,z}$ in the equilibrium equations are

$$[\tilde{\tau}_{xz}, \tilde{\tau}_{yz}]_{e,z} = \Sigma[\tilde{\tau}_{xz}, \tilde{\tau}_{yz}]_{e2n} \cos(z/\lambda_n) \quad (19a)$$
$$[\tilde{\tau}_{xz}, \tilde{\tau}_{yz}]_{b,z} = -\Sigma[\tilde{\tau}_{xz}, \tilde{\tau}_{yz}]_{b2n-1} \sin(z/\lambda_n) \quad (19b)$$

It is convenient to express 2D variables in (18) in the form

$$[u, v]_n = (-1)^n \alpha\, [\psi_{n,x} + \varphi_{n,y}, \psi_{n,y} - \varphi_{n,x}] \quad (20)$$

One gets equations governing 2D variables with [$\tilde{\tau}_{xz}$, $\tilde{\tau}_{yz}$]$_n$ = α [$\tilde{\psi}_{n,x}$, $\tilde{\psi}_{n,y}$]

$$E'[\alpha^2 \Delta u - \tfrac{1}{2}(1+\nu)\,\alpha\,(v_{,x} - u_{,y}]_n = \alpha\, \tilde{\psi}_{n,x} \quad (21a)$$
$$E'[\alpha^2 \Delta v + \tfrac{1}{2}(1+\nu)\,\alpha\,(v_{,x} - u_{,y}]_n = \alpha\, \tilde{\psi}_{n,y} \quad (21b)$$

In (21), odd and even 'n' correspond to bending and extension problems, respectively. From (1, 2), one gets

$$E'\alpha^2 \Delta\, e_n + \sigma_{zn} = 0, \quad \alpha^2 \Delta \omega_{zn} = 0 \quad (22)$$

Replacing e_n with $\alpha^2\Delta\psi$, one gets bi-harmonic equation governing ψ to be solved along with harmonic function φ satisfying three edge conditions.

Expansion of z-distribution of σ_z in terms of either power series or polynomials in z is artificial in the analysis based on PEEES in the energy methods. This is due to ignoring σ_z = q/2 in extension problem and σ_z = z q/2 in the bending problem in the classical and higher order theories. (Even the Poisson's theory of bending of plates is based on neglecting σ_z = z q/2 to resolve Poisson-Kirchhoff boundary conditions paradox).

In bending problems, any *f(z)* function with *f(1)* = 1 can be used but restricted due to priory prescribed practical transverse shear stresses along the segments of the edge of the plate corresponding to the above mentioned linear distribution of σ_z. Higher order theories result only in the series expansion of neglected σ_z = z q/2 in the constitutive relations.

It is clear now that the extended Poisson's theory in conjunction with adapted iterative procedure is the proper procedure for analysis of plates till now to generate proper sequence of 2-D problems converging to 3-D problem within the classical small deformation theory of elasticity.

IV. UNSYMMETRICAL LAMINATES: ALTERNATE FORM OF CLPT

In view of limitations of theories based on plate element equations, we confine here to the use of extended Poisson's theory in a smeared laminate theory for analysis of unsymmetrical laminates. Classical theory of laminated plates (CLPT) is a smeared laminate theory in which number of 2-D variables is independent of number of layers. It offers great advantage in reducing computational effort involved in layer-wise theories. It is useful in local-global approach for analysis of critical areas.

Assumed bending displacements in CLPT are either from Kirchhoff's theory or from First Order Shear Deformation Theory (FSDT). In Kirchhoff's theory, zero transverse shear conditions along faces of the laminate are priory satisfied from strain-displacement relations. *In the case of unsymmetrical laminates, however, transverse shears from integrated equilibrium equations do not satisfy at one of the faces or maintain continuity across mid-plane of the laminate*. Moreover, edge conditions involve integrated stress resultants which have no unique point-wise distribution, thereby the need of post-processing for obtaining transverse stresses. FSDT requires ply-wise shear correction factors and it is shown to be an approximation to the associated torsion problem [8, 9].

In the present work, an alternate form of CLPT is proposed to show that bending stresses in the case of unsymmetrical laminate are uncoupled from primary stresses in the extension problem. Extended Poisson's theory (EPT) is adapted to eliminate post processing for finding transverse stresses and a supplementary problem is formulated to maintain continuity of these stresses across interfaces of the unsymmetrical layup of plies.

