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A Novel Sequential Fractional Order Kalman Filter Considering Colored Noise

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***Abstract*— In this paper, a new fractional order Kalman filter will be provided which is suitable for SIMO (single input multi output) FOSs (fractional order systems). This filter is called sequential fractional order Kalman filter. This is a way of implementing the fractional order Kalman filter without matrix inversion. As a result, the sequential fractional order Kalman filter reduces the computational burden and computational complexity. Furthermore, a novel approach will be proposed for a linear FOS with colored measurement noise by using measurement differencing method. The application of the new algorithm yields more realistic and therefore useful state and covariance information than the standard implementation. Finally, the precision of the proposed algorithms will be examined by using an appropriate and applicable example.**

***Keywords*— Fractional order systems, colored measurement noise, sequential fractional order Kalman filter, measurement differencing**

I. Introduction

In 1695, the concept of fractional calculus was expressed for the first time by Leibniz and L'Hospital. In the late nineteenth century, Riemann and Liouville gave the first definition of fractional derivatives. In recent years, because of their many applications, FOSs (systems that contain fractional derivatives and fractional integrals) have received the attention of many researchers [1]. However, this idea began to be a topic of interest for engineers since 1960, especially insofar as they observed that certain actual systems in which fractional derivatives are used exhibit greater accuracy [2]. Modelling the behaviour of materials such as polymers and rubbers can be considered as an example [3]. Electrochemical processes and robots with flexible arms are also modelled by fractional-order systems [4]. Fractional calculus is also a useful tool for modelling traffic in information networks [5]. Another research topic in the area of FOSs, which is developing rapidly, is fractional order PID controllers [6]. More applications and examples for the FOSs and also the fractional calculus can be found in [7-11].

State estimation is important for controller design and pole placement. Some methods have been reported for linear and nonlinear systems [12-15].

The sequential fractional Kalman filter (sfkf) is a suitable filter for the implementation of the single-input multi-output FOSs. This filter, instead of getting the inversion of a $r \times r$ matrix at every stage, takes r reverse of a scalar value, which significantly reduces the computational burden in implementation.

There are two major advantages of using the sequential filter. The first is that if the output noise covariance R_k is diagonal, using the sequential filter reduces up to fifty percent of the processing time, which depends on the selection of the signal model and other data system. The second reason is that if there is not sufficient time to complete the processing of the data vector, using the sequential filter is better because in simultaneous processing, the data vector will be lost totally; however, in sequential processing, just a part of data vector will be lost [16].

Furthermore, a new fractional order Kalman filter will be proposed in this paper that is suitable for models with colored measurement noise. Measurement differencing method is used in order to provide this novel filter [17]. In fact, by expanding the measurement differencing method, it is modified so that it will be suitable for FOSs. A few researches have been reported so far in the field of state estimation for fractional order systems with colored noise [18-21]. In this paper, using the measurement differencing method, the problem of colored measurement noise will be solved.

The rest of the paper is organized as follows. In Section II, a short review about fractional order state space systems is presented and the sequential fractional order Kalman filter is given in Section III. Section IV provides fractional order Kalman filter with colored measurement noise. In Section V, an example is provided. Finally, Section VI concludes the paper.

II. Fractional Order Systems Summarizd

Consider the following fractional-order discrete time linear stochastic state-space system [22].

$$\Delta^{\Upsilon} X(k+1) = AX(k) + Bu(k) + W(k) \tag{1}$$

$$X(k+1) = \Delta^{\Upsilon} X(k+1) - \sum_{j=1}^{k+1} (-1)^{j} \binom{\Upsilon}{j} X(k+1-j) \tag{2}$$

$$y(k) = HX(k) + v(k) \tag{3}$$

where Υ is the order of the fractional difference $\left(\Upsilon \in R^{+}\right)$ and $X(k)$ is the state vector $\left(X(k) \in R^{n}\right)$. $W(k) = \left[w_1(k) \quad w_2(k) \quad \dots \quad w_n(k)\right]$ and $v(k)$ are the process and measurement white Gaussian noises with zero mean. In addition, $u(k)$ and $y(k)$ are the input and the output of the system, respectively. Furthermore, T denotes the matrix/vector transpose and the symbol $I(I_n)$ shows an identity matrix with appropriate size $(n \times n)$. The system matrices A, B are known.

