ON THE DEFICIENCY INDICES OF PRODUCT DIFFERENTIAL OPERATORS IN DIRECT SUM SPACES

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Abstract. In this paper, we discuss the deficiency index problem for the product differential operators which are generated by a general ordinary quasi-differential expressions $\boldsymbol{\tau_1}, \boldsymbol{\tau_2}, \dots, \boldsymbol{\tau_n}$ each of order $\boldsymbol{n}$ with complex coefficients in the direct sum $\oplus_{p=1}^{N} \boldsymbol{L_w^2(I_p)}$ of spaces of functions defined on each of the separate intervals in the cases of regular and singular end-points. The domains of these operators are described in terms of boundary conditions featuring $\boldsymbol{L_w^2}$ -solutions of the differential equations. These results extend those of formally symmetric expression $\boldsymbol{\tau}$ studied in [1, 2], [8 - 11] and [14,17], and those of general quasi-differential expressions $\boldsymbol{\tau}$ in [3, 4, 7].

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I. INTRODUCTION

The deficiency index problem for ordinary differential operators, at least in the form that we now identity it, goes back to Hermann Weyl [15] around 1910, although in one guise or another it is present in investigations of self-adjoint boundary value problems going back a good deal longer. The operators which fulfill the role that the self-adjoint and maximal symmetric operators play in the case of a formally symmetric expression τ are those which are regularly solvable with respect to the minimal operators $T_0(\tau)$ and $T_0(\tau^+)$ generated by a general ordinary quasi-differential expression τ and its formal adjoint τ^+ respectively, the minimal operators $T_0(\tau)$ and $T_0(\tau^+)$ form an adjoint pair of closed, densely-defined operators in the underlying L_w^2-space, that is $T_0(\tau) \subset [T_0(\tau^+)]^*$. Such an operator S satisfies $T_0(\tau) \subset S \subset [T_0(\tau^+)]^*$ and for some $\lambda \in \mathbb{C}, (S - \lambda I)$ is a Fredholm operator of zero index, this means that S has the desirable Fredholm property that the equation $(S - \lambda I)u = f$ has a solution if and only if f is orthogonal to the solution space of $(S - \lambda I)u = 0$ and furthermore the solution space of $(S - \lambda I)u = 0$ and $(S^* - \bar{\lambda} I)v = 0$ have the same finite dimension. This notion was originally due to Visik [13].

The AGN (Akhiezer, Glazman and Naimark) [1,11] characterized all self-adjoint realizations of linear symmetric (formally self-adjoint) ordinary differential equations in terms of maximal domain functions. These functions depend on the coefficients and this dependence is implicit and complicated. In the regular case an explicit characterization in terms of two-point boundary conditions can be given. In the singular case when the deficiency index d is maximal. The characterization can be made more explicit by replacing the maximal domain functions by a solutions basis for any real or complex value of the spectral parameter λ. In 1986 Sun [9] found a representation of the self-adjoint singular conditions in terms of certain solutions for non real values of λ. This leads to a classification of solutions as limit-point (LP) or limit-circle (LC) in analogy with the celebrated Weyl classification in the second-order case. The LC solutions contribute to the singular boundary conditions, the LP solutions do not. In [4, 7] Evans and Ibrahim extend their results for a general ordinary quasi-differential expression τ of n-th order with complex coefficients in the singular case.

In [5, 17] Everitt and Zettl considered the problem of integrable square solutions of products of differential expressions $\tau_1, \tau_2, \dots, \tau_n$ and investigate the relation ship between the deficiency indices of general symmetric differential expressions $\tau_1, \tau_2, \dots, \tau_n$ and those of the product expression $\prod_{j=1}^{n} \tau_j$ and in [8] Ibrahim considered the problem of the product self-adjoint Sturm-Liouville differential operators in direct sum spaces.

Our objective in this paper is to discuss the deficiency index problem for the product differential operators $\prod_{j=1}^{n} T_0(\tau_j)$ in the direct sum $\oplus_{p=1}^{N} L_w^2(I_p)$ of spaces of functions defined on each of the separate intervals in the cases of regular and singular end-points. The domains of these operators are described in terms of boundary conditions featuring L_w^2-solutions of the equation $\left[\prod_{j=1}^{n} \tau_{jp} - \lambda w\right]u = 0$ and the adjoint equation $\left[\prod_{j=1}^{n} \tau_{jp}^{+} - \bar{\lambda} w\right]v = 0$ $(\lambda \in \mathbb{C})$. These boundary conditions involve τ_{jp} expression on any finite number of intervals I_p, $p = 1,2, \dots, N$.

We shall not attempt to prove some results in detail since the proofs tend to be rather technically complicated and are in analogue to those in [3, 4] and [7].

We deal throughout this paper with a quasi-differential expression τ of arbitrary order n with complex coefficients defined by Shin-Zettl matrices [4,7,17], and the minimal operator $T_0(\tau)$ generated by $w^{-1}\tau[.]$ in $L_w^2(I)$, where w is a positive weight function on the underlying interval $I = (a, b)$. The end-points a and b of I may be regular or singular end-points.

II. NOTATION AND PRELIMINARIES

We begin with a brief survey of adjoint pairs of operators and their associated regularly solvable operators; a full treatment may be found in [3, Chapter III], [4], [7-9], [11] and [17]. The domain and range of a linear operator T acting in a Hilbert space H will be denoted by $D(T)$ and $R(T)$ respectively and $N(T)$ will denote its null space. The nullity of T, written $nul\ (T)$, is the dimension of $N(T)$ and the deficiency of T, written $def(T)$, is the co-dimension of $R(T)$ in H; thus if T is densely defined and $R(T)$ is closed , then $def(T) = nul(T^*)$. The Fredholm domain of T is (in the notation of [3]) the open subset $\Delta_3(T)$ of $\mathbb{C}$ consisting of those values of $\lambda \epsilon \mathbb{C}$ which are such that $(T - \lambda I)$ is a Fredholm operator, where I is the identity operator in H. Thus $\lambda \epsilon \Delta_3(T)$ if and only if $(T - \lambda I)$ has closed range and finite nullity and deficiency. The index of $(T - \lambda I)$ is the number $ind(T - \lambda I) = nul\ (T - \lambda I) - def\ (T - \lambda I)$, this being defined for $\lambda \epsilon \Delta_3(T)$.

Two closed densely defined operators A and B acting in a Hilbert space H are said to form an adjoint pair if $A \subset B^*$ and, consequently, $B \subset A^*$; equivalently , $(Ax, y) = (x, By)$ for all $x \epsilon D(A)$ and $y \epsilon D(B)$, where $(.,.)$ denotes the inner-product on H.

Definition 2.1: The field of regularity $\Pi(A)$ of A is the set of all $\lambda \epsilon \mathbb{C}$ for which there exists a positive constant $K(\lambda)$ such that

$$\|(A - \lambda I)x\| \geq K(\lambda)\|x\| \text{ for all } x \epsilon D(A), \tag{2.1}$$

or, equivalently, on using the Closed Graph Theorem, $nul\ (A - \lambda I) = 0$ and $R(A - \lambda I)$ is closed.

The joint field of regularity $\Pi(A, B)$ of A and B is the set of $\lambda \epsilon \mathbb{C}$ which are such that $\lambda \epsilon \Pi(A)$, $\bar{\lambda} \epsilon \Pi(B)$ and both $def(A - \lambda I)$ and $def(B - \bar{\lambda} I)$ are finite. An adjoint pair A and B is said to be compatible if $\Pi(A, B) \neq \phi$.

Definition 2.2: A closed operator S in H is said to be regularly solvable with respect to the compatible adjoint pair of A and B if $A \subset S \subset B^*$ and $\Pi(A, B) \cap \Delta_4(S) \neq \phi$, where $\Delta_4(S) = \{\lambda : \lambda \epsilon \Delta_3(S),\ ind\ (S - \lambda I) = 0\}$.

Given two operators A and B both acting in a Hilbert space H, we wish to consider the product operator AB. This is defined as follows

$$D(AB) = \{x \epsilon D(B) \mid Bx \epsilon D(A)\} \text{ and}$$
$$(AB)x = A(Bx) \text{ for all } x \epsilon D(AB). \tag{2.2}$$

It may be happen in general that $D(AB)$ contains only the null element of H. However, in the case of many differential operators the domains of the product will be dense in H .

The next result gives conditions under which the deficiency of a product is the sum of the deficiencies of the factors. It is a generalization of that in [8, Theorem A] and [16].

Lemma 2.3 (cf. [8, Lemma 2.3]). Let A and B be closed operators with dense domains in a Hilbert space H. Suppose that $\lambda \epsilon \Pi(A, B)$. Then AB is a closed operator with dense domain and

$$def(AB - \lambda I) = def(A - \lambda I) + def(B - \bar{\lambda} I). \tag{2.3}$$

Evidently Lemma 2.3 extends to the product of any finite number of operators $A_1, A_2, ..., A_n$.

III. Quasi-differential expressions in direct sum spaces

The quasi-differential expressions are defined in terms of a Shin-Zettl matrix F_p on an interval I_p. The set $Z_n(I_p)$ of Shin-Zettl matrices on I_p consists of $n \times n$-matrices $F_p = \{f_{rs}^p\}$, $p = 1,2, ..., N$, whose entries are complex-valued functions on I_p which satisfy the following conditions:

$$f_{rs}^p \epsilon L_{loc}^2(I_p), \quad (1 \leq r, s \leq n,\ n \geq 2)$$
$$f_{r,r+1}^p \neq 0, \quad \text{a. e., on } I_p\ (1 \leq r \leq n - 1) \tag{3.1}$$
$$f_{rs}^p = 0, \text{ a. e., on } I_p,\ (2 \leq r + 1 < s \leq n),\ p = 1, ..., N.$$

For $F_p \epsilon Z_n(I_p)$, the quasi-derivatives associated with F_p are defined by :

$$y^{[0]} := y,$$
$$y^{[r]} := (f_{r,r+1}^p)^{-1}\left\{(y^{[r-1]})' - \sum_{s=1}^{r} f_{rs}^p\, y^{[s-1]}\right\}, \tag{3.2}$$
$$(1 \leq r \leq n - 1)$$
$$y^{[n]} := \left\{(y^{[n-1]})' - \sum_{s=1}^{n} f_{rs}^p\, y^{[s-1]}\right\},$$

where the prime $'$ denotes differentiation.