For simplicity in presentation, a laminate bounded by $0 \leq X \leq a$, $0 \leq Y \leq b$ and $-h_m \leq Z \leq h_n$ with reference to the mid-plane $Z = 0$ in the Cartesian coordinate system (X, Y, Z) is considered. It is convenient to consider positive direction of Z- axis in the upward and downward directions so that $0 \leq Z \leq h_n$ and $0 \leq Z \leq h_m$ in the upper and bottom halves of the laminate, respectively. Here, $h_m = h_n$ but number of plies m need not be equal to n. For convenience, coordinates X, Y, and Z and displacements *U*, *V*, and *W* in non-dimensional form $x= X/L$, $y=Y/L$, $z=Z/h_n$, $u=U/h_n$, $v=V/h_n$, $w=W/h_n$ and half-thickness ratio $\alpha = h_n/L$ with reference to a characteristic length L [mod(x, y) ≤ 1] are utilized. Material of each ply is homogeneous and anisotropic with monoclinic symmetry. Interfaces are given by $z = \alpha_k = h_k/h_n$ (k = 1, 2, …., n-1) in the upper-half and $\alpha_k = h_k/h_m$ (k = 1, 2, …., m-1) in the bottom-half of the laminate, respectively.

Initial sets of solutions in the upper-half and bottom-half of the laminate in the layer-wise theory for analysis of bending of symmetric laminates [1, 2] and a general problem of unsymmetrical laminates in the article submitted elsewhere are unaltered up to the reference plane z = 0. In the case of unsymmetrical laminates, one has to consider continuity of non-zero displacements and transverse stresses across reference plane. Due to unsymmetrical layup, these sets of solutions in the upper-half and bottom-half of the laminate will be different to each other. A novel procedure was proposed to maintain their continuity across z = 0 plane in the above mentioned article in which derived 2-D problems of bending, extension and torsion in each ply are independent of lamination. Interface continuity of displacements and transverse stresses is through solutions of appropriate supplementary problems in the face ply and recurrence relations across interfaces. It is shown that *methods of analysis* of these problems are mutually exclusive to one other.

Vertical displacement is *w(x, y, z)* and in-plane displacements [u, v] are denoted as $[u_i]$, (i = 1, 2). Similarly, stress components $[\sigma_x, \sigma_y, \tau_{xy}]$ and $[\tau_{xz}, \tau_{yz}, \sigma_z]$ are denoted as $[\sigma_i]$, $[\sigma_{3+i}]$, (i = 1, 2, 3), respectively. With the corresponding notation for strains, strain-stress and semi-inverted stress-strain relations with usual summation convention in which repeated suffix indicates summation over its specified range of integers are

$$\varepsilon_i = S_{ij}\,\sigma_j \quad (i, j = 1, 2, 3, 6) \qquad (23)$$
$$\varepsilon_r = S_{rs}\,\sigma_s \quad (r, s = 4, 5) \qquad (24)$$
$$\sigma_i = Q_{ij}[\varepsilon_j - S_{j6}\,\sigma_z] \quad (i, j = 1\text{-}3) \qquad (25)$$
$$\sigma_r = Q_{rs}\,\varepsilon_s \quad (r, s = 4, 5) \qquad (26)$$

S_{ij} and Q_{ij} (i, j = 1, 2, ….6) are elastic compliance and stiffness coefficients, respectively. With σ_i in (25), in-plane equilibrium equations are

$$\alpha\,[Q_{1j}(\varepsilon_j - S_{j6}\,\sigma_z)_{,x} + Q_{3j}(\varepsilon_j - S_{j6}\,\sigma_z)_{,y}] + \tau_{xz,z} = 0 \qquad (27a)$$
$$\alpha\,[Q_{2j}(\varepsilon_j - S_{j6}\,\sigma_z)_{,y} + Q_{3j}(\varepsilon_j - S_{j6}\,\sigma_z)_{,x}] + \tau_{yz,z} = 0 \qquad (27b)$$

Displacements in the classical theories are in the form $[w_0, -\alpha z w_{0,x}, -\alpha z w_{0,y}]$ and $[u_0, v_0]$ in pure bending and extension problems, respectively. Coupling between extension and bending problems is in the case of unsymmetrical laminates. This unsymmetrical lay-up of laminates may be classified into two groups: (1) Number of plies 'm' in the bottom-half is different from the number of plies 'n' in the upper-half of the laminate. (2) Even if m = n, thickness and/or material properties in a k^{th} ply are different from each other in these parts of the laminate. This coupling between extension and bending problems is through 'B' matrix in PEEEs due to thickness-wise integration of products of even and odd z-functions. This 'B' matrix imposes same order of effect on extension and bending displacements.

