Population Dynamics of Harvesting Fishery and Predator

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***Abstract*—In this paper, we have applied brody growth function to describe the population dynamics of fishery without harvesting and then extended to the corresponding harvesting case. It is observed that the harvesting case of fishery too follows brody just similar to non – harvesting case except that the parameters are different. We have also considered a prey – predation problem with an assumption that predator population has little or no effect on the prey population and studied. For this purpose fishery both with and without harvesting are considered as prey and constructed the corresponding growth function to describe the predator population. The predator here considered is completely theoretical and not physically defined. The results show that the predator population size either converges to a finite non negative limit, i.e., either zero or positive or diverges to positive infinity while the fish population size follows brody function and grows to an upper asymptote $[kA/(k+E)]$. In the harvesting case we have shown that the prey converges to a lower asymptote while the predator either (i) converges to a lower positive asymptote or (ii) converges to zero early or (iii) diverges to positive infinity early depending on the three cases studied separately. Numerical simulation of the model, exploration of equilibrium points and stability analysis are also included.**

Keywords—Fish harvesting, Koya-Goshu, Prey –predator, Brody, Simulation, Stability analysis

1. Introduction

Fish and fishing provides a lot many numbers of benefits to human beings including food, employment, business opportunities and recreational activities. However, over fishing has the disadvantage of reducing fish stock and also reducing reproductive aged fish below sustainability. In the management of fishery it is desirable to develop a strategy so that an optimum harvesting rate of fishing is allowed while the fish population is maintained above the sustainable level [1].

The objective of fishery management must be to maintain balance between harvesting of fish and the implications of harvesting on ecology. The availability of fish population varies with place and time across the world, viz., below, above and at par with the sustainable levels due to environmental effects and / or manmade factors. The fish population reproduction rate, fishing or harvesting frequency and living conditions of the fish have a great impact on the dynamics of the fish population system. In fishery management, it is important to fish in such a way that a species is sustainable and without going extinct or without depleted [2-3].

Growth models have been widely studied and applied in many areas especially in plant and animal sciences [5-10]. A generalized mathematical model to describe the dynamics of biological growth is introduced in [7]. The model includes the commonly known growth functions such as Logistic, Richards, Von Bertalanffy and Brody. The model is very flexible in the sense that it can be used for model selection to address a wide range and varieties of growths. Moreover, in [9-10] some theoretical mathematical aspects of predator-prey problem have been introduced. The assumptions include "the interaction of a predation leads to a little or no effect on growth of the prey population". Considering that the prey population grows following Logistic, Von Bertalanffy and Richards functions and the corresponding growth models describing the dynamics of predator population are constructed and studied [9–10]. Brody function is a special case of Richards's growth model with the parameter m taking a value one unit [7].

In the present study we considered the population dynamics of fishery. For this purpose harvesting models of fish population both with and without the presence of predator are considered. We introduce and study harvesting model of fishery resource with predator as an extension of [9 – 10] in the case of Brody.

All the assumptions made and presented in [10] are taken in to consideration as it is in this study. We have considered here that the fish population grows following Brody function together with harvesting term and constructed the corresponding function that describes the predator population. Equilibrium points are found, stability analysis is performed and simulation study is conducted. Important conclusions have been drawn.

The design of the study is divided into mainly two parts harvesting model of fishery with and without predator.

In Section 2, we have presented a growth model to represent fish population without predator and the results are derived and discussed. We have applied brody function to describe population dynamics of fishery and then constructed a model to represent the corresponding harvesting model. It turns out that the harvesting model to follow brody with a change in the parameters. The asymptotic value of harvesting model always lies below that of the non – harvesting model. Pictorial representations in support of this finding are also provided.

In Section 3, stability analysis of harvesting fishery population model without predator is made. The only equilibrium point identified $(x, y) = \left((kE/k + E), 0\right)$ is stable. The asymptotic value of harvesting model lies in the open intervals $(A/2\,, A)$ or in $(0, A/2)$ depending on the parametric representation satisfies $(E < k)$ or $(E > k)$. Further, the asymptotic value converges exactly to $A/2$ whenever $E = k$.

In Section 4, numerical simulations of harvesting model without predator are presented. The simulations are made with varying initial values 20, 30, 60 and 150 and considered parameters satisfying the relations $E < k$, $E = k$ and $E > k$.

Harvesting model of fishery in presence of predator is considered in Section 5. The classical assumptions of prey – predator model are considered relaxed. The case when the population sizes of both fishery and predator reach the same asymptote is studied. Other cases in which predator population converges either zero or positive constant or diverges to positive infinity are discussed.

In Section 6, the stability analysis of fishery harvesting model with predator is considered. The conditions under which the only possible equilibrium point $[kA/(k + E), 0]$ is stable or unstable are considered.

Numerical simulations of harvesting fishery model with predator are considered in Section 7 and the paper ends in section 8 with concluding remarks.

2. Harvesting model of fish population without predator

Let us assume that the dynamics of a fish population in an environment is governed by the Brody equation

$$dx/dt = k(A - x) \tag{1}$$

Here in (1) the constant k represents the growth rate and A represents the natural carrying capacity or asymptotic growth of the fish population. Similarly, the corresponding Brody equation that includes both the growth and the harvesting rates of fish population can be expressed as

$$dx/dt = k(A - x) - Ex \tag{2}$$

Here in (2) the term (Ex) is the harvested yield of fish per unit time and the parameter E is a positive constant measuring the rate of harvest. Clearly, if $E = 0$ is set then the harvesting fishery model (2) reduces to that of the non-harvesting equation (1). In fact the harvesting model represented by (2) also follows a Brody function just similar to (1). This fact can be made evident explicitly by rearranging the harvesting model (2) as $dx/dt = (k + E)\{\,[kA/(k + E)] - x\}$. This is a Brody model with an absolute growth rate of fish population being $(k + E)$ and asymptotic growth value of fish population being $[kA/(k + E)]$. This shows that both the harvesting and non-harvesting models representing the dynamics of fish population are governed by Brody growths function.