The quasi-differential expression τ_p associated with F_p is given by:

$$\tau_p[.] := i^n y^{[n]}, \quad (n \geq 2), \tag{3.3}$$

this being defined on the set :

$$V(\tau_p) := \{y : y^{[r-1]} \epsilon AC_{loc}(I_p), r = 1,2, ..., n\},$$

$p = 1,2, ..., N$, where $AC_{loc}(I_p)$, denotes the set of functions which are absolutely continuous on every compact subinterval of I_p.

The formal adjoint τ_p^+ of τ_p is defined by the matrix F_p^+ given by:

$$\tau_p^+[.] := i^n y_+^{[n]}, \text{ for all } y \epsilon V(\tau_p^+), \tag{3.4}$$
$$V(\tau_p^+) := \left\{y : y_+^{[r-1]} \epsilon AC_{loc}(I_p), r = 1,2, ..., n\right\},$$

$p = 1,2, ..., N$, where $y_+^{[r-1]}$, the quasi-derivatives associated with the matrix F_p^+ in $Z_n(I_p)$,

$$F_p^+ = (f_{rs}^p)^+ = (-1)^{r+s+1} \bar{f}_{n-s+1,n-r+1}^p, \tag{3.5}$$

for each r and s .

Note that: $(F_p^+)^+ = F_p$ and so $(\tau_p^+)^+ = \tau_p$. We refer to [3], [4], [7], [11] and [17] for a full account of the above and subsequent results on quasi-differential expressions.

For $u \epsilon V(\tau_p)$, $v \epsilon V(\tau_p^+)$ and $\alpha, \beta \epsilon I_p$, we have Green's formula,

$$\int_{a_p}^{b_p} \left\{\bar{v}\tau_p[u] - u\,\overline{\tau_p^+[v]}\right\} dx =$$
$$[u, v](b_p) - [u, v](a_p), p = 1,2, ..., N, \tag{3.6}$$

where,

$$[u, v](x) = i^n \left(\sum_{r=0}^{n-1} (-1)^{r+s+1} u^{[r]}(x)\, \overline{v_+^{[n-r-1]}}\,(x)\right)$$
$$= (-i)^n (u, u^{[1]}, ..., u^{[n-1]}) \times J_{n\times n} \begin{pmatrix} \bar{v} \\ \vdots \\ \bar{v}_+^{[n-1]} \end{pmatrix}(x); \tag{3.7}$$

where $J_{n\times n} = ((-1)^r \delta_{r,n-s+1})_{1\leq r,s\leq n}$ is the non-singular matrix, see [1, 3, 4, 7] and [9, 11, 17]. Let the interval I_p have end-points a_p, b_p $(-\infty \leq a_p < b_p \leq \infty)$, and let $w_p : I_p \to \mathbb{R}$ be a non-negative weight function with $w_p \epsilon L_{loc}^1(I_p)$ and $w_p > 0$ (for almost all $x \epsilon I_p$). Then $H_p = L_{w_p}^2(I_p)$ denotes the Hilbert function space of equivalence classes of Lebesgue measurable functions\ such that $\int_{I_p} w_p |f|^2 < \infty$; the inner-product is defined by:

$$(f, g)_p := \int_{I_p} w_p f(x)\overline{g(x)} dx \quad (f, g \epsilon L_{w_p}^2(I_p). \tag{3.8}$$

$p = 1,2, ..., N$).

The equation

$$\tau_p[u] - \lambda w_p u = 0 \quad (\lambda \epsilon \mathbb{C}) \text{ on } I_p,\ p = 1,2, ..., N \tag{3.9}$$

is said to be regular at the left end-point $a_p \epsilon \mathbb{R}$, if for all $X \epsilon (a_p, b_p)$,

$$a_p \epsilon \mathbb{R},\ w_p,\ f_{rs}^p \epsilon L^1(a_p, X), \quad r, s = 1,2, ..., n;$$

$p = 1,2, ..., N$, otherwise (3.9) is said to be singular at a_p.

If (3.9) is regular at both end-points, then it is said to be regular; in this case we have,

$a_p,\ b_p \in \mathbb{R},\ w_p,\ f^p_{rs} \in L^1(a_p,\ b_p), \quad r,s = 1,2,\dots,n;$ $p = 1,2,\dots,N$.

We shall be concerned with the case when a_p is a regular end-point of (3.9), the end-point b_p being allowed to be either regular or singular. Note that, in view of (3.5), an end-point of I_p is regular for (3.9), if and only if it is regular for the equation,

$$\tau_p^+[v] - \bar{\lambda} w_p v = 0 \ (\lambda \in \mathbb{C}) \text{ on } I_p,\ p = 1,2,\dots,N. \quad (3.10)$$

Note that, at a regular end-point a_p, say , $u^{[r-1]}(a_p)\left(v_+^{[r-1]}(a_p)\right)$, $r = 1,2,\dots,n$ is defined for all $u \in V(\tau_p)\ \left(v \in V(\tau_p^+)\right)$. Set:

$$\left.\begin{aligned} D(\tau_p) &:= \{u: u \in V(\tau_p), \\ &\quad u \text{ and } w_p^{-1}\tau_p[u] \in L^2_{w_p}(a_p,\ b_p)\} \\ D(\tau_p^+) &:= \{v: v \in V(\tau_p^+), \\ &\quad v \text{ and } w_p^{-1}\tau_p^+[v] \in L^2_{w_p}(a_p,\ b_p)\} \end{aligned}\right\}. \quad (3.11)$$

The subspaces $D(\tau_p)$ and $D(\tau_p^+)$ of $L^2_{w_p}(a_p,\ b_p)$ are domains of the so-called maximal operators $T(\tau_p)$ and $T(\tau_p^+)$ respectively, defined by:

$T(\tau_p)\,u := w_p^{-1}\tau_p[u]$, $(u \in D(\tau_p))$ and

$T(\tau_p^+)\,v := w_p^{-1}\tau_p^+[v]$, $(v \in D(\tau_p^+))$.

For the regular problem the minimal operators $T_0(\tau_p)$ and $T_0(\tau_p^+)$, $p = 1,2,\dots,N$ are the restrictions of $w_p^{-1}\tau_p[u]$ and $w_p^{-1}\tau_p^+[v]$ to the subspaces:

$$\left.\begin{aligned} D_0(\tau_p) &:= \{u: u \in D(\tau_p), \\ &\quad u^{[r-1]}(a_p) = u^{[r-1]}(b_p) = 0, p = 1,2,\dots,N\}, \\ D_0(\tau_p^+) &:= \{v: v \in D(\tau_p^+), \\ &\quad v_+^{[r-1]}(a_p) = v_+^{[r-1]}(b_p) = 0,\ p = 1,2,\dots,N\} \end{aligned}\right\}, (3.12)$$

respectively. The subspaces $D_0(\tau_p)$ and $D_0(\tau_p^+)$ are dense in $L^2_{w_p}(a_p,\ b_p)$ and $T_0(\tau_p)$ and $T_0(\tau_p^+)$ are closed operators (see [3], [4], [7] and [17, Section 3]).

In the singular problem we first introduce the operators $T_0'(\tau_p)$ and $T_0'(\tau_p^+)$; $T_0'(\tau_p)$ being the restriction of $w_p^{-1}\tau_p[.]$ to the subspace:

$$D_0'(\tau_p) := \{u: u \in D(\tau_p),\ supp\,(u) \subset (a_p,\ b_p),\quad p = 1,2,\dots,N\} \quad (3.13)$$

and with $T_0'(\tau_p^+)$ defined similarly. These operators are densely-defined and closable in $L^2_{w_p}(a_p,\ b_p)$; and we define the minimal operators $T_0(\tau_p)$ and $T_0(\tau_p^+)$ to be their respective closures (see [3], [4], [7], [10,11] and [17]). We denote the domains of $T_0(\tau_p)$ and $T_0(\tau_p^+)$ by $D_0(\tau_p)$ and $D_0(\tau_p^+)$ respectively. It can be shown that:

$$\left.\begin{aligned} u \in D_0(\tau_p) &\Rightarrow u^{[r-1]}(a_p) = 0, r = 1,2,\dots,n, \\ v \in D_0(\tau_p^+) &\Rightarrow v_+^{[r-1]}(a_p) = 0, r = 1,2,\dots,n, \end{aligned}\right\} \quad (3.14)$$

$p = 1,\dots,N$, because we are assuming that a_p is a regular end-point. Moreover, in both regular and singular problems, we have

$$T_0^*(\tau_p) = T(\tau_p^+), \quad T_0^*(\tau_p) = T_0(\tau_p^+), \quad (3.15)$$

$p = 1,2,\dots,N$; see [17, Section 5] in the case when $\tau_p = \tau_p^+$ and compare with treatment in [3, Section III.10.3] and [4] in general case.