An alternate form of CLPT designated as ACLPT is proposed here to differentiate this effect in extension and bending problems. In-plane displacements are assumed, with δ = 1 but zero for symmetric laminates, in the form

$$[u, v] = [u_0, v_0] + (\tfrac{1}{2}\,\delta - z)\,[u_1, v_1] \text{ (bending)} \qquad (28)$$
$$[u, v] = [1 - \delta\,(\tfrac{1}{3} + z)][u_0, v_0] + z\,[u_1, v_1 \text{ (extension)} \qquad (29)$$

Equations governing $[u_1, v_1]$ in bending and $[u_0, v_0]$ in extension problems using extended Poisson's theory are considered here.

A. Bending problem

Transverse stresses from auxiliary problem with $\alpha^2\,\Delta\psi_0 + q_1/2 = 0$ are

$$\sigma_z = z\,q_1/2 \qquad (30a)$$
$$[\bar{\tau}_{xz0}, \bar{\tau}_{yz0}] = \alpha[\psi_{0,x}\ \psi_{0,y}] \qquad (30b)$$

In-plane displacements are represented in the form $[u, v] = -(\tfrac{1}{2}\,\delta - z)\,[u_1, v_1]^*$ in which

$$u_1^* = u_1 + \gamma_{xz0} - \alpha\,w_{0,x} \qquad (31a)$$
$$v_1^* = v_1 + \gamma_{yz0} - \alpha\,w_{0,y} \qquad (31b)$$

Transverse shear strains, with $[\gamma_{xz0}, \gamma_{yz0}] = [(S_{44}\bar{\tau}_{xz0} + S_{45}\bar{\tau}_{yz0}), (S_{55}\bar{\tau}_{yz0} + S_{45}\bar{\tau}_{xz0})]$, are

$$\gamma_{xz}^* = [u_1 + \gamma_{xz0}]\ ,\quad \gamma_{yz}^* = [v_1 + \gamma_{yz0}] \qquad (32)$$

Correspondingly, transverse shear stresses are

$$\tau_{xz}^* = [Q_{44}u_1 + Q_{45}v_1 + \bar{\tau}_{xz0}] \qquad (33a)$$

$$\tau_{yz}^* = [Q_{55}v_1 + Q_{45}u_1 + \hat{\tau}_{yz0}] \quad (33b)$$

Transverse shear stresses associated with $[u_1, v_1]$:

$$\tau_{xz0} = [Q_{44}\, u_1 + Q_{45}\, v_1] \quad (34a)$$
$$\tau_{yz0} = [Q_{55}\, v_1 + Q_{45}\, u_1] \quad (34b)$$

Transverse stresses in each ply:

$$[\tau_{xz}, \tau_{yz}] = [\tau_{xz}, \tau_{yz}]^* + f_2^k\, [\tau_{xz2}, \tau_{yz2}]^k \quad (35)$$
$$\sigma_z = z\, q_1/2 + [f_3\, \sigma_{z3}]^k \quad (36)$$

The function f_2 in Eq. (35) and f_3 in Eq. (36) are

$$f_2 = \tfrac{1}{2}\, [(\alpha_k - z)\, \delta - (\alpha_k^2 - z^2)] \quad (37a)$$
$$f_3 = \tfrac{1}{2}[(\alpha_k z - ½\, z^2)\, \delta - (\alpha_k^2 z - ⅓\, z^3)] \quad (37b)$$

$f_3(z)$ in each ply is replaced with $\breve{f}_3(z)$ given by

$$\breve{f}_3(z) = [(\alpha_k z - ½\, z^2)\, \delta - (\alpha_k^2 z - ⅓\, z^3)] - \beta_1 \alpha_k^2\, z \quad (38)$$

in which $\beta_1 = [½\, \delta - ⅔]$ so that $\breve{f}_3(\alpha_k) = 0$.

From equilibrium equation in transverse stresses, one gets

$$\alpha\, [\tau_{xz0},_x + \tau_{yz0},_y] = \beta_1 \alpha_k^2\, \sigma_{z3} \quad (39a)$$
$$\sigma_{z3} = \alpha\, [\tau_{xz2},_x + \tau_{yz2},_y] \quad (39b)$$

To satisfy integrated equilibrium equations, it is convenient to assume

$$[u_1, v_1] = -\, [(\psi_{1,x} + \varphi_{1,y}), (\psi_{1,y} - \varphi_{1,x})] \quad (40)$$