From the harvesting model (2) it can be observed that there is an equilibrium point. The equilibrium point is obtained on setting the left hand side of equation (2) equal to zero. Thus, we obtain $\{k(A - x) - Ex\} = 0$ and which on solving gives the required point as $x = [kA/(k + E)]$. This is a non trivial equilibrium point. It is clear that this nontrivial equilibrium point is the same as the asymptotic growth of harvesting model. The non-harvesting fishery population following Brody function as described in (1), for every set of parametric values, grows to an upper asymptote A over evaluation of time, A being always a positive quantity. Similarly, the harvesting fishery population following Brody function as described in (2), for every set of parametric values, grows to an upper asymptote $[kA/(k + E)]$ over evaluation of time where k , A and E being always a positive quantities. It can be observed that the inequality $[kA/(k + E)] \leq A$ holds good for all the possible values of the parameters involved. This inequality may be interpreted as follows: the carrying capacity or asymptotic value of harvesting fish population will never be higher than that of non-harvesting fish population. The interpretation is very sensible, reasonable and is as expected. Also the supported plots are provided.

The modeling problem is required to deal with the growth dynamics of fish population. It has to maximize the harvesting yield with a constraint that the fish population in an environment is maintained at or above the sustainable level. That is, existence of the population has to be protected from collapse. Further, the model has the objective to fix the maximum harvesting rate that will never bring the population down from the sustainable level or collapse. Maximum sustainable yield (MSY) is the largest yield that can be taken away from a species stock without causing the population to collapse [1 – 4]. The maximum sustained yield (MSY) denoted by $M(E)$ is obtained as the product the harvesting rate E and the non trivial equilibrium point $x = [kA/(k + E)]$. In other words, the maximum sustainable yielding of harvesting fishery population occurs at the non – trivial equilibrium point and the yielding amount is expressed in terms of the parameters

$$MSY = M(E) = Ex = E[kA/(k + E)] \tag{3}$$

We will use the parametric expression (3) for maximum sustainable yielding of fishery with harvesting for further analysis and simulation in this study.

3. Stability analysis of harvesting fishery population model without predator

The harvesting model of fishery is a first order ordinary deferential equation explicitly expressed in terms of the dependent variable, and that can be expressed as given in (2) as $dx/dt = kx\,[(A/x) - 1] - Ex \equiv f(x)$. The harvesting model has a non-trivial equilibrium point when the fishery population reaches its upper asymptotic value or equivalently the amount of $\boldsymbol{x} = [kA/(k+E)] = [k/(k+E)]A$. Note that the quantity $[k/(k+E)]$ is always less than 1 in presence of harvesting $E \neq 0$. Hence the quantity $[k/(k+E)]A$ is always less than A. This observation supports the argument that the asymptotic value of the fishery population with harvesting always lies below that without harvesting.

Since $\mathrm{f}'(\mathrm{x}) = -\mathrm{k} - \mathrm{E} = -(\mathrm{k} + \mathrm{E}) < 0$ for all physically meaningful values of the parameters involved, i.e., $\mathrm{k} > 0$ and $E > 0$. In particular, $f'([kA/(k+E)]) = -(\mathrm{k} + \mathrm{E}) < 0$, which shows that the equilibrium point $x = [kA/(k+E)]$ is stable see Figure 1(a) **.** The phase diagram given in Figure 1(a) is obtained with the special case of $k = E$ and in this special case the non trivial equilibrium point takes the value $x = A/2$. This shows that the asymptotic growth of harvesting fishery population in case when $k = E$ takes the value $A/2$. That means the asymptotic growth of fishery drops down from A by half to $A/2$ due to the harvesting with a rate equivalent to growth rate of fishery, i.e., $k = E$.

Further, while keeping the asymptotic value for the population size of fishery without harvesting at A, we can bring down the corresponding asymptotic value of population size of fishery with harvesting to any positive value below A even closer to zero but not to zero by suitably setting a value for E. That is, the asymptotic value of fishery without harvesting remains fixed at A as shown in Figure 1(e) while that of fishery with harvest slides in the open interval $(0, A)$ depending on the values set for E in comparison with the value given for k as shown in Figures 1(c) and 1(d). If $k >> E$ then the asymptote of harvest fishery reaches that of non-harvest fishery and similarly if $k << E$ then the asymptote of harvest fishery approaches zero. The phase diagram of the latter case when the parameters satisfy the relation $k << E$ is shown in Figure 1(b).

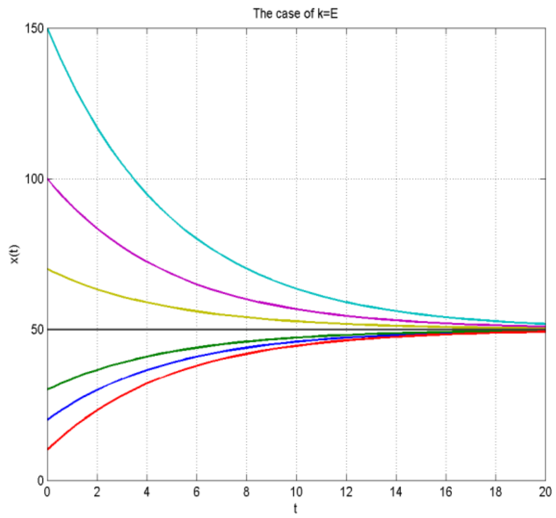

Figure 1(a) Phase diagram of harvesting fishery without predator, the case of $k = E$ upper asymptote is $A/2$

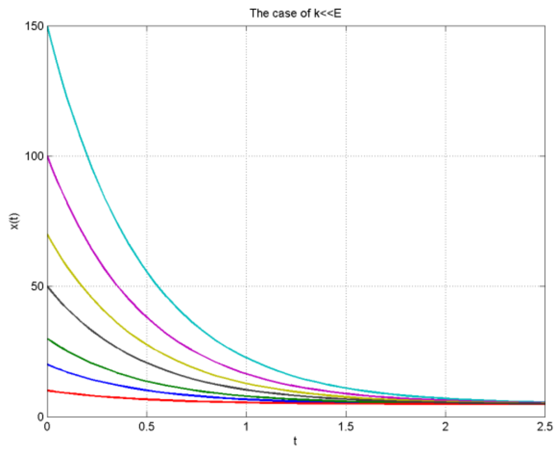

Figure 1(b) Phase diagram of harvesting fishery without predator, the case of $k << E$ ans the upper asymptote is in the open interval $(0,\ A/2)$