In the case of two singular end-points, the problem on $(a_p,\ b_p)$ is effectively reduced to the problems with one singular end-point on the intervals $(a_p,\ c_p]$ and $[c_p,\ b_p)$, where $c_p \in (a_p,\ b_p)$. We denote by $T(\tau_p;\ a_p)$ and $T(\tau_p;\ b_p)$ the maximal operators with domains $D(\tau_p;\ a_p)$ and $D(\tau_p;\ b_p)$ and denote $T_0(\tau_p;\ a_p)$ and $T_0(\tau_p;\ b_p)$ the closures of the operators $T_0'(\tau_p;\ a_p)$ and $T_0'(\tau_p;\ b_p)$ defined by:

$$D_0'(\tau_p;.) := \{u: u \in D(\tau_p;.),\ supp\,(u) \subset (a_p,\ b_p),\ p = 1,2,\dots,N\} \quad (3.16)$$

on the intervals $(a_p,\ c_p]$ and $[c_p,\ b_p)$ respectively, see ([3], [4], [7], [10,11] and [17]).

Let $\widetilde{T}_0'(\tau_p), p = 1,2,\dots,N$, be the orthogonal sum as:

$\widetilde{T}_0'(\tau_p) = T_0'(\tau_p;\ a_p) \oplus T_0'(\tau_p;\ b_p)$ in

$L^2_{w_p}(a_p,\ b_p) = L^2_{w_p}(a_p,\ c_p) \oplus L^2_{w_p}(c_p,\ b_p)$,

$p = 1,2,\dots,N$, $\widetilde{T}_0'(\tau_p)$ is densely-defined and closable in $L^2_{w_p}(a_p,\ b_p)$ and its closure is given by:

$$\widetilde{T}_0(\tau_p) = T_0(\tau_p;\ a_p) \oplus T_0(\tau_p;\ b_p),\quad p = 1,2,\dots,N.$$

Also,

$$nul[\widetilde{T}_0(\tau_p) - \lambda I] = nul[T_0(\tau_p;\ a_p) - \lambda I] + nul[T_0(\tau_p;\ b_p) - \lambda I],$$

$$def[\widetilde{T}_0(\tau_p) - \lambda I] = def[T_0(\tau_p;\ a_p) - \lambda I] + def[T_0(\tau_p;\ b_p) - \lambda I]$$

and $[R[\widetilde{T}_0(\tau_p) - \lambda I]$ is closed if and only if $R[T_0(\tau_p;\ a_p) - \lambda I]$ and $R[T_0(\tau_p;\ b_p) - \lambda I]$ are both closed. These results imply in particular that,

$$\Pi[\widetilde{T}_0(\tau_p)] = \Pi[T_0(\tau_p;\ a_p)] \cap \Pi[T_0(\tau_p;\ b_p)],$$

$p = 1,2,\dots,N$.

We refer to [3, Section 3.10.14], [4] and [7] for more details.

Remark 3.1: If $S_p^{a_p}$ is a regularly solvable extension of $T_0(\tau_p;\ a_p)$ and $S_p^{b_p}$ is a regularly solvable extension of $T_0(\tau_p;\ b_p)$, then $S = \oplus_{p=1}^N (S_p^{a_p} \oplus S_p^{b_p})$ is a regularly solvable extension of $\widetilde{T}_0(\tau)$. We refer to [3], [4, 7] and [17] for more details.

Next, we state the following results; the proof is similar to that in [3, Section 3.10.4], [4] and [7].

Theorem 3.2:

$\widetilde{T}_0(\tau_p) \subset T_0(\tau_p),\quad T(\tau_p) \subset T_0(\tau_p;\ a_p) \ \oplus \ T_0(\tau_p;\ b_p)$

and $\dim(D[T_0(\tau_p)]/D[\widetilde{T}_0(\tau_p)]) = n, \quad p = 1,2,\dots,N$.

If $\lambda \in \Pi[\widetilde{T}_0(\tau_p)] \cap \Delta_3[T_0(\tau_p) - \lambda I]$, then

$$ind[T_0(\tau_p) - \lambda I] = n - def[T_0(\tau_p;\ a_p) - \lambda I] - def[T_0(\tau_p;\ b_p) - \lambda I],$$

and in particular, if $\lambda \in \Pi[T_0(\tau_p)]$,

$$def[T_0(\tau_p) - \lambda I] = def[T_0(\tau_p;\ a_p) - \lambda I] + def[T_0(\tau_p;\ b_p) - \lambda I] - n.$$

Remark 3.3: It can be shown that

$$\left.\begin{aligned} D[\widetilde{T}_0(\tau_p)] &:= \{u:\ u \in D[T_0(\tau_p)], \\ &\quad u^{[r-1]}(c_p) = 0,\ p = 1,2,\dots,N\} \\ D[\widetilde{T}_0(\tau_p^+)] &:= \{v: v \in D[T_0(\tau_p^+)], \\ &\quad v_+^{[r-1]}(c_p) = 0,\ p = 1,2,\dots,N\} \end{aligned}\right\}; \quad (3.17)$$

see [3, Section 3.10.4].

Let H be the direct sum,

$$H = \oplus_{p=1}^{N} H_p = \oplus_{p=1}^{N} L^2_{w_p}(a_p, b_p).$$

The elements of H will be denoted by $\tilde{f} = \{f_1, f_2, \dots, f_N\}$ with $f_1 \epsilon H_1, f_2 \epsilon H_2, \dots, f_N \epsilon H_N$.

Remark 3.4: When $I_i \cap I_j = \emptyset,\ i \neq j;\ i,j = 1,2,\dots,N$, the direct sum space $\oplus_{p=1}^{N} L^2_{w_p}(a_p, b_p)$ can be naturally identified with the space $L^2_w(\cup_{p=1}^{N} I_p)$, where $w_p = w$ on $I_p, p = 1,2,\dots,N$. This remark is of significance when $(\cup_{p=1}^{N} I_p)$, may be taken as a single interval (see [3] and [4]).

We now establish by [4] and [6 - 8] some further notations,

$$\left.\begin{aligned} D_0(\tau) &= \oplus_{p=1}^{N} D_0(\tau_p), D(\tau) = \oplus_{p=1}^{N} D(\tau_p), \\ D_0(\tau^+) &= \oplus_{p=1}^{N} D_0(\tau_p^+), D(\tau^+) = \oplus_{p=1}^{N} D(\tau_p^+) \end{aligned}\right\}, \quad (3.18)$$

$$\begin{aligned} T_0(\tau)f &= \{T_0(\tau_1)f_1, T_0(\tau_2)f_2, \dots, T_0(\tau_N)f_N\}; \\ &\quad f_1 \epsilon D_0(\tau_1),\ f_2 \epsilon D_0(\tau_2), \dots, f_N \epsilon D_0(\tau_N), \\ T_0(\tau^+)g &= \{T_0(\tau_1^+)g_1, T_0(\tau_2^+)g_2, \dots, T_0(\tau_N^+)g_N\}; \\ &\quad g_1 \epsilon D_0(\tau_1^+), g_2 \epsilon D_0(\tau_2^+), \dots, g_N \epsilon D_0(\tau_N^+). \end{aligned}$$

Also,

$$\begin{aligned} T(\tau)f &= \{T(\tau_1)f_1, T(\tau_2)f_2, \dots, T(\tau_N)f_N\}; \\ &\quad f_1 \epsilon D(\tau_1),\ f_2 \epsilon D(\tau_2), \dots, f_N \epsilon D(\tau_N), \\ T(\tau^+)g &= \{T(\tau_1^+)g_1, T(\tau_2^+)g_2, \dots, T(\tau_N^+)g_N\}; \\ &\quad g_1 \epsilon D(\tau_1^+),\ g_2 \epsilon D(\tau_2^+), \dots, g_N \epsilon D(\tau_N^+). \end{aligned}$$

We summarize a few additional properties of $T_0(\tau)$ in the form of a Lemma.

Lemma 3.5: We have,

(i) $[T_0(\tau)]^* = \oplus_{p=1}^{N} [T_0(\tau_p)]^* = \oplus_{p=1}^{N} [T(\tau_p^+)]$,
$[T_0(\tau^+)]^* = \oplus_{p=1}^{N} [T_0(\tau_p^+)]^* = \oplus_{p=1}^{N} [T(\tau_p)]$.

In particular,

$D[T_0(\tau)]^* = D[T(\tau^+)] = \oplus_{p=1}^{N} [T(\tau_p^+)]$,
$D[T_0(\tau^+)]^* = D[T(\tau)] = \oplus_{p=1}^{N} [T(\tau_p)]$.

(ii) $nul[T_0(\tau) - \lambda I] = \sum_{p=1}^{N} nul[T_0(\tau_p) - \lambda I]$,
$nul[T_0(\tau^+) - \bar{\lambda} I] = \sum_{p=1}^{N} nul[T_0(\tau_p^+) - \bar{\lambda} I]$.

(iii) The deficiency indices of $T_0(\tau)$ are given by:
$def[T_0(\tau) - \lambda I] = \sum_{p=1}^{N} def[T_0(\tau_p) - \lambda I]$ for all $\lambda \epsilon \Pi[T_0(\tau_p)]$,
$def[T_0(\tau^+) - \bar{\lambda} I] = \sum_{p=1}^{N} def[T_0(\tau_p^+) - \bar{\lambda} I]$ for all $\lambda \epsilon \Pi[T_0(\tau_p^+)]$.

Proof: Part (a) follows immediately from the definition of $T_0(\tau)$ and from the general definition of an adjoint operator. The other parts are either direct consequences of part (a) or follow immediately from the definitions.