Vertical deflection w_0 in $[u_1, v_1]^*$ is replaced by ψ_1 in [2} so that

$$u_1^* = -\alpha\, (2\psi_{1,x} + \varphi_{1,y}) + \gamma_{xz0} \quad (41a)$$
$$v_1^* = -\alpha(2\psi_{1,y} - \varphi_{1,x})] + \gamma_{yz0}\,] \quad (41b)$$

Reactive transverse stresses are

$$\tilde{\tau}_{xz2} = \alpha\, [Q_{1j}\, (\tilde{\varepsilon}_j - S_{j6}\, \sigma_{z1}),_x + Q_{3j}\, (\tilde{\varepsilon}_j - S_{j6}\, \sigma_{z1}),_y] \quad (42a)$$
$$\tilde{\tau}_{yz2} = \alpha\, [Q_{2j}\, (\tilde{\varepsilon}_j - S_{j6}\, \sigma_{z1}),_y + Q_{3j}\, (\tilde{\varepsilon}_j - S_{j6}\, \sigma_{z1}),_x] \quad (42b)$$
$$\tau_{xz2}^* = \tilde{\tau}_{xz2} + \tau_{xz0}\ ,\quad \tau_{yz2}^* = \tilde{\tau}_{yz2} + \tau_{yz0} \quad (43)$$
$$\sigma_{z3}^* = \sigma_{z3} + \sigma_{z1} \quad (44a)$$
$$\sigma_{z3} = -\, \alpha\, (\tilde{\tau}_{xz2,x} + \tilde{\tau}_{yz2,y}) \quad (44b)$$

Equation governing in-plane displacements (u_1, v_1) is given by

$$\alpha\, \beta_1 \alpha_k^2\, (\tilde{\tau}_{xz2,x} + \tilde{\tau}_{yz2,y}) = \alpha\, [\tau_{xz0},_x + \tau_{yz0},_y] \quad (45)$$

Equation (45) is a fourth order equation in ψ_1 to be solved along with plane Laplace equation $\Delta\varphi_1 = 0$. In the above analysis, ply-wise equilibrium equations are satisfied independent of lamination. In-plane displacements $[u, v]$ thus obtained are dependent on material constants in each ply.

In ACLPT, $[u_1, v_1]$ become dependent on laminate stiffness coefficients in place of ply material constants by assuming that they are same in all plies. Here, it is not convenient to use stationary property of total potential in the energy method. $[u, v]$ are dependent on different type of laminate stiffness coefficients. One needs stress resultants in plate element given by (with sum on k)

$$V_x = -½\, (\alpha_k - \alpha_{k-1})[\delta - (\alpha_k + \alpha_{k-1})]\, (Q_{44}\, u_1 + Q_{45} v_1)^{(k)} \quad (46a)$$
$$V_y = -½\, (\alpha_k - \alpha_{k-1})[\delta - (\alpha_k + \alpha_{k-1})]\, (Q_{55}\, v_1 + Q_{45} u_1)^{(k)} \quad (46b)$$
$$\hat{V}_{x2} = -½\, \alpha\, (\alpha_k - \alpha_{k-1})[\delta - (\alpha_k + \alpha_{k-1})][Q_{1j}(\tilde{\varepsilon}_j - S_{j6}\, \sigma_{z1}),_x + Q_{3j}(\tilde{\varepsilon}_j - S_{j6}\, \sigma_{z1}),_y]^{(k)} \quad (47a)$$
$$\hat{V}_{y2} = -½\, \alpha\, (\alpha_k - \alpha_{k-1})[\delta - (\alpha_k + \alpha_{k-1})][Q_{2j}(\tilde{\varepsilon}_j - S_{j6}\, \sigma_{z1}),_y + Q_{3j}(\tilde{\varepsilon}_j - S_{j6}\, \sigma_{z1}),_x]^{(k)} \quad (47b)$$

Equation governing ψ_1 becomes: $\beta_1\alpha\, (\bar{V}_{x2,x} + \bar{V}_{y2,y}) = \alpha\, (V_{x,x} + V_{y,y})$ which is again a fourth order equation in ψ_1 to be solved along with harmonic function φ_1 subjected to the following three conditions along x = constant edges (and analogue conditions along y = constant edges), with $\hat{M}_i = -½\alpha\, \Sigma\, (\alpha_k - \alpha_{k-1})[\delta - (\alpha_k + \alpha_{k-1})]Q_{ij}(\tilde{\varepsilon}_j - S_{j6}\, \sigma_{z1})]^{(k}$ (i, j = 1, 2, 3),