Furthermore, $\binom{\Upsilon}{j}$ is defined as:

$$\binom{\Upsilon}{j} = \frac{\Gamma(\Upsilon+1)}{\Gamma(j+1)\Gamma(\Upsilon-j+1)} \tag{4}$$

Assuming that $v > 0$, Euler's function Γ is defined as:

$$\Gamma(x) = \int_0^{\infty} e^{-v} v^{x-1} dv \tag{5}$$

$$\Delta^{\Upsilon} X(k+1) = \begin{bmatrix} \Delta^{\gamma_1} x_1(k+1) \\ \vdots \\ \Delta^{\gamma_n} x_n(k+1) \end{bmatrix} \tag{6}$$

where $\gamma_1, \dots, \gamma_n$ are the order of the system equation and n is the number of system equations.

Assumption 1: $v(k)$ and $W(k)$ are two independent white noises with zero mean and covariance matrixes R_k and Q_k, respectively. In other words, we have:

$$E\left[W(k)\right] = 0, E\left[v(k)\right] = 0 \tag{7}$$

$$E\left[W(k)W^{T}(j)\right] = Q_k \delta_{k-j} \tag{8}$$

$$E\left[v(k)v^{T}(j)\right] = R_k \delta_{k-j} \tag{9}$$

$$E\left[W(k)v^{T}(j)\right] = 0, \forall k, j \tag{10}$$

Assumption 2: $X(0)$ is uncorrelated with $v(k)$ and $W(k)$, and

$$E\left[X(0)\right] = \hat{X}(0) \tag{11}$$

$$E\left[(X(0) - \hat{X}(0))(X(0) - \hat{X}(0))^{T}\right] = P_0 \tag{12}$$

By combining the equations (1) and (2), the following equation is obtained:

$$X(k+1) = AX(k) + Bu(k) + W(k) + \sum_{j=1}^{k+1} C_j X(k+1-j) \tag{13}$$

$$C_j = (-1)^{j+1} diag\left[\binom{\gamma_1}{j} \quad \dots \quad \binom{\gamma_n}{j}\right] \tag{14}$$

where the matrices *A*, *B* and the state vector $X(k)$ are defined as follows:

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 \\ a_1 & a_2 & a_3 & \dots & a_n \end{bmatrix}_{n \times n} \quad B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix}_{n \times 1}$$

$$X = \left[x_1 \quad x_2 \quad \cdots \quad x_n\right]^{T}$$

$$C_j = \begin{bmatrix} c_{11,j} & & & 0 \\ & c_{22,j} & & \\ & & \ddots & \\ 0 & & & c_{nn,j} \end{bmatrix}_{n \times n}, H_{1 \times n}$$

In addition, we also assume that:

$$\begin{aligned} u(k) &= 0 \qquad k \le 0 \\ y(k) &= 0 \qquad k \le 0 \\ X(k) &= 0 \qquad k \le 0 \end{aligned} \tag{15}$$

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III. The Sequential Fractional order Kalman Filter

Now in this section, the sequential fractional order Kalman filter is provided. This is a way of implementing the fractional order Kalman filter without matrix inversion [16].

In the standard fractional order Kalman filter, it is required to inverse a $r \times r$ matrix where r is the number of measurements. So, using the sequential fractional order Kalman filter reduces the computational burden and computational complexity. Before providing the sequential fractional order Kalman filter considering the following points is essential.

Suppose that instead of measuring $y(k)$ at time instant k, r separate measurements can be obtained at time k. This means that the firstmeasurement is $y_1(k)$, the second is $y_2(k)$, …, and the final one is $y_r(k)$. In the following, the notations $y_i(k)$ are used for the expression of the *ith* element of the measurement vector $y(k)$. Here, it is assumed that R_k (the measurement noise covariance) is a diagonal matrix as follows:

$$R_k = \begin{bmatrix} R_{1k} & 0 & \dots & 0 \\ 0 & R_{2k} & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & R_{rk} \end{bmatrix} \qquad (16)$$

Furthermore, the notation H_{ik} is used for expressing the *ith* row of H_k and v_{ik} is used for expressing the *ith* element of v_k. Thus, the measurement equation can be rewritten as follows:

$$y_i(k) = H_{ik} X(k) + v_{ik}$$
$$v_{ik} \sim (0, R_{ik}) \qquad (17)$$

Thus, instead of processing a measurement vector at time k, implement the fractional order Kalman filter measurement-update equation with one measurement at a time.