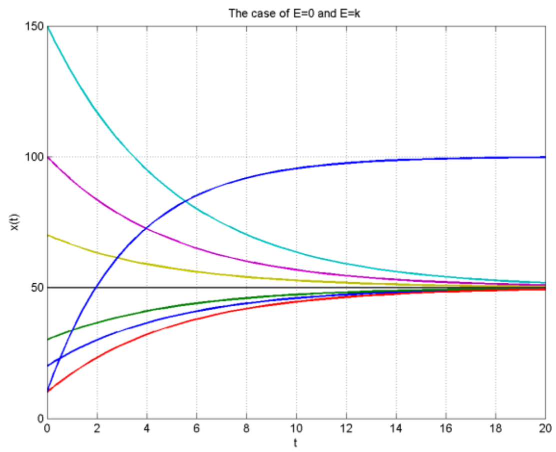

Figure 1(c) Phase diagram of harvesting fishery without predator, the cases of $E = 0$ and $E = k$

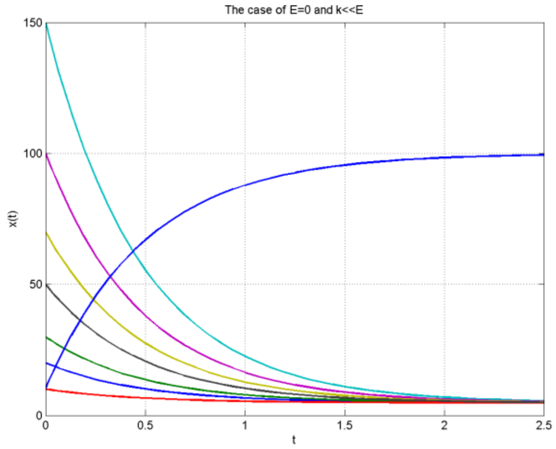

Figure 1(d) Phase diagram of harvesting fishery without predator, the cases of $E = 0$ and $k << E$

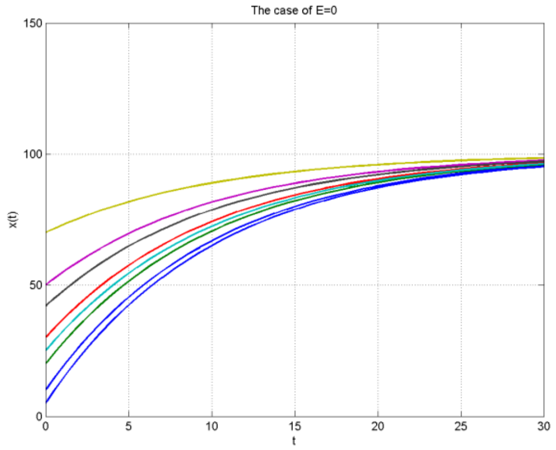

Figure 1(e) Phase diagram of harvesting fishery without predator, the case of $E = 0$ and the upper asymptote is at A

The fish population starting with an initial value grows following brody function as time progresses and ultimately reaches an upper asymptote A in absence of harvesting and $[kA/(k + E)]$ in presence of harvesting. But the quantity $[kA/(k + E)]$ is always less than A for all physically meaningful parametric values. Thus, it is expected to have upper asymptote of harvesting model lies below that of non – harvesting model. The plot 1(a) and 1(e) support this situation.

4. Numerical simulation of harvesting model of fish without predator

In this section we consider numerical study of population dynamics of fishery with variable combinations of values of parameters k and E. This study is aimed to understand the comparison of the asymptotic values of fishery with and without harvesting. As stated earlier, the asymptotic value of fishery with harvesting slides between zero and A while that of fishery without harvesting remains fixed at A . This fact is pictorially illustrated here in this section.

For simplicity we consider four cases of the harvesting parameter E: (i) $E = 0$, (ii) $E = (k/2)$, (iii) $E = k$ and (iv) $E = 2k$. It is clear that the assumption $E = 0$ leads to no harvest and the harvesting model described by (2) reduces to the non-harvesting Brody model given in equation (1). The other parameters are set fixed at as $A = 100, k = 0.1$ and $x(0) = 20$. The pictorial representations are given in Figure 2.

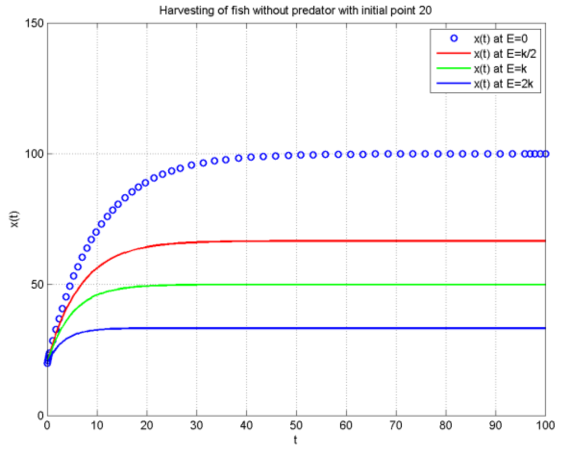

Figure 1 Numerical simulation of harvesting fishery model without predator, with initial value 20, growth rate $\boldsymbol{k = 0.1}$, carrying capacity $\boldsymbol{A = 100}$ and the harvesting rate is varied from top to bottom according as $\boldsymbol{E = 0, 0.05, 0.1}$ and $\boldsymbol{0.2}$

From both the phase diagrams given in Figure 1 and the numerical solutions given in Figure 2 it can be observed as evident that all the solution curves of fishery with harvesting converge to the non-trivial equilibrium point or asymptotic growth value $[kA/(k + E)]$ which lies in the open interval $(0, A)$ while those of fishery without converge to the fixed value A.

The Brody growth function assumes that the asymptotic growth value of a population is independent of the initial value considered. This fact can be verified through simulation study. For this purpose we solve the harvesting and non – harvesting models with different initial values and the results are given in Figure 3. Further, even if the initial population sizes are more than the asymptotic growth values all solution cures of both the harvesting and non – harvesting fishery sizes converge to their respective asymptotic growth values.

