

is the domain of an operator S which is regularly solvable with respect to $[\prod_{j=1}^n T_0(\tau_j)]$ and $[\prod_{j=1}^n T_0(\tau_j^+)]$ and the set

$$\{v: v \in D[\prod_{j=1}^n T(\tau_j^+)], [\psi_j, v](b) - [\psi_j, v](a) = 0, \\ j = 1, 2, \dots, r\} \quad (5.2)$$

is the domain of the operator S^* ; moreover $\lambda \in \Delta_4(S)$.

Conversely, if S is regularly solvable with respect to $[\prod_{j=1}^n T_0(\tau_j)]$ and $[\prod_{j=1}^n T_0(\tau_j^+)]$ and $\lambda \in \Pi[\prod_{j=1}^n [T_0(\tau_j)], \prod_{j=1}^n [T_0(\tau_j^+)]] \cap \Delta_4(S)$, then with r and s defined by (4.5) and there exist functions ψ_j ($j = 1, 2, \dots, r$), Φ_k ($k = r + 1, \dots, m$) which satisfy (i) and (ii) and are such that (5.1) and (5.2) are the domains of S and S^* respectively.

S is self-adjoint if, and only if, $\prod_{j=1}^n(\tau_j) = \prod_{j=1}^n(\tau_j^+)$, $r = s$ and $\Phi_k = \psi_{k-r}$ ($k = r + 1, \dots, m$); S is J -self-adjoint if $\prod_{j=1}^n(\tau_j^+) = J \prod_{j=1}^n(\tau_j) J$ (J is a complex conjugate), $r = s$ and $\Phi_k = \bar{\psi}_{k-r}$ ($k = r + 1, \dots, m$)

Proof: The proof is entirely similar to that of [3], [4], [7], [8], [12] and [17], and therefore omitted.

We shall now consider our interval is $I = [a, b]$ and we characterize all the operators which are regularly solvable with respect to $\prod_{j=1}^n [T_0(\tau_j)]$, and $\prod_{j=1}^n [T_0(\tau_j^+)]$ in terms of boundary conditions featuring L_w^2 -solutions of the equations in (4.7). This generalizes an analogue of a results of Evans and Ibrahim [4], Sun Jiong's result [9] in the special case of symmetric operators with equal deficiency indices and Zai-ju Shang's results [12] for J -symmetric operators.

Theorem 5.2 (cf. [7, Theorem 4.1]):

Let $\lambda \in \Pi[\prod_{j=1}^n [T_0(\tau_j)], \prod_{j=1}^n [T_0(\tau_j^+)]]$, let r, s and m be defined by (4.5), and let x_i ($i = 1, 2, \dots, m$), y_j ($j = 1, 2, \dots, m$) be defined by (4.8) and (4.9) and arranged to satisfy (4.16). Let $K_{r \times n^2 N}$, $L_{r \times (m-n^2 N)}$, $M_{s \times n^2 N}$ and $N_{s \times (m-n^2 N)}$ be numerical matrices which satisfy the following conditions:

- (i) $\text{rank}(K_{r \times n^2 N} \oplus L_{r \times (m-n^2 N)}) = r$,
 $\text{rank}(M_{s \times n^2 N} \oplus N_{s \times (m-n^2 N)}) = s$,
(ii) $L_{r \times (m-n^2 N)} E_{(m-n^2 N) \times (m-n^2 N)}^{1,2} (N_{s \times (m-n^2 N)})^T$
 $+ (-i)^{n^2 N} K_{r \times n^2 N} J_{n^2 N \times n^2 N} (M_{s \times n^2 N})^T = 0_{r \times s}$.

Then the set of all $u \in D[\prod_{j=1}^n T(\tau_j)]$ such that

$$M_{s \times n^2 N} \begin{pmatrix} u(a) \\ \vdots \\ u^{[n^2 N-1]}(a) \end{pmatrix} - N_{s \times (m-n^2 N)} \begin{pmatrix} [u, y_{n^2 N+1}](b) \\ \vdots \\ [u, y_m](b) \end{pmatrix} = 0_{s \times 1} \quad (5.3)$$

is the domain of an operator S which is regularly solvable with respect to $\prod_{j=1}^n T_0(\tau_j)$ and $\prod_{j=1}^n T_0(\tau_j^+)$ and $D(S^*)$ is the set of all $v \in D[\prod_{j=1}^n T(\tau_j^+)]$ which are such that

$$K_{r \times n^2 N} \begin{pmatrix} \bar{v}(a) \\ \vdots \\ \bar{v}^{[n^2 N-1]}(a) \end{pmatrix} - L_{r \times (m-n^2 N)} \begin{pmatrix} [x_1, v](b) \\ \vdots \\ [x_{m-n^2 N}, v](b) \end{pmatrix} = 0_{r \times 1}. \quad (5.4)$$