(i) $u_1(y) = 0$ or $\hat{M}_x = ⅓T_{x1}(y)$ (48a)
(ii) $v_1(y) = 0$ or $\hat{M}_{xy} = ⅓T_{xy1}(y)$ (48b)
(iii) $\psi_1(y) = 0$ or $V_x = ½T_{xz0}(y)$ (48c)

Denote the in-plane displacements $[u_1, v_1]$ thus obtained by $[\bar{u}_1, \bar{v}_1]$ which are continuous across interfaces except across z = 0 plane. Moreover, $[\bar{\tau}_{xz}, \bar{\tau}_{yz}] = \beta_1[\bar{\tau}_{xz2}, \bar{\tau}_{yz2}]$ are simply in terms of $[\bar{u}_1, \bar{v}_1]$ and $\beta_1\ \bar{\sigma}_{z3} = \alpha\ [\bar{\tau}_{xz},_x + \bar{\tau}_{yz},_y]$. (Post processing through equilibrium equations in CLPT for finding transverse stresses is eliminated. Note that this commonly used procedure does not ensure satisfaction of either face conditions or continuity conditions across reference plane.)

Bending displacements and transverse stresses in the above analysis from ACLPT are

$$[u, v] = -\, (\tfrac{1}{2}\, \delta - z)\, [\bar{u}_1, \bar{v}_1] \quad (49)$$
$$[\tau_{xz}, \tau_{yz}] = [\hat{\tau}_{xz0}, \hat{\tau}_{yz0}] + \tfrac{1}{2}\, [(\alpha_k - z)\, \delta - (\alpha_k^2 - z^2)]\, [\bar{\tau}_{xz2}, \bar{\tau}_{yz2}] \quad (50)$$
$$\sigma_z = z\, q_1/2 + \tfrac{1}{2}\, [(\alpha_k z - ½\, z^2)\, \delta - (\alpha_k^2 z - ⅓\, z^3)]\, \bar{\sigma}_{z3} \quad (51)$$
$$\alpha w_0(x, y) = -\int [\bar{u}_1 dx + \bar{v}_1 dy] \quad (52)$$

4.1.1 Discontinuities across z = 0 plane

Bending displacements and transverse shear stresses along z = 0 plane are

$$[u, v]^{(u, b)} = -\tfrac{1}{2}\,\delta[\bar{u}_1, \bar{v}_1]^{(u, b)} \quad (53)$$
$$\alpha w_0^{(u, b)} = -\int[\bar{u}_1 dx + \bar{v}_1 dy]^{(u, b)} \quad (54)$$
$$\tau_{xz}^{(u, b)} = \hat{\bar{\tau}}_{xz0} + \tfrac{1}{2} t_1(\delta - t_1)\,\bar{\tau}_{xz2}^{(u, b)} \quad (55a)$$
$$\tau_{yz}^{(u, b)} = \hat{\bar{\tau}}_{yz0} + \tfrac{1}{2} t_1(\delta - t_1)\,\bar{\tau}_{yz2}^{(u, b)} \quad (55b)$$

It can be seen from equations (53, 54, 55), continuity of displacements and transverse stresses requires only continuity of $[\bar{u}_1, \bar{v}_1]$ and $[\bar{\tau}_{xz2}, \bar{\tau}_{yz2}]$ in the adjacent plies of the reference plane. For this purpose, one has to consider first the problem of reference plane subjected to

$$\tau_{xz}^{(u, b)} = \tfrac{1}{2}\{t_1(\delta - t_1)\,\bar{\tau}_{xz2}^{(b, u)} - \tfrac{1}{2}\{t_1(\delta - t_1)\,\bar{\tau}_{xz2}^{(u, b)} \quad (56a)$$
$$\tau_{yz}^{(u, b)} = \tfrac{1}{2}\{t_1(\delta - t_1)\,\bar{\tau}_{yz2}^{(b, u)} - \tfrac{1}{2}\{t_1(\delta - t_1)\,\bar{\tau}_{yz2}^{(u, b)} \quad (56b)$$

It is convenient to introduce the coordinate z' = (1− z) for (z ≥ 0) so that the reference plane z = 0 corresponds to z' = 1. Consequently, $h_k' = 1 - h_k$, $\alpha_k' = (1 - \alpha_k)$ and $[\tau_{xz}, \tau_{yz}]$ are $[\tau_{xz}, \tau_{yz}]'$. Here, q = 0 along z' = 1 and the faces z'= 0 are free of transverse stresses.

Replace $[u_1, v_1]$ with $[u_1', v_1']$ and δ = 0 and determine $[u_1', v_1']$, as before, in terms of laminate stiffness coefficients with three conditions $M_x' = 0$, $M_{xy}' = 0$, $V_x' = T_{xz}'$ (y) along x = constant edges (with analogue conditions along y = constant edges).