We can also define the following notations:

$K_i(k)$ denotes the Kalman gain that is obtained from the *ith* measurement at time k. $\hat{x}_i(k)$ denotes the state estimation that is obtained for the *ith* measurement at time *k*. In addition, P_{ik} denotes the covariance matrix that is obtained from the *ith* measurement at time *k.*

The sequential fractional order Kalman filter algorithm is expressed in the following steps.

1- The system and measurement equations are given as:

$$\Delta^{\Upsilon} X(k+1) = AX(k) + Bu(k) + W(k) \qquad (18)$$

$$X(k+1) = \Delta^{\Upsilon} X(k+1) + \sum_{j=1}^{k+1} C_j X(k+1-j) \qquad (19)$$

$$y(k) = HX(k) + v(k) \qquad (20)$$

$$w_k \sim (0, Q_k)$$
$$v_k \sim (0, R_k) \qquad (21)$$

The measurement noise covariance R_k is a diagonal matrix and can be considered as follows:

$$R_k = diag(R_{1k}, R_{2k}, \dots, R_{ik})$$

2- The filter is initialized as:

$$\hat{X}(0) = E(X(0))$$
$$P_0 = E\left[(X(0) - \hat{X}(0))(X(0) - \hat{X}(0))^T \right] \qquad (22)$$

3- At each time step *k,* the time-update equations are given as:

$$\tilde{P}_k = (A + C_1) P_{k-1} (A + C_1)^T + \sum_{j=2}^{k} C_j P_{k-j} C_j^T + Q_{k-1} \qquad (23)$$

$$\Delta^{\Upsilon} \tilde{X}(k) = A\hat{X}(k) + Bu(k) \qquad (24)$$

$$\tilde{X}(k) = \Delta^{\Upsilon} \tilde{X}(k) + \sum_{j=1}^{k} C_j \hat{X}(k-j) \qquad (25)$$

This is the same as the standard fractional Kalman filter.

4- Measurement update equations at each time step *k* are given as follows:

(a) Initialize the a posteriori estimate and covariance as:

$$\hat{x}_0(k) = \tilde{x}(k)$$
$$P_{0k} = \tilde{P}_k \qquad (26)$$

(b) For $i = 1, \dots, r$ (where r is the number of measurements), we act as follows:

$$K_i(k) = \frac{P_{i-1,k} H_{ik}^T}{H_{ik} P_{i-1,k} H_{ik}^T + R_{ik}} \qquad (27)$$

$$\hat{x}_i(k) = \hat{x}_{i-1}(k) + K_i(k)\left(y_i(k) - H_{ik} \hat{x}_{i-1}(k) \right) \qquad (28)$$

$$P_{ik} = (I - K_i(k) H_{ik}) P_{i-1,k} \qquad (29)$$

(c) Finally, assign the estimate and covariance as follows:

$$\hat{X}(k) = \hat{x}_r(k)$$
$$P_k = P_{rk} \qquad (30)$$

In the above process, it is assumed that the measurement noise covariance R_k is diagonal.

IV. Fractional Order Kalman Filter with Colored Measurement Noise

The Kalman filter introduced in section III was exposed to uncorrelated process and measurement noises. In this section, a novel fractional order Kalman filter will be proposed to estimate the states of a fractional order state space system with colored measurement noise. A fractional order state space system with colored measurement noise is expressed as follows:

$$\Delta^{\Upsilon} X(k+1) = AX(k) + Bu(k) + W(k) \tag{31}$$

$$X(k+1) = \Delta^{\Upsilon} X(k+1) + \sum_{j=1}^{k+1} C_j X(k+1-j) \tag{32}$$

$$y(k) = HX(k) + v(k) \tag{33}$$

$$v(k) = \psi v(k-1) + \zeta_{k-1} \tag{34}$$

where ζ is a white Gaussian noise with zero mean and $v(k)$ is a colored measurement noise.