Lemma 3.6: For $\lambda \epsilon \Pi[T_0(\tau), T_0(\tau^+)]$,

$def[T_0(\tau) - \lambda I] + def[T_0(\tau^+) - \bar{\lambda} I]$ is constant and

$$0 \leq def[T_0(\tau) - \lambda I] + def[T_0(\tau^+) - \bar{\lambda} I] \leq 2nN.$$

In the problem with one singular end-point,

$$nN \leq def[T_0(\tau) - \lambda I] + def[T_0(\tau^+) - \bar{\lambda} I] \leq 2nN,$$

for all $\lambda \epsilon \Pi[T_0(\tau), T_0(\tau^+)]$.

In the regular problem,

$$def[T_0(\tau) - \lambda I] + def[T_0(\tau^+) - \bar{\lambda} I] = 2nN,$$

for all $\lambda \epsilon \Pi[T_0(\tau), T_0(\tau^+)]$.

Proof: The proof is similar to that in [4], [6] and [8], and therefore omitted.

Lemma 3.7: Let $T_0(\tau) = \oplus_{p=1}^{N} T_0(\tau_p)$ be a closed densely-defined operator on H. Then,

$$\Pi[T_0(\tau)] = \cap_{p=1}^{N} [T_0(\tau_p)]$$

Proof: The proof follows from Lemma 3.5 and since $R[T_0(\tau) - \lambda I]$ is closed if and only if $R[T_0(\tau_p) - \lambda I]$, $p = 1,2,\dots,N$ are closed.

Remark 3.8: If $S_p^{a_p}, p = 1,2,\dots,N$ is a regularly solvable extension of $T_0(\tau_p; a_p)$ is a regularly solvable extension of $S_p^{b_p}$ then $T_0(\tau_p; b_p)$ is regularly solvable extension of $S = \oplus_{p=1}^{N}(S_p^{a_p} \oplus S_p^{b_p})$. We refer to [3], [4], [6], [7], [10, 11] and [17] for more details.

IV. The product operators in direct sum spaces

The proof of general theorems will be based on the results in this section. We start by listing some properties and results of quasi-differential expressions $\tau_1, \tau_2, \dots, \tau_n$. For proofs the reader is referred to [5], [8] and [16].

$$\begin{aligned} (\tau_1 + \tau_2)^+ &= \tau_1^+ + \tau_2^+ \\ (\tau_1 \tau_2)^+ &= \tau_2^+ \tau_1^+,\ (\lambda\tau)^+ = \bar{\lambda}\tau^+ \end{aligned} \quad (4.1)$$

for λ a complex number .

A consequence of Properties (4.1) is that if $\tau^+ = \tau$ then $(P(\tau))^+ = P(\tau^+)$ for P any polynomial with complex coefficients. Also we note that the leading coefficients of a product is the product of the leading coefficients. Hence the product of regular differential expressions is regular.

Lemma 4.1: (cf. [8]). Suppose τ_j is a regular differential expression on the interval $[a, b]$ and $\lambda \epsilon \Pi[T_0(\tau_1 \tau_2 \dots \tau_n), T_0(\tau_1 \tau_2 \dots \tau_n)^+]$, then we have,

(i) The product operator $\prod_{j=1}^{n} T_0(\tau_j)$ is closed, densely-defined, and
$def[\prod_{j=1}^{n} T_0(\tau_j) - \lambda I] = \sum_{j=1}^{n} def[T_0(\tau_j) - \lambda I]$,
$def[\prod_{j=1}^{n} T_0(\tau_j^+) - \bar{\lambda} I] = \sum_{j=1}^{n} def[T_0(\tau_j^+) - \bar{\lambda} I]$.

(ii) $T_0(\tau_1 \tau_2 \dots \tau_n) \subseteq \prod_{j=1}^{n} [T_0(\tau_j)]$ and
$T_0(\tau_1 \tau_2 \dots \tau_n)^+ \subseteq \prod_{j=1}^{n} [T_0(\tau_j^+)]$.

Note in part (ii) that the containment may be proper, i.e., the operators $T_0(\tau_1 \tau_2 \dots \tau_n)$ and $\prod_{j=1}^{n} [T_0(\tau_j)]$ are not equal in general.

Lemma 4.2: Let $\tau_1, \tau_2, \dots, \tau_n$ be a regular differential expressions on $[a, b)$ and suppose that $\lambda \epsilon \Pi[T_0(\tau_1 \tau_2 \dots \tau_n), T_0(\tau_1 \tau_2 \dots \tau_n)^+]$. Then

$$[T_0(\tau_1 \tau_2 \dots \tau_n)] = \prod_{j=1}^{n} [T_0(\tau_j)], \quad (4.2)$$

if and only if the following **partial separation conditions** is satisfied:

$$\left.\begin{array}{l} f \epsilon L^2_w(a,b), f^{[s-1]} \epsilon AC_{loc}[a,b), \text{where s is the} \\ \text{order of product expression } (\tau_1 \tau_2 \dots \tau_n) \text{ and} \\ (\tau_1 \tau_2 \dots \tau_n)^+ f \epsilon L^2_w(a,b) \text{ together imply that} \\ (\prod_{j=1}^{k} (\tau_j^+)) f \epsilon L^2_w(a,b),\ k = 1, \dots, n-1. \end{array}\right\}. \quad (4.3)$$

Furthermore,

$T_0(\tau_1 \tau_2 \dots \tau_n) = \prod_{j=1}^{n} [T_0(\tau_j)]$ and
$T_0(\tau_1 \tau_2 \dots \tau_n)^+ = \prod_{j=1}^{n} [T_0(\tau_j^+)]$,

if and only if ,

$def[T_0(\tau_1 \tau_2 \dots \tau_n) - \lambda I] = \sum_{j=1}^{n} def[T_0(\tau_j) - \lambda I]$,
$def[T_0(\tau_1 \tau_2 \dots \tau_n)^+ - \bar{\lambda} I] = \sum_{j=1}^{n} def[T_0(\tau_j^+) - \bar{\lambda} I]$.

We will say that the product $\tau_1, \tau_2, \dots, \tau_n$ is partially separated expressions in $L^2_w(a,b)$ whenever Property (4.3) holds.

Lemma 4.3: For $\lambda \epsilon \Pi[T_0(\tau_1 \tau_2 \dots \tau_n), T_0(\tau_1 \tau_2 \dots \tau_n)^+]$, we have

$\Pi[T_0(\tau_1\,\tau_2\,...\,\tau_n), T_0(\tau_1\,\tau_2\,...\,\tau_n)^+] =$
$$\Pi\left[\prod_{j=1}^{n}[T_0(\tau_j)], \prod_{j=1}^{n}[T_0(\tau_j^+)]\right]. \quad (4.4)$$

Proof: Let $\lambda \in \Pi[T_0(\tau_1\,\tau_2\,...\,\tau_n), T_0(\tau_1\,\tau_2\,...\,\tau_n)^+]$, then from definition of the field of regularity we have $\lambda \in \Pi[T_0(\tau_1\,\tau_2\,...\,\tau_n)]$ and $\overline{\lambda} \in \Pi[T_0(\tau_1\,\tau_2\,...\,\tau_n)^+]$, *i.e.*, each of the operators $T_0(\tau_1\,\tau_2\,...\,\tau_n)$ and $T_0(\tau_1\,\tau_2\,...\,\tau_n)^+$ has closed range and densely-defined on H with finite deficiency indices. Consequently by Lemma 4.2 each of the operators $\left[\prod_{j=1}^{n}T_0(\tau_j) - \lambda I\right]$ and $\left[\prod_{j=1}^{n}T_0(\tau_j^+) - \overline{\lambda} I\right]$ has closed range and their deficiency indices are finite, i.e., $\lambda \in \Pi\left[\prod_{j=1}^{n}[T_0(\tau_j)], \prod_{j=1}^{n}[T_0(\tau_j^+)]\right]$. The rest of the proof follows from definition and Lemma 4.2.

Corollary 4.4: Let τ_j is a regular differential expressions on $[a,b)$ for $j = 1,2,\dots,n$ If all solutions of the differential equation $(\tau_j - \lambda I)u = 0$ and $(\tau_j^+ - \overline{\lambda} I)v = 0$ on $[a,b)$ are in $L_w^2(a,b)$ for $j = 1,2,\dots,n$ and $\lambda \in \mathbb{C}$; then all solutions of $[\prod_{j=1}^{n}\tau_j - \lambda I]u = 0$ and $(\prod_{j=1}^{n}\tau_j^+ - \overline{\lambda} I)v = 0$ on $[a,b)$ are in $L_w^2(a,b)$ for all $\lambda \in \mathbb{C}$.

Proof: Let $\mathrm{n} = n_j$ = order of τ_j = order of τ_j^+ for $j = 1,2,\dots,n$. Then by Lemma 3.5, we have
$$def[T_0(\tau) - \lambda I] = def\left[T_0(\tau^+) - \overline{\lambda} I\right] = n$$
for all $\lambda \in \Pi[T_0(\tau), T_0(\tau^+)]$. Hence, by Lemma 4.1, we have
$$\begin{aligned} def[T_0(\tau_1\,\tau_2\,...\,\tau_n)^+ - \lambda I] &= def\left[\prod_{j=1}^{n}T_0(\tau_j^+) - \overline{\lambda} I\right] \\ &= \sum_{j=1}^{n} n_j = n^2 = \text{order of } (\tau_1\,\tau_2\,...\,\tau_n) \\ &= \text{order of } (\tau_1\,\tau_2\,...\,\tau_n)^+ . \end{aligned}$$
Thus $def[T_0(\tau_n^+\,...\,\tau_2^+\,\tau_1^+) - \lambda I] =$ order of $(\tau_1\,\tau_2\,...\,\tau_n)^+$ and consequently all solutions of the equations $[\prod_{j=1}^{n}\tau_j - \lambda I]u = 0$ and $[\prod_{j=1}^{n}\tau_j^+ - \overline{\lambda} I]v = 0$ are in $L_w^2(a,b)$. Repeating this argument with τ_j^+ replaced by τ_j, we conclude that all solutions of $(\prod_{j=1}^{n}\tau_j^+ - \overline{\lambda} I)v = 0$ are in $L_w^2(a,b)$.