Proof: The proof follows by using (3.7), Theorem 5.1, Lemmas 4.9, 4.10 and by considering:

$$M_{s \times n^2 N} J_{n^2 N \times n^2 N}^{-1} = -i^{n^2 N} (\alpha_{jk})_{\substack{r+1 \leq j \leq s, m \\ 1 \leq k \leq n^2 N}}, \\ N_{s \times (m-n^2 N)} = (\beta_{jk})_{\substack{r+1 \leq j \leq m \\ n^2 N+1 \leq k \leq m}}, \quad (5.5)$$

$$K_{r \times n^2 N} J_{n^2 N \times n^2 N}^{-1} = -i^{n^2 N} (\gamma_{jk})_{\substack{1 \leq j \leq r \\ 1 \leq k \leq n^2 N}}, \\ L_{r \times (m-n^2 N)} = (\epsilon_{jk})_{\substack{1 \leq j \leq r \\ 1 \leq k \leq m-n^2 N}}. \quad (5.6)$$

The following is the converse of Theorem 5.2:

Theorem 5.3 (cf. [7, Theorem 4.2]): Let the operator S be a regularly solvable with respect to $\prod_{j=1}^n T_0(\tau_j)$ and $\prod_{j=1}^n T_0(\tau_j^+)$, let $\lambda \in \Pi[\prod_{j=1}^n [T_0(\tau_j)], \prod_{j=1}^n [T_0(\tau_j^+)]] \cap \Delta_4(S)$, let r, s and m be defined by (4.5), and suppose that (4.16) is satisfied. Then there exist numerical matrices $K_{r \times n^2 N}$, $L_{r \times (m-n^2 N)}$, $M_{s \times n^2 N}$ and $N_{s \times (m-n^2 N)}$ such that the conditions (i) and (ii) in Theorem 5.1 are satisfied and $D(S)$ is the set of all $u \in D[\prod_{j=1}^n T(\tau_j)]$ satisfying (5.3) while $D(S^*)$ is the set of all $v \in D[\prod_{j=1}^n T(\tau_j^+)]$ satisfying (5.4).

Remark 5.4: Theorems 5.2 and 5.3 follows from the following results for the case of one singular end points in [4], for the case of two singular end-points in [7] and for any finite number of intervals in [8].

Remark 5.5: In the case when $\prod_{j=1}^n(\tau_j)$ is formally J -symmetric, that is $\prod_{j=1}^n(\tau_j^+) = J \prod_{j=1}^n(\tau_j) J$, where J is complex conjugation. The operator $\prod_{j=1}^n [T_0(\tau_j)]$ is then J -symmetric and $\prod_{j=1}^n [T_0(\tau_j)]$ and $\prod_{j=1}^n [T_0(\tau_j^+)] = J \prod_{j=1}^n [T_0(\tau_j)] J$ form an adjoint pair with $\Pi[\prod_{j=1}^n T_0(\tau_j), \prod_{j=1}^n T_0(\tau_j^+)] = \Pi[\prod_{j=1}^n T_0(\tau_j)]$. Since $\prod_{j=1}^n(\tau_j) [u] = \lambda w u$ if and only if $\prod_{j=1}^n(\tau_j^+) [\bar{u}] = \bar{\lambda} w \bar{u}$ it follows that from (4.6) that for all $\lambda \in \prod_{j=1}^n [T_0(\tau_j)]$, $\text{def}[\prod_{j=1}^n [T_0(\tau_j)] - \lambda I] = \text{def}[\prod_{j=1}^n [T_0(\tau_j^+)] - \bar{\lambda} I]$ is constant ℓ , say, and so in (4.5), $r = s = \ell$ with $0 \leq \ell \leq n^2 N$.

Remark 5.6: If S is a J -self-adjoint extension of, then $S^* = JSJ$ and consequently $v \in D(S^*)$ if and only if, $\bar{v} \in D(S)$. In this case when $\prod_{j=1}^n(\tau_j)$ is formally J -symmetric for a complex conjugation J , Theorems 5.2 and 5.3 include Theorem 5.5 of Zai-Jiu-Shang [12].

VI. Discussion.

Everitt and Zettl [6] discussed the possibility of generating self-adjoint operators which are not expressible as the direct sum of self-adjoint operators defined in the separate intervals. In this section we extend this case of general ordinary differential operators, i.e., we discuss the possibility of the regularly solvable operators which are not expressible as the direct sums of regularly solvable operators defined in the separate intervals $I_p = (a_p, b_p)$, $p = 1, 2, 3, 4$. We shall refer to these operators as "new regularly solvable operators".