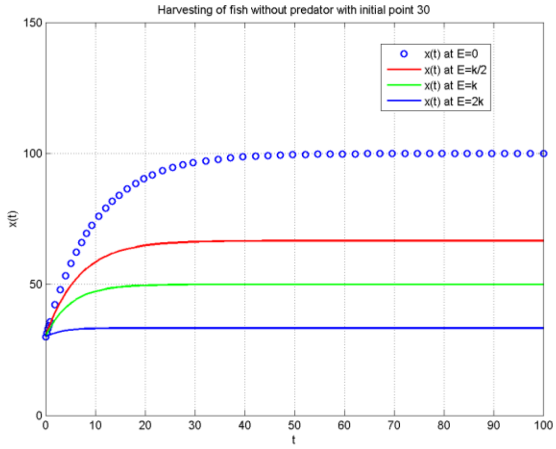

Figure 3(a) Numerical solution of harvesting and non – harvesting model without predator and with initial value $x(0) = 30$, growth rate $\mathrm{k} = 0.1$, carrying capacity $\mathrm{A} = 100$ and varied harvesting rates $\mathrm{E} = 0, 0.05, 0.1$ and 0.2

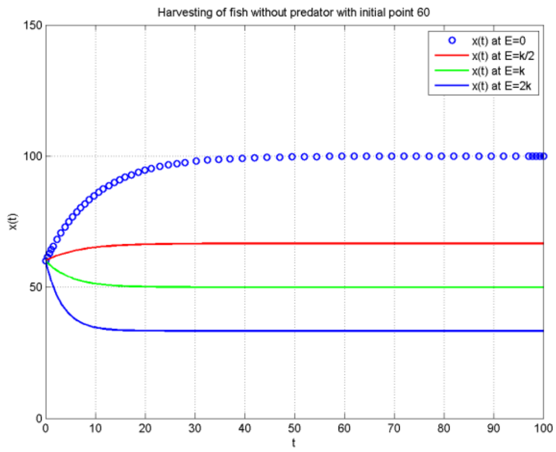

Figure 3(b) Numerical solution of harvesting and non – harvesting model without predator and with initial value $x(0) = 60$, growth rate $\mathrm{k} = 0.1$, carrying capacity $\mathrm{A} = 100$ and varied harvesting rates $\mathrm{E} = 0, 0.05, 0.1$ and 0.2

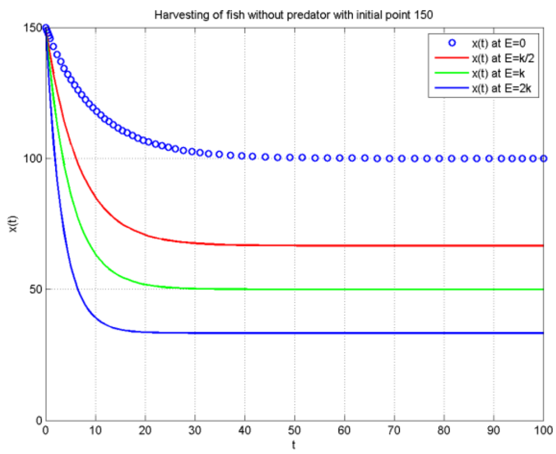

Figure 3(c) Numerical solution of harvesting and non – harvesting model without predator and with initial value $x(0) = 150$, growth rate $\mathrm{k} = 0.1$, carrying capacity $\mathrm{A} = 100$ and varied harvesting rates $\mathrm{E} = 0, 0.05, 0.1$ and 0.2

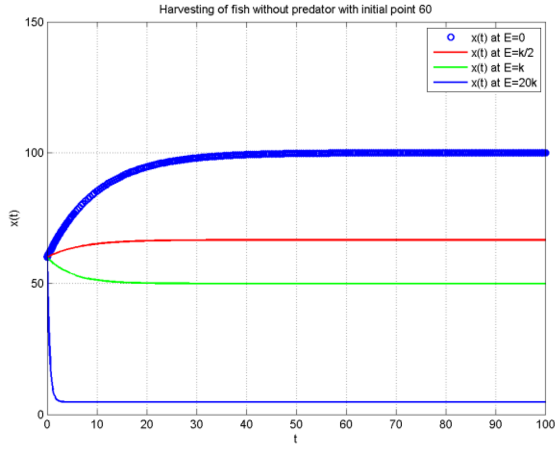

Figure 3(d) Numerical solution of harvesting and non – harvesting model without predator and with initial value $x(0) = 60$, growth rate $\mathrm{k} = 0.1$, carrying capacity $\mathrm{A} = 100$ and varied harvesting rates $\mathrm{E} = 0, 0.05, 0.1$ and 2

We observe from the results shown in Figure 2 and Figure 3 if (i) $\boldsymbol{k} > \boldsymbol{E}$, (ii) $\boldsymbol{k} < \boldsymbol{E}$ and (iii) $\boldsymbol{k} = \boldsymbol{E}$ then the harvesting fish population size converges respectively to an asymptotic value (i) laying in the open interval $(\boldsymbol{A}/\mathbf{2}, \boldsymbol{A})$, (ii) laying in the open interval $(\mathbf{0}, \boldsymbol{A}/\mathbf{2})$ and (iii) exactly at $\boldsymbol{A}/\mathbf{2}$.

5. Harvesting model of fish population with predator

We now consider a prey – predator model with fishery population as prey represented by $x(t)$ and unknown predator population represented by $y(t)$. Let as also make an assumption that the interaction between the prey and predator populations leads to a little or no effect on the growth of the prey population. We further assume that the prey population growth follows Brody function together with the inclusion of a harvesting term as described earlier in (2).

The first order linear differential equation (2) is easy to solve analytically using the method of separation of variables. Hence, the solution for harvesting model given in (2) is obtained as

$$x(t) = [A\,k/(k+E)]\left[1 - Be^{-(k+E)t}\right] \tag{4}$$

Here in (4), the new parameter B represents the parametric expression $B = \{1 - (A_0/A)[1 + (E/k)]\}$ where $A_0 = x(0)$ denotes the initial prey population size. The harvesting curve (4) has no point of inflection just similar to the corresponding non – harvesting curve given in (2), see [8].