The special case of Corollary 4.4 when $\tau_j = \tau$ for $j = 1,2,\dots,n$ and τ is symmetric was established in [5]. In this case it is easy to see that the converse also holds. If all solutions of $(\tau^n - \lambda I)u = 0$ are in $L_w^2(a,b)$, then all solutions of $(\tau - \lambda I)u = 0$ must be in $L_w^2(a,b)$. In general, if all solutions of $[(\tau_1\,\tau_2\,...\,\tau_n) - \lambda I]u = 0$ are in $L_w^2(a,b)$, then all solutions of $(\tau_n - \lambda I)u = 0$ are in $L_w^2(a,b)$ since these are also solutions of $[(\tau_1\,\tau_2\,...\,\tau_n) - \lambda I]u = 0$. If all solutions of the adjoint equation $[(\tau_1\,\tau_2\,...\,\tau_n)^+ - \overline{\lambda} I]v = 0$ are also in $L_w^2(a,b)$, then it follows similarly that all solutions of $(\tau_j^+ - \overline{\lambda} I)v = 0$ are in $L_w^2(a,b)$.

Next, we consider our interval is $I = [a,b]$ and denote by $T_0(\tau_1\,\tau_2\,...\,\tau_n)$ and $T(\tau_1\,\tau_2\,...\,\tau_n)$ the minimal and maximal operators. We see from (3.15) and Lemma 4.2 that
$$T_0(\tau_1\,\tau_2\,...\,\tau_n) \subset T(\tau_1\,\tau_2\,...\,\tau_n) \subset [T_0(\tau_1\,\tau_2\,...\,\tau_n)^+]^*$$
and hence $T_0(\tau_1\,\tau_2\,...\,\tau_n)$ and $T_0(\tau_1\,\tau_2\,...\,\tau_n)^+$ form an adjoint pair of closed densely defined operators in $L_w^2(a,b)$. From Lemmas 3.5 and 4.1 we have the following:

Lemma 4.5: For $\lambda \in \Pi\left[\prod_{j=1}^{n}[T_0(\tau_j)], \prod_{j=1}^{n}[T_0(\tau_j^+)]\right]$ we have

(i) $$\begin{aligned} \left[\prod_{j=1}^{n}T_0^*(\tau_j)\right] &= \oplus_{p=1}^{N}\left[\prod_{j=1}^{n}T_0^*(\tau_{jp})\right] \\ &= \oplus_{p=1}^{N}\left[\prod_{j=1}^{n}T(\tau_{jp}^+)\right], \end{aligned}$$
$$\begin{aligned} \left[\prod_{j=1}^{n}T_0^*(\tau_j^+)\right] &= \oplus_{p=1}^{N}\left[\prod_{j=1}^{n}T_0^*(\tau_{jp}^+)\right] \\ &= \oplus_{p=1}^{N}\left[\prod_{j=1}^{n}T(\tau_{jp})\right]. \end{aligned}$$

(ii) $$\begin{aligned} nul\left[\prod_{j=1}^{n}T_0(\tau_j) - \lambda I\right] &= \sum_{p=1}^{N} nul\left[\prod_{j=1}^{n}T_0(\tau_{jp}) - \lambda I\right] \\ &= \sum_{p=1}^{N}\left(\sum_{j=1}^{n} nul\left[T_0(\tau_{jp}) - \lambda I\right]\right), \end{aligned}$$
$$\begin{aligned} nul\left[\prod_{j=1}^{n}T_0(\tau_j^+) - \overline{\lambda} I\right] &= \sum_{p=1}^{N} nul\left[\prod_{j=1}^{n}T_0(\tau_{jp}^+) - \overline{\lambda} I\right] \\ &= \sum_{p=1}^{N}\left(\sum_{j=1}^{n} nul\left[T_0(\tau_{jp}^+) - \overline{\lambda} I\right]\right). \end{aligned}$$

(iii) The deficiency indices of $\prod_{j=1}^{n}T_0(\tau_j)$ and $\prod_{j=1}^{n}T_0(\tau_j^+)$ are given by:
$$\begin{aligned} def\left[\prod_{j=1}^{n}T_0(\tau_j) - \lambda I\right] &= \sum_{p=1}^{N} def\left[\prod_{j=1}^{n}T_0(\tau_{jp}) - \lambda I\right] \\ &= \sum_{p=1}^{N}\left(\sum_{j=1}^{n} def\left[T_0(\tau_{jp}) - \lambda I\right]\right), \end{aligned}$$
$$\begin{aligned} def\left[\prod_{j=1}^{n}T_0(\tau_j^+) - \overline{\lambda} I\right] &= \sum_{p=1}^{N} def\left[\prod_{j=1}^{n}T_0(\tau_{jp}^+) - \overline{\lambda} I\right] \\ &= \sum_{p=1}^{N}\left(\sum_{j=1}^{n} def\left[T_0(\tau_{jp}^+) - \overline{\lambda} I\right]\right). \end{aligned}$$

Lemma 4.6: For $\lambda \in \Pi\left[\prod_{j=1}^{n}[T_0(\tau_j)], \prod_{j=1}^{n}[T_0(\tau_j^+)]\right]$,
$$def\left[\prod_{j=1}^{n}[T_0(\tau_j)] - \lambda I\right] + def\left[\prod_{j=1}^{n}[T_0(\tau_j^+)] - \overline{\lambda} I\right]$$
is constant and
$$0 \le def\left[\prod_{j=1}^{n}[T_0(\tau_j)] - \lambda I\right] + def\left[\prod_{j=1}^{n}[T_0(\tau_j^+)] - \overline{\lambda} I\right] \le 2n^2N .$$
In the problem with one singular end-point,
$$n^2N \le def\left[\prod_{j=1}^{n}[T_0(\tau_j)] - \lambda I\right] + def\left[\prod_{j=1}^{n}[T_0(\tau_j^+)] - \overline{\lambda} I\right] \le 2n^2N .$$
for all $\lambda \in \Pi\left[\prod_{j=1}^{n}[T_0(\tau_j)], \prod_{j=1}^{n}[T_0(\tau_j^+)]\right]$.
In the regular problem,
$$def\left[\prod_{j=1}^{n}[T_0(\tau_j)] - \lambda I\right] + def\left[\prod_{j=1}^{n}[T_0(\tau_j^+)] - \overline{\lambda} I\right] = 2n^2N,$$
for all $\lambda \in \Pi\left[\prod_{j=1}^{n}[T_0(\tau_j)], \prod_{j=1}^{n}[T_0(\tau_j^+)]\right]$.

Proof: The proof is similar to that in [3, 4], [7] and [17], and therefore omitted.

For $\lambda \in \Pi\left[\prod_{j=1}^{n}T_0(\tau_j), \prod_{j=1}^{n}T_0(\tau_j^+)\right]$, we define r, s and m as follows :
$$\left.\begin{aligned} r &= r(\lambda) := def\left[\prod_{j=1}^{n}T_0(\tau_j) - \lambda I\right] \\ &= \sum_{p=1}^{N} def\left[\prod_{j=1}^{n}T_0(\tau_{jp}) - \lambda I\right] = \sum_{p=1}^{N} r_p \\ s &= s(\lambda) := def\left[\prod_{j=1}^{n}T_0(\tau_j^+) - \overline{\lambda} I\right] \\ &= \sum_{p=1}^{N} def\left[\prod_{j=1}^{n}T_0(\tau_{jp}^+) - \overline{\lambda} I\right] = \sum_{p=1}^{N} s_p \\ m &:= r + s = \sum_{p=1}^{N}(r_p + s_p) \end{aligned}\right\}. \quad (4.5)$$
Also,
$$0 \le m \le 2n^2N . \quad (4.6)$$

For $\lambda \in \Pi\left[\prod_{j=1}^{n}T_0(\tau_j), \prod_{j=1}^{n}T_0(\tau_j^+)\right]$, define r, s and m as in (4.1), let $\psi_i (i = 1,2,\dots,s)$, $\Phi_l (l = s+1,\dots,m)$ be bases for $N[\prod_{j=1}^{n}T(\tau_j) - \lambda I]$ and $N[\prod_{j=1}^{n}T(\tau_j^+) - \overline{\lambda} I]$ respectively, thus ψ_j, $\Phi_k \in L_w^2(I)$ for ($i = 1,2,\dots,s$; $l = s+1,\dots,m$) and
$$\prod_{j=1}^{n}(\tau_j)\,[\psi_i] = \lambda w \psi_i , \quad \prod_{j=1}^{n}(\tau_j^+)\,[\Phi_l] = \overline{\lambda} w \Phi_l \quad (4.7)$$
Since $[\prod_{j=1}^{n}T_0(\tau_j^+) - \overline{\lambda} I]$ has closed range , so does its adjoint $\left[\prod_{j=1}^{n}T(\tau_j) - \lambda I\right]$ and moreover
$$R\left[\prod_{j=1}^{n}T(\tau_j) - \lambda I\right]^{\perp} = N\left[\prod_{j=1}^{n}T_0(\tau_j^+) - \overline{\lambda} I\right] = \{0\}.$$
Hence $R\left[\prod_{j=1}^{n}T(\tau_j) - \lambda I\right] = L_w^2(I)$ and $R\left[\prod_{j=1}^{n}T(\tau_j^+) - \overline{\lambda} I\right] = L_w^2(I)$. We refer to [4], [7] and [17] for more details.