Displacements in the extension problem coupled with bending displacements now become

$$[\bar{u}_0, \bar{v}_0] = [u_0, v_0] - \tfrac{1}{2}\,\delta\,[(u_1 + u_1'), (v_1 + v_1')] \quad (57)$$

Modified in-plane displacements $[\bar{u}_0, \bar{v}_0]$ are obtained from the earlier smeared laminate theory by replacing $[u_0, v_0]$ with $[\bar{u}_0, \bar{v}_0]$ in the differential equations and edge conditions. The displacements thus obtained along with face variable $w_0(x, y)$ from top and bottom halves of the laminate are continuous across the reference plane z = 0.

B. Extension problem

We consider [u, v] such that they do not effect displacements $z[u_1, v_1]$ (ignoring their relations with w_0) in the presence of bending loads by assuming z-distribution orthogonal to z in the form

$$[u, v] = [1 - \delta\,(\tfrac{1}{3} + z)]\,[u_0, v_0] \quad (58)$$

Transverse strains are zero and

$$[\epsilon_x, \epsilon_y] = [1 - \delta\,(\tfrac{1}{3} + z)]\,\alpha\,[u_{0,x}, v_{0,y}] \quad (59a)$$

$$\gamma_{xy} = [1 - \delta\,(\tfrac{1}{3} + z)]\,\alpha\,(v_{0,x} + u_{0,y}) \quad (59b)$$

Stress components in k[th] ply are, with (i, j = 1, 2, 3)

$$\sigma_i^{(k)} = [1 - \delta\,(\tfrac{1}{3} + z)]\,Q_{ij}^{(k)}\varepsilon_{j0}^{(k)} \quad (60)$$

In smeared laminate theories, stress resultant in plate element is sum of ply-wise stress resultant in k plies with k equal to ‘n’ and ‘m’ in the upper and bottom halves of the laminate, respectively. Hence, summation sign on k in each half implies that k varies up to ‘n’ in the upper-half and up to ‘m’ in the bottom-half of the laminate. Accordingly, stress resultants added together with z = − z in the bottom-half in the smeared laminate theory and PEEEs with (i, j = 1, 2, 3) are

$$N_i = \Sigma\,[(1 - \tfrac{1}{6}\delta)^2 t_k - 2\,\delta(1 - \delta)(\alpha_k - \alpha_{k-1}) + \delta^2(\alpha_k^2 - \alpha_{k-1}^2)]Q_{ij}^{(k}\,\varepsilon_{j0}^{(k)} \quad (61)$$

$$N_{x,x} + N_{xy,y} = 0\,,\quad N_{y,y} + N_{xy,x} = 0 \quad (62)$$

Solution of the above equilibrium equations along with appropriately modified edge conditions gives $[u_0, v_0]$ denoted as $[\bar{u}_0, \bar{v}_0]$ same in all plies, thereby, continuous across interfaces.

Transverse stresses dependent on material constants and sub-laminate stiffness coefficients are obtained from the usual post processing from integration of equilibrium equations in each ply maintaining continuity across inter faces of each ply. However, the procedure from one face to the other face is at the expense not satisfying face conditions. If the process is used separately in each half, then these stresses are not continuous across the reference plane z = 0.

The above transverse stresses from post process in each ply adjacent to the reference plane z = 0, with

$$f_1(z) = [(1 - \tfrac{1}{3}\delta)\,(\alpha_1 - z) + \delta\,(\alpha_1^2 - z^2)/2]$$
$$f_2(z) = \{(1 - \tfrac{1}{3}\delta)[\alpha_1(\alpha_1 - z) - (\alpha_1^2 - z^2)/2] + \tfrac{1}{2}\delta\,[(\alpha_1^2\,(\alpha_1 - z) - (\alpha_1^3 - z^3)/3]\}$$

are

$$\bar{\tau}_{xz}^{(u, b)} = -\,\alpha f_1(z)[(\sigma_{x,x} + \sigma_{xy,y})^{(1)}]^{(u, b)} \quad (63a)$$
$$\bar{\tau}_{yz}^{(u, b)} = -\,\alpha f_1(z)[(\sigma_{y,y} + \sigma_{xy,x})^{(1)}]^{(u, b)} \quad (63b)$$
$$\bar{\sigma}_z^{(u, b)} = \alpha^2 f_2(z)\,(\bar{\tau}_{xz,x} + \bar{\tau}_{yz,y})^{(u, b)} \quad (64)$$

Note that in-plane distributions of the above stress components are in terms of $[\bar{u}_0, \bar{v}_0]$.