Measurement differencing is a method that can be used when there is a colored measurement noise. However, this method is applicable to integer order systems. Hence, a generalization to this method is necessary to be applicable for fractional order systems. So, a complementary signal denoted by $z(k)$ is defined as:

$$z(k-1) = y(k) - \psi y(k-1) \tag{35}$$

Therefore, a new equivalent system is formed as follows:

$$\begin{aligned} z(k-1) &= H'X(k-1) + HBu(k-1) \\ &\quad + H\left(\sum_{j=1}^{k} C_j X(k-j)\right) + v'(k-1) \end{aligned} \tag{36}$$

where H' and $v'(k-1)$ are defined as follows:

$$H' = HA - \psi H \tag{37}$$

$$v'(k-1) = HW(k-1) + \zeta_{k-1} \tag{38}$$

It is clear that a new measurement equation is now introduced for $z(k)$. Therefore, the equivalent system can be formulated as below:

$$\Delta^{\Upsilon} X(k+1) = AX(k) + Bu(k) + W(k) \tag{39}$$

$$X(k+1) = \Delta^{\Upsilon} X(k+1) + \sum_{j=1}^{k+1} C_j X(k+1-j) \tag{40}$$

$$\begin{aligned} z(k) &= H'X(k) + HBu(k) \\ &\quad + H\left(\sum_{j=1}^{k+1} C_j X(k+1-j)\right) + v'(k) \end{aligned} \tag{41}$$

The covariance for the new measurement noise v' and its cross-covariance with the process noise W can be calculated as:

$$E\left[W(k)W^T(j)\right] = Q_k \delta_{k-j} \tag{42}$$

$$E\left[\zeta_k \zeta_j^T\right] = Q_{\zeta k} \delta_{k-j} \tag{43}$$

$$\begin{aligned} E\left[v'(k)v'^T(k)\right] &= E\left[\left(HW(k)+\zeta_k\right)\left(HW(k)+\zeta_k\right)^T\right] \\ &= HQ_k H^T + Q_{\zeta k} = R'_k \end{aligned} \tag{44}$$

$$\begin{aligned} E\left[W(k)v'^T(k)\right] &= E\left[W(k)\left(HW(k)+\zeta_k\right)^T\right] \\ &= E\left[W(k)\left(W^T(k)H^T + \zeta_k^T\right)\right] = Q_k H^T = M_k \end{aligned} \tag{45}$$

As it can be seen in Eq. 45, there is a correlation between measurement noise of the new equivalent system and the process noise. Therefore, to estimate the states of the new equivalent system, a fractional Kalman filter is required. As a result, a fractional order Kalman filter algorithm suitable for systems with correlated noises can be introduced.

This Kalman filter is expressed for each time instance $k = 1, 2, \ldots$ as following:

1. The system and measurement equations are given by Eqs. 39 - 41.

2. H' , v' and $z(k)$ are defined by Eqs. 36 and 38.

3. At each time step, execute the following equations to update the state estimate:

$$\tilde{P}_k = (A + C_1) P_{k-1} (A + C_1)^T + \sum_{j=2}^{k} C_j P_{k-j}^{+} C_j^T + Q_{k-1} \tag{46}$$

$$K(k) = \left(\tilde{P}_k H^T + M_k\right)\left(H\tilde{P}_k H^T + HM_k + M_k^T H^T + R'_k\right)^{-1} \tag{47}$$

$$\tilde{X}(k+1) = A\hat{X}(k+1) + Bu(k) + \sum_{j=1}^{k+1} C_j \hat{X}(k+1-j) \tag{48}$$

$$\hat{X}(k) = \tilde{X}(k) + K(k)\left(z(k) - H'\tilde{X}(k)\right) \tag{49}$$

$$\begin{aligned} P_k &= \left(I - K(k)H\right)\tilde{P}_k \left(I - K(k)H\right)^T \\ &\quad + K(k)\left(HM_k + M_k^T H^T + R'_k\right)K^T(k) \\ &\quad - M_k K^T(k) - K(k) M_k^T \\ &= \tilde{P}_k - K(k)\left(H\tilde{P}_k + M_k^T\right) \end{aligned} \tag{50}$$

The above mentioned algorithm is a novel method for estimating the states of fractional order systems with colored measurement noise.

V. Simulation Results

To show the accuracy of the algorithm expressed in Sections III and IV, a FOS with one input and four outputs have been used. The system has two states.