We can therefore define the following functions x_j , y_j , $j = 1,2,\dots,m$:

$$\left.\begin{aligned} x_j &= \psi_j, \quad (j = 1,2,\dots,s) \\ [\prod_{j=1}^n T(\tau_j^+) - \bar{\lambda} I]\, x_j &= \Phi_j, \quad (j = s+1,\dots,m) \end{aligned}\right\}, \tag{4.8}$$

$$\left.\begin{aligned} [\prod_{j=1}^n T(\tau_j^+) - \bar{\lambda} I]\ y_j &= \psi_j, \quad (j = 1,2,\dots,s) \\ y_j &= \Phi_j, \quad (j = s+1,\dots,m) \end{aligned}\right\}. \tag{4.9}$$

Lemma 4.7: **(cf. [4, Lemma 4.1]).**

The set $\{x_j : j = 1,2,\dots,m\}$ is a basis of $N([\prod_{j=1}^n T(\tau_j^+) - \bar{\lambda} I][\prod_{j=1}^n T(\tau_j) - \lambda I]\ ])$ and $\{y_j : j = 1,2,\dots,m\}$ is a basis of $N([\prod_{j=1}^n T(\tau_j) - \lambda I]\ ][\prod_{j=1}^n T(\tau_j^+) - \bar{\lambda} I\ ])$.

On applying [3, Theorem III.3.1] we obtain the following :

Lemma 4.8: (cf. [4, Corollary 4.2]).

Any $z \in D[\prod_{j=1}^n T(\tau_j)]$ and $z^+ \in D[\prod_{j=1}^n T(\tau_j^+)]$ have a unique representation

$$z = z_0 + \sum_{j=1}^m a_j x_j \quad (z_0 \in D[\prod_{j=1}^n T_0(\tau_j)]), \tag{4.10}$$

$$z^+ = z_0^+ + \sum_{j=1}^m b_j y_j \quad (z_0^+ \in D[\prod_{j=1}^n T_0(\tau_j^+)]), \tag{4.11}$$

$a_j,\ b_j \in \mathbb{C}$

A central rule in the argument is played by

Lemma 4.9: (cf. [4, Lemma 4.3]). Let

$$E_{m\times m} := ([x_j, y_j](b))_{\substack{1\le i\le m \\ 1\le j\le m}} \tag{4.12}$$

and

$$E^{1,2}_{s\times r} := ([x_j, y_j](b))_{\substack{1\le i\le s \\ s+1\le j\le m}} . \tag{4.13}$$

Then

$$rank\ E^{1,2}_{s\times r} = rank\ E_{m\times m} = m - n^2 N\ . \tag{4.14}$$

In view of Lemma 4.9 and since $r, s \ge m - n^2 N$, we may suppose, without loss of generality, that the matrix

$$E^{1,2}_{(m-n^2N)\times(m-n^2N)} = ([x_j, y_k](b))_{\substack{1\le j\le (m-n^2N) \\ n^2N+1\le k\le m}} . \tag{4.15}$$

satisfies

$$rank\ E^{1,2}_{(m-n^2N)\times(m-n^2N)} = m - n^2 N\ . \tag{4.16}$$

If we partition $E_{m\times m}$ **:**

$$E_{m\times m} = \begin{pmatrix} E^{1,1}_{(m-n^2N)\times n^2N} & \vdots & E^{1,2}_{(m-n^2N)\times(m-n^2N)} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ E^{2,1}_{n^2N\times n^2N} & \vdots & E^{2,2}_{n^2N\times(m-n^2N)} \end{pmatrix} \tag{4.17}$$

and set

$$\left.\begin{aligned} E^1_{(m-n^2N)\times m} &= E^{1,1}_{(m-n^2N)\times n^2N} \oplus E^{1,2}_{(m-n^2N)\times(m-n^2N)} \\ E^2_{n^2N\times m} &= E^{2,1}_{n^2N\times n^2N} \oplus E^{2,2}_{n^2N\times(m-n^2N)} \end{aligned}\right\} \tag{4.18}$$

$$\left.\begin{aligned} F^1_{m\times n^2N} &= E^{1,1}_{(m-n^2N)\times n^2N} \oplus^T E^{2,1}_{n^2N\times n^2N} \\ F^2_{m\times(m-n^2N)} &= E^{1,2}_{(m-n^2N)\times(m-n^2N)} \oplus^T E^{2,2}_{n^2N\times(m-n^2N)} \end{aligned}\right\} \tag{4.19}$$

Then (4.16) yields the results

$$\begin{aligned} rank\ E^1_{(m-n^2N)\times m} &= rank\ F^2_{m\times(m-n^2N)} \\ &= m - n^2 N\ . \end{aligned} \tag{4.20}$$

Lemma 4.10: **(cf. [4, Lemma 4.5]).**

Let z_i $(i = 1,2,\dots,n^2N)$ be a function in $D[\prod_{j=1}^n T(\tau_j)]$ which satisfy

$$\left.\begin{aligned} z_i^{[k-1]}(a) = \delta_{ik}\,, \quad z_i^{[k-1]}(c) = 0 \\ z_i(t) = 0 \quad for \quad t \ge c\,. \end{aligned}\right\} \tag{4.21}$$

and suppose that (4.14) is satisfied. Then x_i $(i = m - n^2N + 1,\dots,m)$ has a unique representation

$$x_i = x_{i0} + \sum_{j=1}^{n^2N} b_{ij} z_j + \sum_{j=1}^{m-n^2N} c_{ij} x_j \tag{4.22}$$

where $x_{i0} \in D[\prod_{j=1}^n T_0(\tau_j)]$, and the b_{ij} and c_{ij} are complex constants .

Lemma 4.11: **(cf. [4, Lemma 4.6]).** Let $D_1[\prod_{j=1}^n T(\tau_j)]$ be the linear span of $\{z_i : i = 1,2,\dots,n^2N\}$ where $z_i \in D[\prod_{j=1}^n T(\tau_j)]$ satisfy (4.21) and let $D_2[\prod_{j=1}^n T(\tau_j)]$ be the linear span of $\{x_i : i = 1,2,\dots,m - n^2N\}$ with (4.16) satisfied . Then

$$\begin{aligned} D[\prod_{j=1}^n T(\tau_j)] = D[\prod_{j=1}^n T_0(\tau_j)] &\dotplus D_1[\prod_{j=1}^n T(\tau_j)] \\ &\dotplus D_2[\prod_{j=1}^n T(\tau_j)]. \end{aligned}$$

If z_i^+ $(i = 1,2,\dots,n^2N)$ are function in $D[\prod_{j=1}^n T(\tau_j^+)]$ which for $k = 1,2,\dots,n^2N$ and some $c \in I$, satisfy

$$\left.\begin{aligned} (z_i^+)_+^{[k-1]}(a) = \delta_{ik}\,, \quad (z_i^+)_+^{[k-1]}(c) = 0 \\ z_i^+(t) = 0 \quad , \quad for \quad t \ge c\,. \end{aligned}\right\} \tag{4.23}$$

and (4.16) is satisfied, then we have that each y_i $(i = 1,2,\dots,n^2N)$ has a unique representation

$$y_i = y_{i0} + \sum_{j=1}^{n^2N} d_{ij} z_j^+ + \sum_{j=n^2N+1}^{m} e_{ij} x_j\ , \tag{4.24}$$

where $y_{i0} \in D[\prod_{j=1}^n T_0(\tau_j^+)]$, and the d_{ij} and e_{ij} are complex constants. Also, if $D_1[\prod_{j=1}^n T(\tau_j^+)]$ and $D_2[\prod_{j=1}^n T(\tau_j^+)]$ are the linear spans of $\{z_i^+ : i = 1,2,\dots,n^2N\}$ and $\{y_i : i = n^2N + 1,\dots,m\}$ respectively then

$$\begin{aligned} D[\prod_{j=1}^n T(\tau_j^+)]] = D[\prod_{j=1}^n T_0(\tau_j^+)] &\dotplus D_1[\prod_{j=1}^n T(\tau_j^+)] \\ &\dotplus D_2[\prod_{j=1}^n T(\tau_j^+)] \end{aligned} \tag{4.25}$$

V. Boundary conditions featuring L^2_w-solutions

For $\Pi[\prod_{j=1}^n T_0(\tau_j), \prod_{j=1}^n T_0(\tau_j^+)] \ne \emptyset$ the domains of the operators which are regularly solvable with respect to the product operators $\prod_{j=1}^n [T_0(\tau_j)]$ and $\prod_{j=1}^n [T_0(\tau_j^+)]$ on $[a, b)$ are characterized by the following theorem which proved for a general quasi-differential operator in [3, Theorem 10.15], [4], [7] and [17].

Theorem 5.1: For $\lambda \in \Pi[\prod_{j=1}^n T_0(\tau_j), \prod_{j=1}^n T_0(\tau_j^+)]$. Let r, s and m be defined by (4.5), and let ψ_j $(j = 1,2,\dots,r)$, $\Phi_k (k = r+1,\dots,m)$ be arbitrary functions satisfying:

(i) ψ_j $(j = 1,2,\dots,r) \subset D[\prod_{j=1}^n T(\tau_j)]$ are linearly independent modulo $D[\prod_{j=1}^n T_0(\tau_j)]$ and $\Phi_k (k = r+1,\dots,m) \subset D[\prod_{j=1}^n T(\tau_j^+)]$ are linearly independent modulo $D[\prod_{j=1}^n T_0(\tau_j^+)]$.

(ii) $[\psi_j, \Phi_k](b) - [\psi_j, \Phi_k](a) = 0$, $j = 1,2,\dots,r$; $k = r+1,\dots,m$.