Transverse stresses $[\tau_{xz}, \tau_{yz}, \sigma_z]$ along the reference plane z = 0 from upper-half and bottom-half of the laminate with

$$f_1(0) = [(1- \delta\, \alpha_1(\tfrac{1}{3} - \tfrac{1}{2}\alpha_1],$$
$$f_2(0) = [(1- \delta \tfrac{1}{3}(\tfrac{1}{2} + \tfrac{1}{3}\alpha_1)\, \alpha_1^2]$$

are

$$[\tau_{xz}, \tau_{yz}] = f_1(0)[\tau_{xz}, \tau_{yz}]^{(u,\, b)} \qquad (65a)$$
$$\sigma_z = f_2(0)\, \sigma_z^{(u,\, b)} \qquad (65b)$$

It can be seen from above equations, continuity of transverse stresses requires only continuity of $[\bar{\boldsymbol{\tau}}_{xz0}, \bar{\boldsymbol{\tau}}_{yz0}]$ in the adjacent plies of the reference plane. For this purpose, one has to consider first the problem of reference plane subjected to

$$\tau_{xz}^{(u,\, b)} = f_1(0)(\bar{\boldsymbol{\tau}}_{xz2}^{\ (b,\, u)} - \bar{\boldsymbol{\tau}}_{xz2}^{\ (u,\, b)}) \qquad (66a)$$
$$\tau_{yz}^{(u,\, b)} = f_1(0)(\bar{\boldsymbol{\tau}}_{yz2}^{\ (b,\, u)} - \bar{\boldsymbol{\tau}}_{yz2}^{\ (u,\, b)}) \qquad (66b)$$

It is convenient to introduce the coordinate z' = (1− z) for (z ≥ 0) so that the reference plane z = 0 corresponds to z' = 1. Consequently, $h_k' = 1- h_k$, $\alpha_k' = (1- \alpha_k)$ and $[\tau_{xz}, \tau_{yz}, \sigma_z]$ are $[\tau_{xz}, \tau_{yz}, \sigma_z]'$ which are zero along z'= 0 faces.

Replace $[u_0, v_0]$ with $[u_0', v_0']$ and δ = 0 in equations (61-65) and determine $[u_1', v_1']$, as before, in terms of laminate stiffness coefficients with three conditions

$$N_x' = 0 \quad , \quad N_{xy}' = 0 \quad , \quad V_x' = T_{xz}'(y) \qquad (67)$$

at x = constant edges (with analogue conditions along y = constant edges). The displacements un-coupled with bending displacements now become

$$u_0 = [1 - \delta\,(\tfrac{1}{3} + z)]\,(\bar{\mathbf{u}}_0 + \delta u_0') \qquad (68a)$$
$$v_0 = [1 - \delta\,(\tfrac{1}{3} + z)]\,(\bar{\mathbf{v}}_0 + \delta v_0') \qquad (68b)$$

Displacements $[u_0, v_0]$, thereby, $[\tau_{xz}, \tau_{yz}, \sigma_z]$ thus obtained are continuous not only across z = 0 plane but also across all interfaces of plies.

From the above analysis, $[u, v]_e$ in extension problem (denoted with suffix '$_e$') from smeared laminate theory are

$$u_e = [1 - \delta\,(\tfrac{1}{3} + z)]\,(\bar{\mathbf{u}}_{0e} + \delta\, u_{0e}' + z\, \bar{\mathbf{u}}_{1e}) \qquad (69a)$$
$$v_e = [1 - \delta\,(\tfrac{1}{3} + z)]\,(\bar{\mathbf{v}}_{0e} + \delta v_{0e}' + z\, \bar{\mathbf{v}}_{1e}) \qquad (69b)$$

C. Torsion associated with bending loads

For determination of w_0(x, y) which is a primary variable in bending (associated torsion) problems, all three static equations (1) and (2) are required and they are different from integrated equations. Kirchhoff's theory is based on integrated equations normally used in bending problems. Author's recent investigations, however, indicate that sum of τ_{xy} in bending and τ_{xy} in torsion is zero in the exact solutions of 3D equations. This is due to the use of w_0 as domain variable in torsion problems and as face variable in bending problems. Poisson's theory recently proposed by the author brings out this distinction in the analysis of these problems.