Consider a SIMO fractional-order state-space system with the following matrices:

$$A=\begin{bmatrix}0 & 1\\ a_{21} & a_{22}\end{bmatrix},B=\begin{bmatrix}b_1\\ b_2\end{bmatrix},$$

$$H=\begin{bmatrix}1 & 0\\ 0.9 & 0\\ 0.8 & 0\\ 0.7 & 0\end{bmatrix},\ \Upsilon=\begin{bmatrix}\gamma_1 & \gamma_2\end{bmatrix}$$

$$a_{21}=-0.5,\ \ a_{22}=-0.9$$
$$b_1=-0.3,\ \ b_2=-0.7$$
$$\gamma_1=0.9,\ \ \gamma_2=0.3$$

The number of elements in equation (2) should be limited – here, the value is equal to *L*, which would simplify and reduce the number of calculations. Although it will cause a bit of error, by considering a reasonable value for *L*, the error value would be very small and negligible. Accordingly, equation (2) can be written as follows:

$$\mathrm{X}(k+1)=\Delta^{\Upsilon}\mathrm{X}(k+1)-\sum_{j=1}^{L}(-1)^j\binom{\Upsilon}{j}\mathrm{X}(k+1-j) \quad (51)$$

The state equations are then given as:

$$\Delta^{\Upsilon}\mathrm{X}(k+1)=\begin{bmatrix}0 & 1\\ -0.5 & -0.9\end{bmatrix}\mathrm{X}(k)+\begin{bmatrix}-0.3\\ -0.7\end{bmatrix}u(k)+W(k)$$

$$\mathrm{X}(k+1)=\Delta^{\Upsilon}\mathrm{X}(k+1)-\sum_{j=1}^{L}(-1)^j\begin{pmatrix}\binom{0.9}{j} & 0\\ 0 & \binom{0.3}{j}\end{pmatrix}\mathrm{X}(k-1+j)$$

$$y(k)=\begin{bmatrix}1 & 0\\ 0.9 & 0\\ 0.8 & 0\\ 0.7 & 0\end{bmatrix}\mathrm{X}(k)+v(k)$$

and

$$E\left[W(k)W^T(k)\right]=\begin{bmatrix}0.1 & 0\\ 0 & 0.1\end{bmatrix}$$

$$E\left[v(k)v^T(k)\right]=\begin{bmatrix}0.02 & 0 & 0 & 0\\ 0 & 0.02 & 0 & 0\\ 0 & 0 & 0.02 & 0\\ 0 & 0 & 0 & 0.02\end{bmatrix}$$

Here, $u(k)$ and $y(k)$ are the input and the output of the system, respectively. In this simulation, the input $\{u(k)\}$ is an uncorrelated signal with variance 1.

The initial values at time *k=1* are considered as follows:

$$P_0=10^6,\hat{\mathrm{X}}(1)=0$$

$$R=\begin{bmatrix}0.02 & 0 & 0 & 0\\ 0 & 0.02 & 0 & 0\\ 0 & 0 & 0.02 & 0\\ 0 & 0 & 0 & 0.02\end{bmatrix},Q=\begin{bmatrix}0.1 & 0\\ 0 & 0.1\end{bmatrix}$$

The original and the estimated state variables x_1 and x_2 are also shown in Figs 1 and 2, respectively. As it is seen, the proposed method can estimate the state variables accurately. It is assumed that $L=50$ in these figures. Furthermore, Fig. 3 shows the input and first output signals from the plant with white noise.

The above example is re-analyzed with the below dynamics and colored measurement noise.

$$v(k)=\psi v(k-1)+\zeta(k-1)$$
$$\psi=0.2$$

where ζ is a white Gaussian noise with zero mean.

Instead of the Kalman filter introduced in section III which was suitable for uncorrelated white noise, the new Kalman filter introduced in section IV is applied which is applicable for systems with colored measurement noise. The simulation results for the estimated states x_1 and x_2 with colored measurement noise are depicted in Figs. 4 and 5. It is expectable that accuracy of the estimations is a little lesser with colored measurement noise but yet it is acceptable.

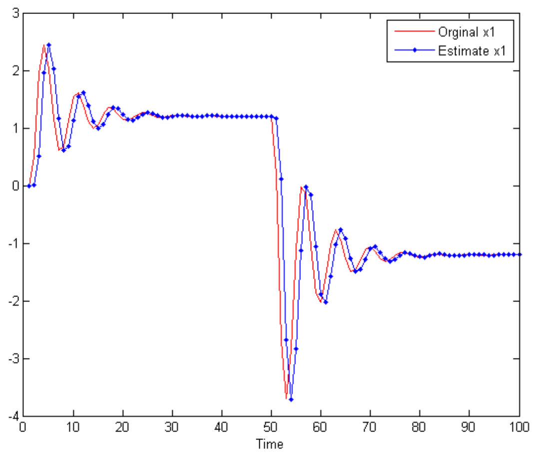

Fig. 1: Original and estimated state variable x_1 for $L=50$ with white noise.