Then the set

$$\{u : u \in D[\prod_{j=1}^n T(\tau_j)],\ [u, \Phi_k](b) - [u, \Phi_k](a) = 0,\ k = r+1,\dots,m\}, \tag{5.1}$$

is the domain of an operator S which is regularly solvable with respect to $[\prod_{j=1}^{n} T_0(\tau_j)]$ and $[\prod_{j=1}^{n} T_0(\tau_j^+)]$ and the set

$$\{v : v \in D[\prod_{j=1}^{n} T(\tau_j^+)],\ [\psi_j, v](b) - [\psi_j, v](a) = 0,\ j = 1, 2, \dots, r\} \quad (5.2)$$

is the domain of the operator S^* ; moreover $\lambda \in \Delta_4(S)$.

Conversely, if S is regularly solvable with respect to $[\prod_{j=1}^{n} T_0(\tau_j)]$ and $[\prod_{j=1}^{n} T_0(\tau_j^+)]$ and $\lambda \in \Pi[\prod_{j=1}^{n} [T_0(\tau_j)], \prod_{j=1}^{n}[T_0(\tau_j^+)]] \cap \Delta_4(S)$, then with r and s defined by (4.5) and there exist functions ψ_j $(j = 1,2,\dots,r), \Phi_k (k = r+1,\dots,m)$ which satisfy (i) and (ii) and are such that (5.1) and (5.2) are the domains of S and S^* respectively.

S is self-adjoint if, and only if, $\prod_{j=1}^{n}(\tau_j) = \prod_{j=1}^{n}(\tau_j^+)$, $r = s$ and $\Phi_k = \psi_{k-r}$ $(k = r+1,\dots,m)$; S is J −self-adjoint if $\prod_{j=1}^{n}(\tau_j^+) = J\prod_{j=1}^{n}(\tau_j)J$ (J is a complex conjugate), $r = s$ and $\Phi_k = \overline{\psi}_{k-r}$ $(k = r+1,\dots,m)$

Proof: The proof is entirely similar to that of [3], [4], [7], [8], [12] and [17], and therefore omitted.

We shall now consider our interval is $I = [a, b)$ and we characterize all the operators which are regularly solvable with respect to $\prod_{j=1}^{n} [T_0(\tau_j)]$, and $\prod_{j=1}^{n}[T_0(\tau_j^+)]$ in terms of boundary conditions featuring L_w^2-solutions of the equations in (4.7). This generalizes an analogue of a results of Evans and Ibrahim [4], Sun Jiong's result [9] in the special case of symmetric operators with equal deficiency indices and Zai-ju Shang' ~~s~~ results [12] for J-symmetric operators.

Theorem 5.2 (cf. [7, Theorem 4.1]):

Let $\lambda \in \Pi[\prod_{j=1}^{n} [T_0(\tau_j)], \prod_{j=1}^{n}[T_0(\tau_j^+)]]$, let r, s and m be defined by (4.5), and let x_i $(i = 1,2\dots,m)$, y_j $(j = 1,2\dots,m)$ be defined by (4.8) and (4.9) and arranged to satisfy (4.16). Let $K_{r\times n^2N}$, $L_{r\times(m-n^2N)}$, $M_{s\times n^2N}$ and $N_{s\times(m-n^2N)}$ be numerical matrices which satisfy the following conditions:

(i) $rank\left(K_{r\times n^2N} \oplus L_{r\times(m-n^2N)}\right) = r,$
$rank\left(M_{s\times n^2N} \oplus N_{s\times(m-n^2N)}\right) = s,$

(ii) $L_{r\times(m-n^2N)}\, E^{1,2}_{(m-n^2N)\times(m-n^2N)} \left(N_{s\times(m-n^2N)}\right)^T + (-i)^{n^2N} K_{r\times n^2N}\, J_{n^2N\times n^2N}\, (M_{s\times n^2N})^T = 0_{r\times s}$.

Then the set of all $u \in D[\prod_{j=1}^{n} T(\tau_j)]$ such that

$$M_{s\times n^2N}\begin{pmatrix} u(a) \\ \vdots \\ u^{[n^2N-1]}(a) \end{pmatrix} - N_{s\times(m-n^2N)}\begin{pmatrix} [u, y_{n^2N+1]}](b) \\ \vdots \\ [u, y_{m]}](b) \end{pmatrix} = 0_{s\times 1} \quad (5.3)$$

Is the domain of an operator S which is regularly solvable with respect to $\prod_{j=1}^{n} T_0(\tau_j)$ and $\prod_{j=1}^{n}T_0(\tau_j^+)$ and $D(S^*)$ is the set of all $v \in D[\prod_{j=1}^{n} T(\tau_j^+)]$ which are such that

$$K_{r\times n^2N}\begin{pmatrix} \bar{v}(a) \\ \vdots \\ \bar{v}^{[n^2N-1]}(a) \end{pmatrix} - L_{r\times(m-n^2N)}\begin{pmatrix} [x_1, v](b) \\ \vdots \\ [x_{m-n^2N}, v](b) \end{pmatrix} = 0_{r\times 1}. \quad (5.4)$$

Proof: The proof follows by using (3.7), Theorem 5.1, Lemmas 4.9 , 4.10 and by considering :

$$M_{s\times n^2N}\, J^{-1}_{n^2N\times n^2N} = -i^{n^2N}(\alpha_{jk})_{\substack{r+1\le j\le m \\ 1\le k\le n^2N}},$$
$$N_{s\times(m-n^2N)} = (\beta_{jk})_{\substack{r+1\le j\le m \\ n^2N+1\le k\le m}}, \quad (5.5)$$
$$K_{r\times n^2N}\, J^{-1}_{n^2N\times n^2N} = -i^{n^2N}(\gamma_{jk})_{\substack{1\le j\le r \\ 1\le k\le n^2N}},$$
$$L_{r\times(m-n^2N)} = (\varepsilon_{jk})_{\substack{1\le j\le r \\ 1\le k\le m-n^2N}}. \quad (5.6)$$

The following is the converse of Theorem 5.2:

Theorem 5.3 (cf. [7, Theorem 4.2]): Let the operator S be a regularly solvable with respect to $\prod_{j=1}^{n} T_0(\tau_j)$ and $\prod_{j=1}^{n}T_0(\tau_j^+)$, let $\lambda \in \Pi[\prod_{j=1}^{n} [T_0(\tau_j)], \prod_{j=1}^{n}[T_0(\tau_j^+)]] \cap \Delta_4(S)$, let r, s and m be defined by (4.5) , and suppose that (4.16) is satisfied. Then there exist numerical matrices $K_{r\times n^2N}$, $L_{r\times(m-n^2N)}$, $M_{s\times n^2N}$ and $N_{s\times(m-n^2N)}$ such that the conditions (i) and (ii) in Theorem 5.1 are satisfied and $D(S)$ is the set of all $u \in D[\prod_{j=1}^{n} T(\tau_j)]$ satisfying (5.3) while $D(S^*)$ is the set of all $v \in D[\prod_{j=1}^{n} T(\tau_j^+)]$ satisfying (5.4) .

Remark 5.4: Theorems 5.2 and 5.3 follows from the following results for the case of one singular end points in [4], for the case of two singular end-points in [7] and for any finite number of intervals in [8].

Remark 5.5: In the case when $\prod_{j=1}^{n}(\tau_j)$ is formally $J-$ symmetric , that is $\prod_{j=1}^{n}(\tau_j^+) = J\prod_{j=1}^{n}(\tau_j)J$, where J is complex conjugation. The operator $\prod_{j=1}^{n} [T_0(\tau_j)]$ is then $J-$symmetric and $\prod_{j=1}^{n} [T_0(\tau_j)]$ and $\prod_{j=1}^{n} [T_0(\tau_j^+)] = J\prod_{j=1}^{n} [T_0(\tau_j)]J$ form an adjoint pair with $\Pi[\prod_{j=1}^{n} T_0(\tau_j), \prod_{j=1}^{n}T_0(\tau_j^+)] = \Pi[\prod_{j=1}^{n} T_0(\tau_j),]$. Since $\prod_{j=1}^{n}(\tau_j)[u] = \lambda wu$ if and only if $\prod_{j=1}^{n}(\tau_j^+)[\bar{u}] = \bar{\lambda}w\bar{u}$ it follows that from (4.6) that for all $\lambda \in \prod_{j=1}^{n} [T_0(\tau_j)]$, $def[\prod_{j=1}^{n} [T_0(\tau_j)] - \lambda I] = def[\prod_{j=1}^{n}[T_0(\tau_j^+)] - \bar{\lambda}I]$ is constant ℓ , say, and so in (4.5), $r = s = \ell$ with $0 \le \ell \le n^2N$.

Remark 5.6: If S is a $J-$self-adjoint extension of, then $S^* = JSJ$ and consequently $v \in D(S^*)$ if and only if, $\bar{v} \in D(S)$. In this case when $\prod_{j=1}^{n}(\tau_j)$ is formally $J-$ symmetric for a complex conjugation J, Theorems 5.2 and 5.3 include Theorem 5.5 of Zai-Jiu-Shang [12].

VI. Discussion.

Everitt and Zettl [6] discussed the possibility of generating self-adjoint operators which are not expressible as the direct sum of self-adjoint operators defined in the separate intervals. In this section we extend this case of general ordinary differential operators, i.e., we discuss the possibility of the regularly solvable operators which are not expressible as the direct sums of regularly solvable operators defined in the separate intervals $I_p = (a_p, b_p)$, $p = 1, 2, 3, 4$. We shall refer to these operators as " new regularly solvable operators" .