Displacements $[w_0, u_0, v_0]$ in the classical theory of unsymmetrical laminates (CLPT) are coupled through B matrix in plate element equilibrium equations (PEEEs) arising due to thickness-wise integration of products of even and odd z-functions. This 'B' matrix imposes same order of effect on $[u_0, v_0]$ and α $[w_{0,x}, w_{0,y}]$. Displacements in CLPT are assumed in the form $w = w_0(x, y)$, $u = u_0 - z\, \alpha\, w_{0,x}$, $v = v_0 - z\, \alpha\, w_{0,y}$. Here, w_0 is a domain variable and the coupling is between extension and torsion problems. An alternate form of CLPT denoted as ACLPT shows that the effect of un-symmetry in bending displacements is independent of $[u_0, v_0]$. Extended Poisson's theory is used satisfying both static and integrated equilibrium equations. A secondary problem is formulated governing induced second order displacements of extension problem for the purpose of maintaining continuity of transverse stresses across interfaces of plies. Same analysis is applicable here also by replacing domain variable ψ_1 in bending problem with domain variable w_0(x, y) in torsion problem. Hence, the corresponding analysis is not presented here. Associated torsion-type problem in extension problem requires the use of $[u_2, v_2]$ and the analysis of their influence on bending displacements needs to be carried out in future investigation.

V. CONCLUDING REMARKS WITH SUGGESTIONS FOR FUTURE WORK

Set of polynomials generated in z are necessary in satisfying both static and integrated equilibrium equations. Poisson's theory and Extended Poisson's theory are based on satisfaction of both static and integrated equilibrium equations. *One significant feature of the present work is that the ply analysis is independent of lamination.* This feature needs exploitation in investigations on optimum ply lay-up, its utility in the analysis of associated eigen-value problems of free vibration and buckling of plates, and even in the area of fracture mechanics. However, polynomials in z are not adequate for proper solutions of 3-D problems. Solution of a supplementary problem based on appropriate trigonometric function in z representing each of displacement and stress components is required. Solution of additional similar problem is required in the analysis of unsymmetrical laminates.

Sequence of 2D problems converging to 3D problem is through extended Poisson's theory in conjunction with an auxiliary problem and a sequence of supplementary problems.

An alternate form of classical laminate theory is proposed for analysis of unsymmetrical laminated plates using extended Poisson's theory. There is no coupling between extension and problems. Effect of un-symmetry is through linear variation of transverse stresses independent of each other in these problems.

Analysis in extension and bending problems in ACLPT is based on assumed displacements (28) and (29). There is a need to quantify amount of discontinuity in transverse stresses across reference plane before using solution of auxiliary problem. It is dependent on location of reference plane which, in principle, can be any z = constant plane except either face of the laminate. Use of discontinuity in the corresponding strain energy density for this purpose gives wide scope for future investigations in finding the location of reference plane with either minimum or minimum of maximum discontinuity.

Coupling due to 'B' matrix in CLPT imposes same order of effect on extension and bending displacements but this coupling appears to be different in ACLPT. It is worthy of consideration in future investigations.

Number of 2D displacement variables is limited to minimum number mainly because the in-plane distributions of these variables are not ply dependent and can never be equal to ply dependent distributions in layer-wise theories. Normal requirement is two term representation of each in-plane displacement variable in the analysis of unsymmetrical laminates like in the present ACLPT. In CLPT, bending displacements are in terms of a single variable $w_0(x, y)$ due to its use as a domain variable and two (instead of three required in 3D problem) edge conditions are prescribed in each of extension and bending problems.

In ACLPT, three edge conditions in bending are prescribed due to use of $w_0(x, y)$ as face variable in Poisson's theory and extended Poisson's theory. Corresponding in-plane displacements are determined by satisfying both static and integrated equilibrium equations. Such facility is absent in extension problems since the even prescribed $\sigma_z = q_0/2$ along faces of the laminate does not disturb the equilibrium equations. If transverse shear stresses are prescribed along faces of the laminate, they have to be asymmetric in z and have to satisfy equilibrium equation in z-direction even in the case of the above prescribed $\sigma_z = q_0/2$. They are governed by static equilibrium equation. One gets from integrated equation $\sigma_z = f_2(z)\sigma_{z2}$ with unknown σ_{z2}. Its determination is dependent on second order $[u_2, v_2]$ displacements which have to be obtained from Poisson's theory satisfying both static and integrated equilibrium equations.

From the above observations, it is clear that one requires one term representation in bending and two term representation in extension problems with w from integration of ε_z from constitutive relation. Higher order displacement terms are to improve in-plane distributions whose utility in global-local approach in the analysis of critical areas of unsymmetrical laminates may not be of much important.

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