Further, it is assumed that the predator population declines naturally in absence of prey and grows with a rate proportional to a function of both x and y in presence of prey. Thus, the rate of change of predator population with respect to time *t* is given by

$$\frac{dy}{dt} = -vy + sx \quad (5)$$

Here in (5), the parameters *v* and *s* are positive constants representing natural death and growth due to the presence of prey rates of predator respectively. Here our objective is to construct the predator model corresponding to the harvesting fishery or prey model given in (4). After substituting (4) in (5), the corresponding predator population growth function is derived in a straight forward manner and given in (6). The complete derivation of (6) from (4) and (5) is given in Appendix.

$$y(t) = y_o e^{\{[Ask/(k+E)]-v\}t} . e^{-\{[Ask/(k+E)^2]\,[1-Be^{-(k+E)t}]\}} . e^{\{[Ask/(k+E)^2](A_0/A)[1+(E/k)]\}} \quad (6)$$

On keen observation of predator growth function (6), it can be interpreted that the asymptotic predator growth converges to a non – negative value, positive or zero, or diverges to infinity on the positive side. These three cases essentially depend on the values assigned to the expression $\{[As\,k/(k+E)] - v\}$ viz., zero, negative or positive. These three cases are discussed in the following:

Case I In this case the model parameters satisfy the condition $As[k/(k+E)] = v$. Thus, in this case the predator population growth model described in (6) takes the form $y(t) = y_0$ $e^{-\{[Ask/(k+E)^2]\,[1-Be^{-(k+E)t}]\}} * e^{\{[Ask/(k+E)^2](A_0/A)[1+(E/k)]\}}$ and its asymptotic growth can be computed easily and is given by a non – zero positive quantity $y(\infty) = y_0 e^{-[Ask/(k+E)^2]\,[1-(A_0/A)[1+(E/k)]]}$. It can be interpreted that the predator population size decays to a lower asymptotic value or grows to an upper asymptotic value as given by the quantity $y(\infty)$, while the prey population grows following brody function and reaches the upper asymptotic value $\{A(k/(k+E))\}$. Furthermore, in this case there is a situation when the asymptotic values of the populations viz., prey and predator, converge to same size if birth rate of predator satisfies the condition

$$s = [(k+E)^2/kA]\left[\frac{\log y_0 - \log\{A\,k/(k+E)\}}{1 - [A_0/\ \{A\,k/(k+E)\}]}\right]$$

Results of the simulation study of Case I are given in Figure 4.

Case II In this case the model parameters satisfy the condition $As[k/(k+E)] < v$. The predator population decays irrespective of the initial size set to any value and converges to zero while the prey population remains to follows Brody growth model and approaches an upper asymptotic value $[A(k/(k+E))]$. Results of the simulation study of Case II are given Figure 5.

Case III In this case the model parameters satisfy the condition $As[k/(k+E)] > v$. The predator population declines for some time and then grows higher and higher without bound and diverges to infinity on the positive side while the prey population follows brody model and grows to a finite positive upper asymptotic value $[A(k/(k+E))]$. Results of the simulation study of Case III are given Figure 6.

6. Stability analysis of fishery harvesting model with predator

The equilibrium point of prey – predator model is obtained from equations (2) and (5) by equating right hand sides of the equations to zero and solving them simultaneously. The result shows that the single equilibrium point of the system is given by $\big((Ak/(k+E)), 0\big)$.

The Jacobean matrix of the system at any arbitrary point (x, y) can be constructed as $J(x, y) = \begin{bmatrix} -(k+E) & 0 \\ ys & -v+sx \end{bmatrix}$.

The Jacobean matrix at the single equilibrium point is given by $J([A(k/(k+E))]\,,0) = \begin{bmatrix} -(k+E) & 0 \\ 0 & -v+(Ask/(k+E)) \end{bmatrix}$ and its eigenvalues are computed as $\lambda_1 = -(k+E)$ and $\lambda_2 = -v + (Ask/(k+E))$. Here note that the first eigenvalue λ_1 is always negative since $-(k+E) < 0$ as $(k+E) > 0$ while the second eigenvalue λ_2 can be negative or zero or positive. Hence, here arise three sets of signs of the eigenvalues which are studied in the following three cases:

Case (i) Let $\lambda_2 = -v + [As(k/(k+E))] = 0$. In this case the first eigenvalue λ_1 is negative while the second eigenvalue λ_2 is zero. Hence, the equilibrium point $([A(k/(k+E))], 0)$ is stable.

Case (ii) let $\lambda_2 = -v + [As(k/(k+E))] < 0$. In this case both the eigenvalues λ_1 and λ_2 are negative. Hence, the equilibrium point $([A(k/(k+E))], 0)$ is stable.

Case (iii) let $\lambda_2 = -v + [As(k/(k+E))] > 0$. In this case the first eigenvalue λ_1 is negative while the second eigenvalue λ_2 is positive. Hence, the equilibrium point $([A(k/(k+E))], 0)$ is unstable.

7. Numerical solutions of harvesting model of fish with predator

Here we consider the numerical simulation of fish harvesting model (6). For simplicity we will consider the three cases I, II and III separately. In each simulation we have one prey curve and four predator curves corresponding to $E = 0$, $E = k/2$, $E = k$ and $E = 2k$. In each case the results are obtained by specifying the prey and predator parameters as follows: Prey parameters are $A = 100$, $A_0 = x(0) = 20$ and $k = 0.1$. The initial size of predator is set according as $y_0 = 1.5A$. The other parameters E, s and v are considered to be varied case wise as follows:

Case I $[\boldsymbol{As(k/(k+E))] = v}$. In this case, we now consider the numerical simulation study of comparative population growths of prey and predator as follows:

(i) For $E = 0$: s=0.0005 & v=0.05; s=0.0022 & v=0.22; s=0.01& v=1; s=10^-10 & v=10^-8.
(ii) For $E = k/2$: s=0.0011 & v=0.0755; s=0.011 & v=0.733; s=0.015 & v=1; s= 10^-8 & v=7 ×10^-7.
(iii) For E=k: s=0.0075 & v=0.375; s=0.018 & v=0.9; s=0.02 & v=1; s= 10^-8 & v=5 ×10^-7.
(iv) For E=2k: s=0.035 & v=1.1667; s=0.06 & v=2; s=0.08 & v=2.67; s= 10^-8 & v=3.33 ×10^-7.

