Existence & Convergent Criterion for the Numerical Solution of the Duffing's Equation by the One Step Euler Scheme

ADEWOLE O.O
Department of Physics & Electronics,
Ajayi Crowther University, Oyo, Nigeria
*Correspondence author
mayowaadewole@hotmail.com

LAWRENCE M.O, and Agbo-Ajala O
Department of Computer Science,
Ajayi Crowther University, Oyo, Nigeria

ALLI S.G
Department of Mathematics & Statistics,
The Polytechnic Ibadan, Nigeria

Fajemiroye J.A
Department of Physics with Electronics,
The Polytechnic, Ibadan, Nigeria

***Abstract*—The existence conditionality for the existence and uniqueness of a differential equation is quite crucial and pertinent in obtaining its solution. This has been discussed and the criterion for the convergence of the Duffin's equation with an approximate expression for the numerical solution by a one step or Euler scheme has been established, and the Picard's iteration procedure for a sequence of solution of an ordinary differential equation has been succinctly discussed.**

Keywords—Duffing's equation, Cauchy-Lipschitz conditions, One Step Scheme, Picard's iteration, Convergence.

1.0 Introduction

Existence and uniqueness theorem is the tool which makes it possible for us to conclude that there exists only one solution to a first order differential equation which satisfies a given initial condition. Thus, it would be interesting to see the main ideas behind this, first let us state the theorem itself.

Theorem I. Let *f*(*x*,*y*) be a real valued function which is continuous on the rectangle

$$R = \{(x, y); |x - x_0| \le a\ ,\ |y - y_0| \le b\} \quad \text{(i)}$$

Assume *f* has a partial derivative with respect to *y* and that $\frac{\partial y}{\partial y}$ is also continuous on the rectangle *R*. Then there exists an interval

$$I = [x_0 - h, x_0 + h]$$

(with $h \le a$) such that the initial value problem

$$\begin{cases} y' = f(x, y)\,, \\ y(x_0) = y_0 \end{cases} \quad \text{(ii)}$$

has a unique solution *y*(*x*) defined on the interval *I*.

Note that the number *h* may be smaller than *a*. In order to understand the main ideas behind this theorem, assume the conclusion is true. Then if *y*(*x*) is a solution to the initial value problem, we must have

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))dt \quad \text{(iii)}$$

It is not hard to see in fact that if a function *y*(*x*) satisfies the equation (called functional equation)

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))dt \quad \text{(iv)}$$

on an interval *I*, then it is solution to the initial value problem

$$y' = f(x, y) \quad y(x_0) = y_0 \quad \text{(v)}$$

Picard was among the first to look at the associated functional equation. The method he developed to find *y* is known as the **method of successive approximations** or **Picard's iteration method**. This is how it goes:

Step 1. Consider the constant function

$$y_0(x) = y_0 \quad \text{(vi)}$$

Step 2. Once the function $y_n(x)$ is known, define the function

$$y_{n+1}(x) = y_0 + \int_{x_0}^{x} f(t, y_n(t))dt \quad \text{(vii)}$$

Step 3. By induction, we generate a sequence of functions $\{y_n(x)\}$ which, under the assumptions made on *f*(*x*,*y*), converges to the solution *y*(*x*) of the initial value problem

$$y' = f(x, y) \quad y(x_0) = y_0 \quad \text{(viii)}$$

We are interested in the approximate solution of the system,

$$\dot{u}(t) = f(t, u(t))\ , \forall_t \in [t_0, T] \quad \text{(ix)}$$

with the initial condition given on

,with the initial condition

$u(t_0) = u_0,$ (x)

We will always suppose that f satisfies the Cauchy-Lipschitz conditions.

Indeed, often it is very hard to solve differential equations, but we do have a numerical process that can approximate the solution. This process is known as the **Picard iterative proc**ess, consider the IVP

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0 . \qquad \text{(xi)}$$

It is not hard to see that the solution to this problem is also given as a solution to (called the integral associated equation)

$$y(x) = y_0 + \int_{x_0}^{x} f(s, y(s)ds . \qquad \text{(xii)}$$

The Picard iterative process consists of constructing a sequence (y_n) of functions which will get closer and closer to the desired solution. This is how the process works:

(1)

$y_0(x) = y_0$ for every *x*;

(2)

then the recurrent formula holds

$$y_{n+1}(x) = y_0 + \int_{x_0}^{x} f(t, y_n(t))dt, \qquad \text{(xiii)}$$

for n≥ 1.