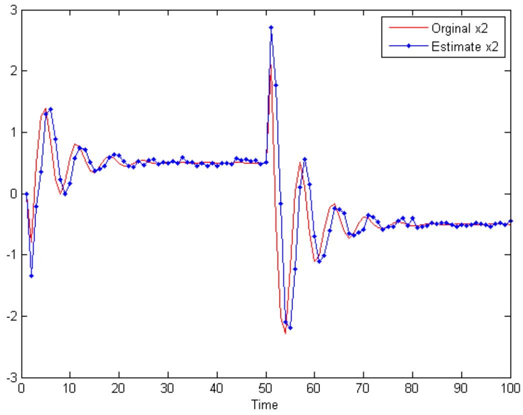

Fig. 2: Original and estimated state variable x_2 for $L=50$ with white noise.

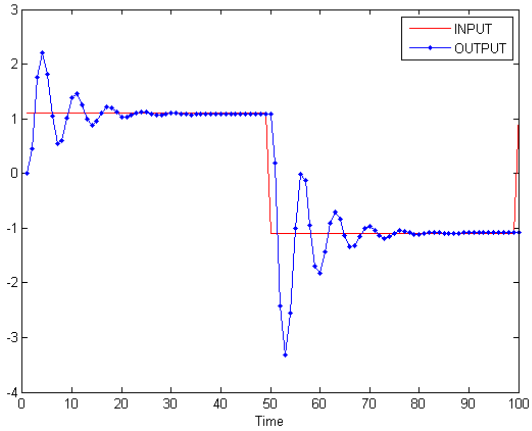

Fig. 3: Input and output signals from the plant with white noise.

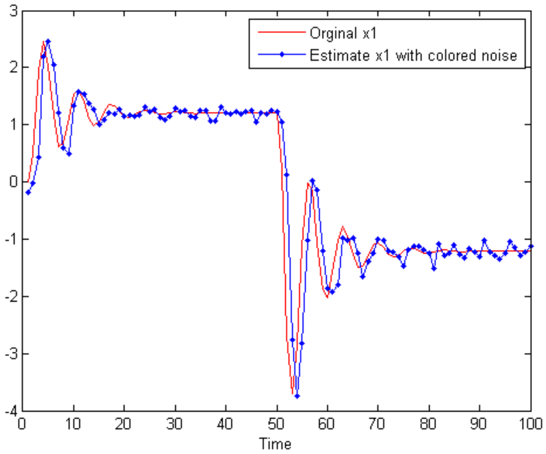

Fig. 4: Original and estimated state variable x_1 for $L=50$ with colored noise.

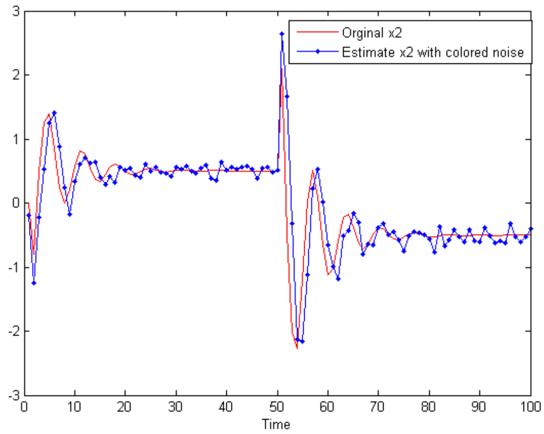

Fig. 5: Original and estimated state variable x_2 for $L=50$ with colored noise.

VI. Conclusion

In this paper, two new algorithms were provided. First, a novel sequential fractional order Kalman filter was introduced. The simulation results show that this filter is a suitable filter for the implementation of the single-input multi-output FOSs. Secondly, a new fractional order Kalman filter was proposed in this paper that is suitable for models with colored measurement noise. The results show that these two methods have had better performances than previous methods.

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