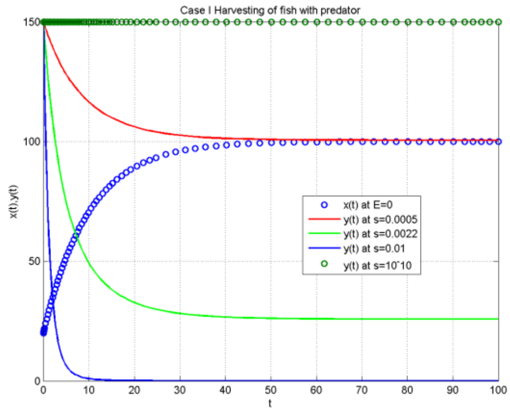

Figure 4(a) Numerical simulation fishery harvesting model in presence of predator, the case I with $E = 0$

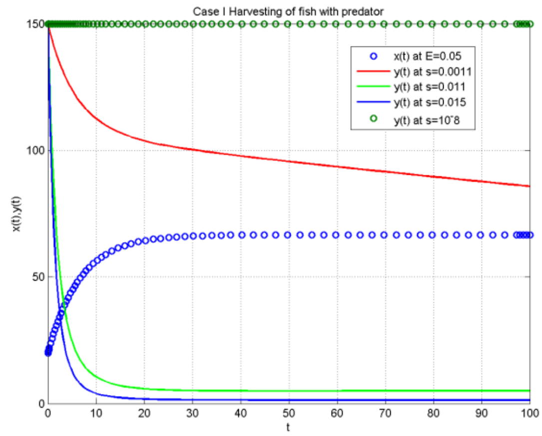

Figure 4(b) Numerical simulation fishery harvesting model in presence of predator, the case I with $E = 0.05$

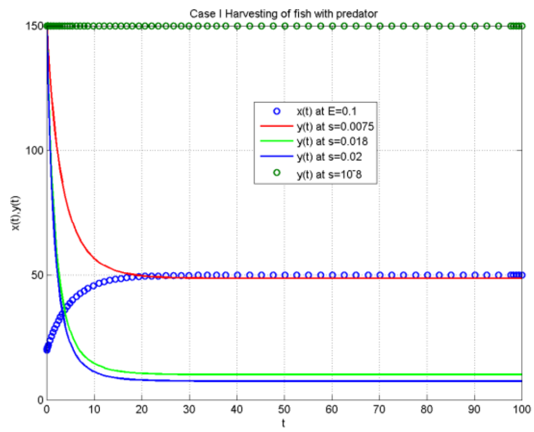

Figure 4(c) Numerical simulation fishery harvesting model in presence of predator, the case I with $E = 0.1$

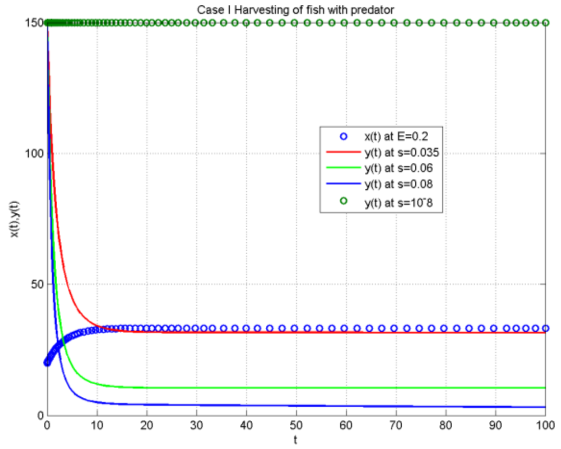

Figure 4(d) Numerical simulation fishery harvesting model in presence of predator, the case I with $E = 0.2$

From the numerical simulation study presented in the Figures (4) it can be concluded that asymptotic population sizes of both the prey and predator decrease as the harvesting rate increases.

Case II $[As(k/(k+E))] < v$**.** In this case, we now consider the numerical simulation study of comparative population growths of prey and predator as follows:

(i) For E=0: s=0.0005 & v=0.099; s=0.00032 & v=0.09; s=0.01 & v=1.5; s=0.001 & v=0.2.
(ii) For E=k/2: s=0.0005 & v=0.1; s=0.00032 & v=0.15; s=0.015 & v=1.3; s=0.001 & v=0.3.
(iii) For E=k: s=0.005 & v=0.3; s=0.0032 & v=0.25; s=0.02 & v=1.2; s=0.001 & v=0.3.
(iv) For E=2k: s=0.005 & v=0.25; s=0.003 & v=0.2; s=0.03 & v=1.25; s=0.001 & v=0.3.

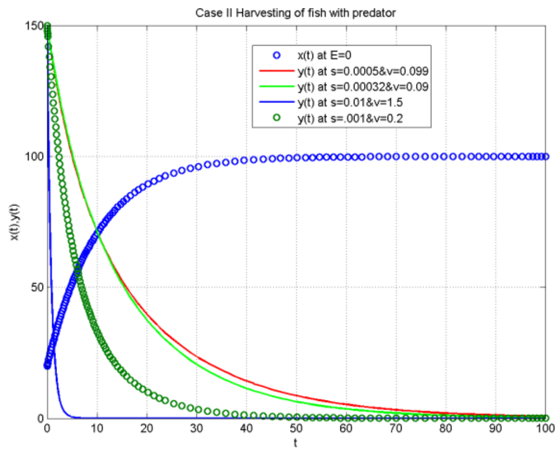

Figure 5(a) Numerical simulation fishery harvesting model in presence of predator, the case II with $E = 0$