If a_p is a regular end-point and b_p is singular, then

$$def\left[\prod_{j=1}^{n} [T_0(\tau_j)] - \lambda I\right] + def\left[\prod_{j=1}^{n}[T_0(\tau_j^+)] - \bar{\lambda} I\right] = 4n^2$$

for all $\lambda \in \Pi\left[\prod_{j=1}^{n} [T_0(\tau_j)], \prod_{j=1}^{n}[T_0(\tau_j^+)]\right]$ if and only if, the terms in (5.1) at the end-points b_p is zero. By Lemma 4.6, for all $\lambda \in \Pi\left[\prod_{j=1}^{n} [T_0(\tau_j)], \prod_{j=1}^{n}[T_0(\tau_j^+)]\right]$, we get in all cases

$$0 \le def\left[\prod_{j=1}^{n} [T_0(\tau_j)] - \lambda I\right] + def\left[\prod_{j=1}^{n}[T_0(\tau_j^+)] - \bar{\lambda} I\right] \le 8n^2 \qquad (6.1)$$

While

$$2n^2 \le def\left[\prod_{j=1}^{n} [T_0(\tau_j)] - \lambda I\right] + def\left[\prod_{j=1}^{n}[T_0(\tau_j^+)] - \bar{\lambda} I\right] \le 8n^2 \qquad (6.2)$$

when each interval has at least one singular end-point, and

$$def\left[\prod_{j=1}^{n} [T_0(\tau_j)] - \lambda I\right] + def\left[\prod_{j=1}^{n}[T_0(\tau_j^+)] - \bar{\lambda} I\right] = 8n^2 \qquad (6.3)$$

for the case when all end-points are regular.

Let

$$def\left[\prod_{j=1}^{n} [T_0(\tau_j)] - \lambda I\right] + def\left[\prod_{j=1}^{n}[T_0(\tau_j^+)] - \bar{\lambda} I\right] = d,$$

and

$$def\left[\prod_{j=1}^{n} [T_0(\tau_{jp})] - \lambda I\right] + def\left[\prod_{j=1}^{n}[T_0(\tau_{jp}^+)] - \bar{\lambda} I\right] = d_p\,,\ p = 1,2,3,4\,.$$

Then, by part (iii) in Lemma 4.5, we have that

$$d = \sum_{p=1}^{4} d_p \quad .$$

We now consider some of the possibilities.

Example 1: $d = 0$. This is the minimal case in (6.1) and can only occur when all eight end-points are singular. In this case $\prod_{j=1}^{n} [T_0(\tau_j)]\,|$ is itself regularly solvable and has no proper regularly solvable extensions; see [3, Chapter III] and [4] for more details.

Example 2: $d = n^2$ with one of d_p, $p = 1,2,3,4$ is equal to n^2 and all the others are equal to zero. We assume that $d_1 = n^2$ and $d_2 = d_3 = d_4 = 0$. The other possibilities are entirely similar. In this case we must have seven singular end-points and one regular end-point. There are no new regularly solvable extensions and we have that $S = S_1 \oplus_{p=2}^{4} \prod_{j=1}^{n} [T_0(\tau_{jp})]$, where S_1 is regularly solvable extension of $\prod_{j=1}^{n} [T_0(\tau_{j1})]$, i.e., all regularly solvable extensions of $\prod_{j=1}^{n} [T_0(\tau_j)]$ can be obtained by forming sums of regularly solvable extensions of $\prod_{j=1}^{n} [T_0(\tau_{jp})]$, $p = 1,2,3,4$. These are obtained as in the case of one interval.

Example 3: six singular end-points and $d = 2n^2$. We consider two cases.

(i) One interval has two regular end-points, say I_1, and each one of the others has two singular end-points. Then, $S = S_1 \oplus_{p=2}^{4} \prod_{j=1}^{n} [T_0(\tau_{jp})]$, where S_1 is regularly solvable extension of $\prod_{j=1}^{n} [T_0(\tau_{j1})]$, generates all regularly solvable extensions of the product differential operator $\prod_{j=1}^{n}[T_0(\tau_j)]$.

(ii) There are two intervals, say I_1 and I_2 each has one singular and one regular end-points, and each one of the others has two singular end-points. In this case

$$S = S_1 \oplus S_2 \oplus \prod_{j=1}^{n} [T_0(\tau_{j3})] \oplus \prod_{j=1}^{n} [T_0(\tau_{j4})],$$

and $S_1 \oplus S_2$ generates all regularly solvable extensions of the product operator $\prod_{j=1}^{n} [T_0(\tau_j)]$. The other possibilities in the cases (i) and (ii) are entirely similar.

Example 4: Five end-points and $d = 3n^2$. We consider two cases.

(i) There are two intervals, say I_1 and I_2 such that I_1 has two regular end-points and I_2 has one regular and one singular end-points, and each one of the others has two singular end-points. In this case $d_1 = 2n^2$ and $d_2 = n^2$, then $S = S_1 \oplus S_2 \oplus \prod_{j=1}^{n} [T_0(\tau_{j3})] \oplus \prod_{j=1}^{n} [T_0(\tau_{j4})]$ and $S_1 \oplus S_2$ generates all regularly solvable extensions of the product operator $\prod_{j=1}^{n} [T_0(\tau_j)]$ which is similar to the case (ii) of Example 3.

(ii) There are three intervals, say I_1, I_2 and I_3 each one has one regular and one singular end-points and the fourth has two singular end-points. In this case $d_1 = d_2 = d_3 = n^2$ and $d_4 = 0$, then $S = S_1 \oplus S_2 \oplus S_3 \oplus \prod_{j=1}^{n} [T_0(\tau_{j4})]$ and $S_1 \oplus S_2 \oplus S_3$ generates all regularly solvable extensions of the product operator $\prod_{j=1}^{n}[T_0(\tau_j)]$. The other possibilities are entirely similar.

Example 5: Four singular end-points and $d = 4n^2$. We consider three cases.

(i) There are two intervals, say I_1 and I_2 each one has two regular end-points, and each one of the others has two singular end-points. In this case $d_1 = d_2 = 2n^2$, $d_3 = d_4 = 0$, then $S = S_1 \oplus S_2 \oplus \prod_{j=1}^{n} [T_0(\tau_{j3})] \oplus \prod_{j=1}^{n} [T_0(\tau_{j4})]$, and $S_1 \oplus S_2$ generates all regularly solvable extensions of $\prod_{j=1}^{n} [T_0(\tau_j)]$.

(ii) There are two intervals, say I_1 and I_2 each one has one regular and one singular end-points, and the others I_3 and I_4 has two regular and singular end-points respectively. In this case $d_1 = d_2 = n^2$, $d_3 = 2n^2$ and $d_4 = 0$, then $S = S_1 \oplus S_2 \oplus S_3 \oplus \prod_{j=1}^{n} [T_0(\tau_{j4})]$ as the case (ii) of Example 4.

(iii) Each interval has one regular and one singular end-points. In this case, $d_p = n^2$, $p = 1,2,3,4$. Then "mixing" can occur and we get **new regularly solvable extensions** of the product operator $\prod_{j=1}^{n} [T_0(\tau_j)]$. For the sake of definiteness assume that the end-points a_1, b_2, a_3 and b_4 are singular end-points and b_1, a_2, b_3 and a_4 are regular end-points. The other possibilities are entirely similar.

For $u \in D[\prod_{j=1}^{n} T(\tau_j)]$ and $\Phi_k \in D[\prod_{j=1}^{n} T(\tau_j^+)]$ where $u = \{u_1, u_2, u_3, u_4\}$ and $\Phi_k = \{\Phi_{1j}, \Phi_{2j}, \Phi_{3j}, \Phi_{4j}\}$ the conditions (5.1) reads:

$$0 = [u, \Phi_k] = \sum_{j=1}^{4}\{[u, \Phi_{kj}](b) - [u, \Phi_{kj}](a)\}, \quad (k = 1,2,3,4). \qquad (6.4)$$

Also, for $v \epsilon D[\prod_{j=1}^{n} T(\tau_j^+)]$ and $\psi_i \epsilon D[\prod_{j=1}^{n} T(\tau_j)]$ where $v = \{v_1, v_2, v_3, v_4\}$ and $\psi_i = \{\psi_{1i}, \psi_{2i}, \psi_{3i}, \psi_{4i}\}$ the conditions (5.2) reads:

$$0 = [\psi_i, v] = \sum_{p=1}^{4}\{[\psi_{ip}, v](b) - [\psi_{ip}, v](a)\}, \tag{6.5}$$

$i = 1, 2, 3, 4$ and condition (ii) in Theorem 4.7 reads

$$0 = [\psi_i, \Phi_k] = \sum_{i=1}^{4}\{[\psi_i, \Phi_k](b) - [\psi_i, \Phi_k](a)\}.$$

By [3, Theorem III.10.13], the terms involving the singular end-points a_1, b_2, a_3 and b_4 are zero, such that (6.4), (6.5) and (6.6) reduces to

$$[u, \Phi_{k2}](b_2) - [u, \Phi_{k1}](a_1) - [u, \Phi_{k3}](a_3) - [u, \Phi_{k4}](b_4) = 0,$$

$$[\psi_{i2}, v](b_2) - [\psi_{i1}, v](a_1) - [\psi_{i3}, v](a_3) - [\psi_{i4}, v](b_4) = 0$$

and

$$[\psi_{i2}, \Phi_{k2}](b_2) - [\psi_{i1}, \Phi_{k1}](a_1) - [\psi_{i3}, \Phi_{k3}](a_3) - [\psi_{i4}, \Phi_{k4}](b_4) = 0,$$

$(i, k = 1, 2, 3, 4)$ respectively. Thus the boundary conditions are not separated for the four intervals and hence the regularly solvable operator can not be expressed as a direct sum of regularly solvable operators defined in the separate intervals $I_p, p = 1, 2, 3, 4$.

We refer to Everitt and Zettl's papers [6] and [8] for more examples and more details.

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