Definition. The approximation of equations (i x) and (x) is defined by a one step scheme.

$U_{k+1} = U_k + F(U_{k+1}, U_k, h),$ (xiv)

,is said to be convergent if for any initial U_0 ,

$lim_{h\to 0\, max}|U(t_k) - U_k| = 0,$ (xv)

Indeed, there could be a truncation error in the initial condition as we do not suppose that the scheme has the exact initial condition of the ODE.

It's extremely important to sought for veritable numerical schemes in diverse applications spanning across physical sciences, applied mathematics, etc, thus there grows huge urge to explore a viable numerical scheme for each application that arises.

Diverse numerical methods are iterative in nature, and thus it is tremendously important to sought for swift convergence. While at present, we centered on existence and convergence, future investigations would be expanded to elucidate more other interesting features and mathematical properties.

2.0 Discussion

Cauchy-Lipschitz Existence Theorem

Theorem II. (Cauchy-Lipschitz).

Suppose that $[t_1, t_2]$ is a compact interval and f is a continous function from $[t_1, t_2] \times |R^d$ which satisfies the following property: there exists a constant L such that;

$|f(t, v) - f(t, w)| \le |v - w|, \forall_t \in [t_1, t_2], \forall_{v,w} \in |R^d$ (xvi)

Here |. |denotes some norms on $|R^d$. then for any t_0 in $[t_1, t_2]$, and u_0 in $|R^d$, there exists a unique continuously differentiable function u from $[t_1, t_2]$ to $|R^d$ which satisfies equations (i) and (ii).

Euler's Scheme to a Model Equation

For the model equation

$y' = \lambda y$; (xvii)

$y_{n+1} = y_n + \lambda h y_n = (1 + \lambda h) y_n$ (xviii)

This equation indicates obviously that one step or Euler scheme is of first order accuracy.

For stability;

$|\sigma| \ll l$, where $\sigma = (1 + \lambda h),$ (xix)

Duffin's Equation.

The Duffin's equation arises out of the motion of a pendulum, it is expressed as;

$\check{\phi} + asin\phi = cost, \phi(0) = \phi(\pi) = 0,$ (xx)

The differential equation has an integral equivalent or equation representation viz;

$\phi(t) - \int_0^t (t - y)\left\{a(\phi(y)) - \frac{1}{6}\phi^3(y)\right\} dy = ct$, (xxi)

,where c is the unknown value of $\phi'(0)$ which is called the shape parameter.

Based on the Taylor's series approximation, sinϕ elsewhere could be replaced by; $\phi - \frac{1}{6}\phi^3$ for the first few terms.

Adopting the Euler or one step scheme (iii), an expression for an approximate numerical solution of the Duffin's equation is given by;

$\Phi_{n+1} = \Phi_n - h(asin\boldsymbol{\Phi} - cost),$ (xxii)a

$\Phi_{n+1} - \Phi_n = -h(asin\Phi - cost),$ (xxiii)b

The approximate expression for the numerical solution of the Duffin's equation would converge if equations (xi)b satisfies the scheme in (iv), and obviously this would be practically achievable by choosing a relatively small step size, h such enough, could be close to zero and thus (xxii)b would approach zero faster.

3.0 Conclusion

The existence and uniqueness of a differential equation has been discussed. The criterion for the convergence of the Duffin's equation with an approximate expression for the numerical solution by a one step or Euler scheme has been established, with discussion of the Picard iteration procedure. The Euler scheme is a one step numerical scheme of first order accuracy. Despite the availability of known higher order numerical schemes like the Runge-Kutta method, Adams-Bashforth, Milne schemes etc, the one step scheme could still be adopted by choosing appropriate relatively small step sizes. Convergence is often crucial in the field of numerical methods, as these methods are often iterative in nature and thus the need for convergence among other pertinent factors comprising computation efficiency and costs. The existence of solution of ordinary differential equation is quite extremely important, the Cauchy-Lipschitz condition defines the existence, with the Picard's iteration procedure a sequence of solution can be constructed and obviously this is suitable for our numerical one step Euler Scheme.

References

[1] K.E Atkinson, An introduction to Numerical Analysis (2^{nd} edition). John Wiley & Sons, New York, 1989.

[2] R.M Cohen, Numerical Analysis, MC Graw-Hill, 1973.