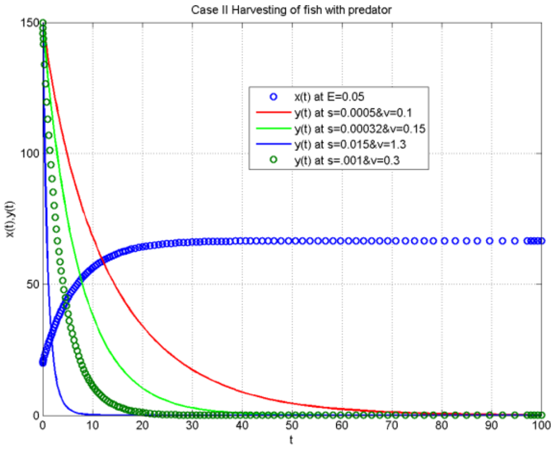

Figure 5(b) Numerical simulation fishery harvesting model in presence of predator, the case II with $E = 0.05$

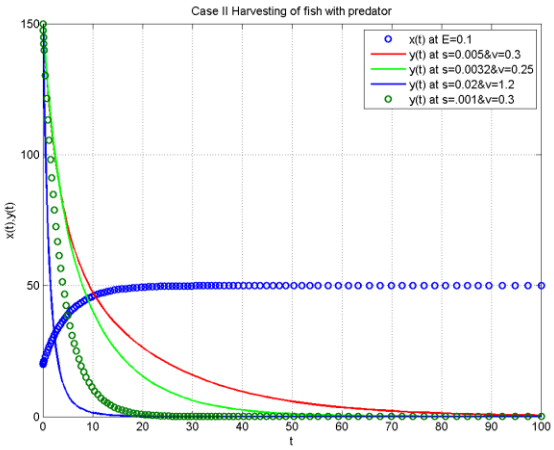

Figure 5(c) Numerical simulation fishery harvesting model in presence of predator, the case II with $E = 0.1$

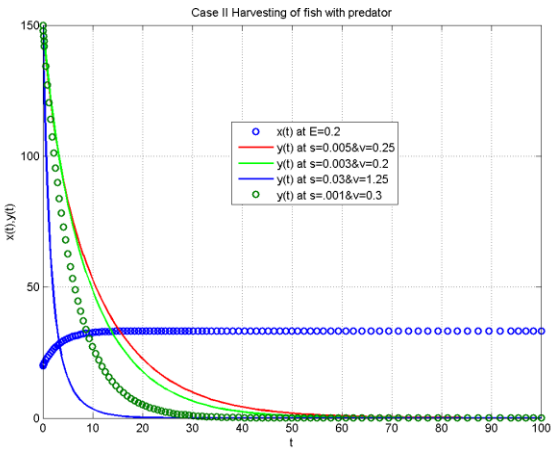

Figure 5(d) Numerical simulation fishery harvesting model in presence of predator, the case II with $E = 0.2$

From the numerical simulation study presented in the Figures (5) it can be concluded that asymptotic population sizes of the prey converges to lower positive values while that of the predator converges to zero faster with the increase the harvesting rates.

Case III $[As(k/(k+E))] > v$**.** In this case, we now consider the numerical simulation study of comparative population growths of prey and predator as follows:

(i) For E=0: s=0.001 & v=0.085; s=0.001 & v=0.088; s=0.001 & v=0.09; s=0.001 & v=0.08.
(ii) For E=k/2: s=0.001 & v=0.061; s=0.001 & v=0.062; s=0.001 & v=0.063; s=0.001 & v=0.06.
(iii) For E=k: s=0.002 & v=0.095; s=0.002 & v=0.0956; s=0.002 & v=0.0959; s=0.002 & v=0.094.
(iv) For E=2k: s=0.003 & v=0.096; s=0.003 & v=0.0965; s=0.003 & v=0.097; s=0.003 & v=0.095.

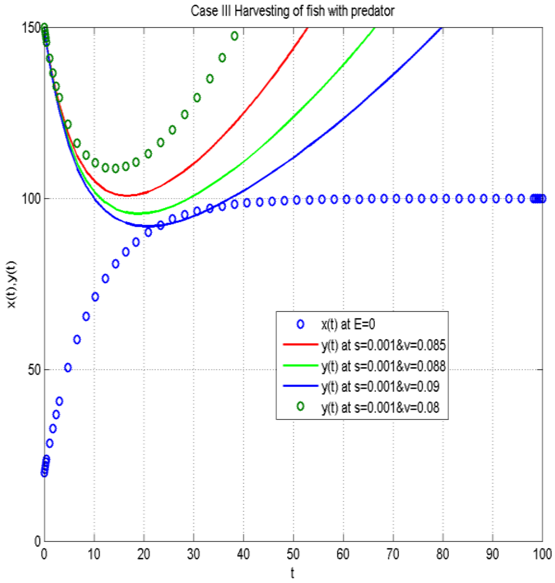

Figure 6(a) Numerical simulation fishery harvesting model in presence of predator, the case III with $E = 0$

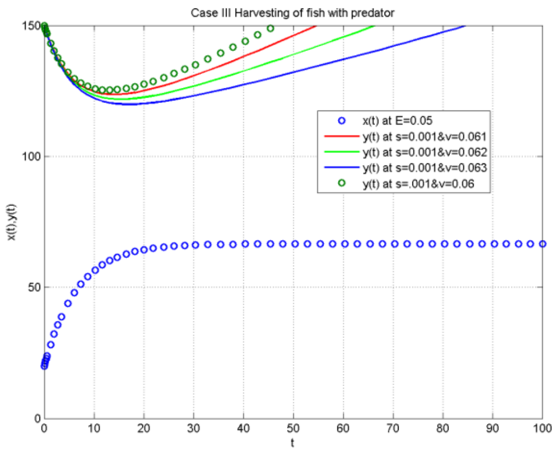

Figure 6(b) Numerical simulation fishery harvesting model in presence of predator, the case III with $E = 0.05$

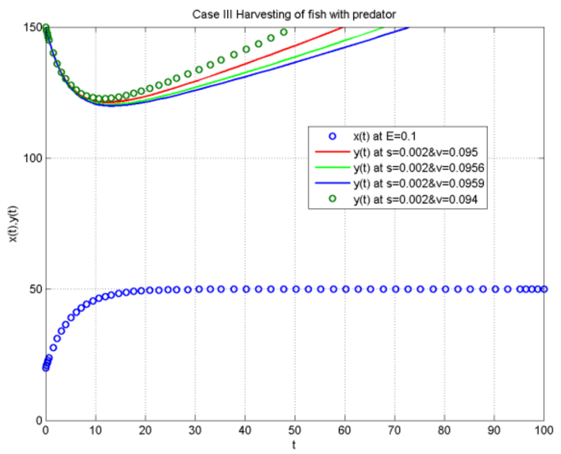

Figure 6(c) Numerical simulation fishery harvesting model in presence of predator, the case III with $E = 0.1$

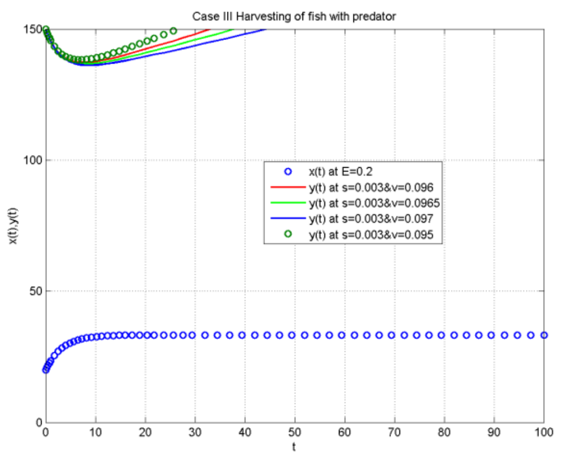

Figure 6(d) Numerical simulation fishery harvesting model in presence of predator, the case III with $E = 0.2$

From the numerical simulation study presented in the Figures (6) it can be concluded that asymptotic population sizes of the prey converges to lower positive values while that of the predator diverges to infinity on the positive side faster with the increase the harvesting rates

Further, in case III the predator population decreases for some time and then increases and ultimately diverges to positive infinity as shown in Figure 6. The minimum point of the cure is donated by t_{min} and has the expression $t_{min} = \left[\frac{1}{k+E}\right] \log\left[\frac{\{1-(A_0(k+E)/kA)\}}{\{1-(v(k+E)/ksA)\}}\right]$.

8. Conclusions

Population dynamics of fish with and without harvesting and with and without predator have been considered and studied. It is assumed that the fish population without harvesting grows following brody model with growth parameter k and asymptotic growth A. Interestingly it is found that the corresponding harvesting model also follows brody function with absolute growth parameter $(k + E)$ and asymptotic growth $[A(k/(k + E))]$. In both the case of harvesting and non – harvesting all the solution curves of fish population converge to the asymptotic growth $x = [\,A\,k/(k + E)]$. In case the harvesting rate E is much more greater than the growth rate k that is, $k << E$, then the asymptotic growth of fish approaches closer to zero but remains as a non – zero positive value.

The harvesting case of prey with predator is also studied based on the assumption that the interaction of a predation leads to a little or no effect on growth of the prey population. The predator models corresponding to harvesting fishery is derived and analyzed. The simulation studies and further analysis of the models show that the predator population grows and either converges to a positive finite limit or zero or diverges to positive infinity, while the prey population grows following brody curve and reaches the upper asymptote $[A(k/(k + E))]$. There is a situation at which both the prey populations and predator populations converge to the same size. There is also a situation where the predator population declines for some time and then starts to increase and diverges to infinity without bound. Moreover, in both cases of fish harvesting with and without predator equilibrium points are identified, which are stable only under some specific conditions. In general, the analytic and simulation studies have revealed some insights to the problem addressed in this paper so that the models obtained can be applied to the real-world situations.

Dedication The first author Kinfe Hailemariam Hntsa dedicates his contribution to the fond memory of his father (Late) priest Hailemariam Hntsa.

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Appendix

Derivation of predator model given that prey (fish) follows Brody model with the inclusion of harvesting

Consider the predator equation given in (5) as $[dy/dt] = -vy + sxy$ which can be rearranged as $[dy/y] = [-v + sx]dt$ and the latter on integration results in $\log y = -vt + s\int x\,dt$. To evaluate the integral in the second term on the left hand side of the fore going equation we now substitute for $x(t)$ the brody function with the inclusion of harvesting fish (prey) growth given in (4). Thus, on substituting (4) in (5), we get

$$\log y = -vt + s[A(k/(k+E))] \int \left[1 - Be^{-(k+E)t}\right] dt$$ (i)

To evaluate the integral in (i) let us introduce a new variable w as $w = 1 - Be^{-(k-E)t}$ and thus we obtain $dw = -(k+E)Be^{-(k+E)t}dt = -(k+E)(w-1)dt$ or equivalently $dt = -[1/(k+E)(w-1)]\, d$. Now, on substituting w and dw the equation (i) takes the following form

$$\log y = -vt - [skA/(k+E)^2] \int w/(w-1)\, dw$$

$$= -vt - [skA/(k+E)^2] \int [1 + (1/w-1)]dw$$

$$= -vt - [skA/(k+E)^2][w + \log|w-1|] + \log D$$

$$= -vt - [skA/(k+E)^2]\left[1 - Be^{-(k-E)t} + \log\left(Be^{-(k+E)t}\right)\right] + \log D$$ (ii)

On taking antilogarithms on both sides, the equation (ii) takes the form $y = De^{-vt}e^{-[ska/(k+E)^2][1-Be^{-(k-E)t}+\log B-(k+E)t]}$ (iii)

The parameter D in (iii) denotes an integral constant and is determined by using the initial condition. The substitution of the initial condition on predation $y(0) = y_0$ in (iii) and after some simplification steps we get

$$D = y_0 e^{[skA/(k+E)^2][1-B+\log B]}$$ (iv)

Up on substituting (iv) in (iii) we obtain

$$y(t) = y_0 e^{[skA/(k+E)^2][1-B+\log B]}$$
$$e^{-vt}e^{-[skA/(k+E)^2][1-Be^{-(k-E)t}+\log B-(k+E)t]}$$

Or equivalently

$$y(t) = y_0 e^{[(skA/(k+E))-v]t}$$
$$e^{-[skA/(k+E)^2][1-Be^{-(k-E)t}]}$$
$$e^{[skA/(k+E)^2][A_0/(kA/(k+E))]}$$ (v)

Therefore (v) is the required solution representing population growth of the predator.