If a_p is a regular end-point and b_p is singular, then

$$def[\prod_{j=1}^n [T_0(\tau_j)] - \lambda I] + def[\prod_{j=1}^n [T_0(\tau_j^+)] - \bar{\lambda} I] = 4n^2$$

for all $\lambda \in \Pi[\prod_{j=1}^n [T_0(\tau_j)], \prod_{j=1}^n [T_0(\tau_j^+)]]$ if and only if, the terms in (5.1) at the end-points b_p is zero. By Lemma 4.6, for all $\lambda \in \Pi[\prod_{j=1}^n [T_0(\tau_j)], \prod_{j=1}^n [T_0(\tau_j^+)]]$, we get in all cases

$$0 \leq def[\prod_{j=1}^n [T_0(\tau_j)] - \lambda I] + def[\prod_{j=1}^n [T_0(\tau_j^+)] - \bar{\lambda} I] \leq 8n^2 \quad (6.1)$$

While

$$2n^2 \leq def[\prod_{j=1}^n [T_0(\tau_j)] - \lambda I] + def[\prod_{j=1}^n [T_0(\tau_j^+)] - \bar{\lambda} I] \leq 8n^2 \quad (6.2)$$

when each interval has at least one singular end-point, and

$$def[\prod_{j=1}^n [T_0(\tau_j)] - \lambda I] + def[\prod_{j=1}^n [T_0(\tau_j^+)] - \bar{\lambda} I] = 8n^2 \quad (6.3)$$

for the case when all end-points are regular.

Let

$$def[\prod_{j=1}^n [T_0(\tau_j)] - \lambda I] + def[\prod_{j=1}^n [T_0(\tau_j^+)] - \bar{\lambda} I] = d,$$

and

$$def[\prod_{j=1}^n [T_0(\tau_{jp})] - \lambda I] + def[\prod_{j=1}^n [T_0(\tau_{jp}^+)] - \bar{\lambda} I] = d_p, \quad p = 1, 2, 3, 4.$$

Then, by part (iii) in Lemma 4.5, we have that

$$d = \sum_{p=1}^4 d_p.$$

We now consider some of the possibilities.

Example 1: $d = 0$. This is the minimal case in (6.1) and can only occur when all eight end-points are singular. In this case $\prod_{j=1}^n [T_0(\tau_j)]$ is itself regularly solvable and has no proper regularly solvable extensions; see [3, Chapter III] and [4] for more details.

Example 2: $d = n^2$ with one of d_p , $p = 1, 2, 3, 4$ is equal to n^2 and all the others are equal to zero. We assume that $d_1 = n^2$ and $d_2 = d_3 = d_4 = 0$. The other possibilities are entirely similar. In this case we must have seven singular end-points and one regular end-point. There are no new regularly solvable extensions and we have that $S = S_1 \oplus_{p=2}^4 \prod_{j=1}^n [T_0(\tau_{jp})]$, where S_1 is regularly solvable extension of $\prod_{j=1}^n [T_0(\tau_{j1})]$, i.e., all regularly solvable extensions of $\prod_{j=1}^n [T_0(\tau_j)]$ can be obtained by forming sums of regularly solvable extensions of $\prod_{j=1}^n [T_0(\tau_{jp})]$, $p = 1, 2, 3, 4$. These are obtained as in the case of one interval.

Example 3: six singular end-points and $d = 2n^2$. We consider two cases.

- (i) One interval has two regular end-points, say I_1 , and each one of the others has two singular end-points. Then, $S = S_1 \oplus_{p=2}^4 \prod_{j=1}^n [T_0(\tau_{jp})]$, where S_1 is regularly solvable extension of $\prod_{j=1}^n [T_0(\tau_{j1})]$, generates all regularly solvable extensions of the product differential operator $\prod_{j=1}^n [T_0(\tau_j)]$.
- (ii) There are two intervals, say I_1 and I_2 each has one singular and one regular end-points, and each one of the others has two singular end-points. In this case

$$S = S_1 \oplus S_2 \oplus \prod_{j=1}^n [T_0(\tau_{j3})] \oplus \prod_{j=1}^n [T_0(\tau_{j4})],$$

and $S_1 \oplus S_2$ generates all regularly solvable extensions of the product operator $\prod_{j=1}^n [T_0(\tau_j)]$. The other possibilities in the cases (i) and (ii) are entirely similar.

Example 4: Five end-points and $d = 3n^2$. We consider two cases.

- (i) There are two intervals, say I_1 and I_2 such that I_1 has two regular end-points and I_2 has one regular and one singular end-points, and each one of the others has two singular end-points. In this case $d_1 = 2n^2$ and $d_2 = n^2$, then $S = S_1 \oplus S_2 \oplus \prod_{j=1}^n [T_0(\tau_{j3})] \oplus \prod_{j=1}^n [T_0(\tau_{j4})]$ and $S_1 \oplus S_2$ generates all regularly solvable extensions of the product operator $\prod_{j=1}^n [T_0(\tau_j)]$ which is similar to the case (ii) of Example 3.
- (ii) There are three intervals, say I_1, I_2 and I_3 each one has one regular and one singular end-points and the fourth has two singular end-points. In this case $d_1 = d_2 = d_3 = n^2$ and $d_4 = 0$, then $S = S_1 \oplus S_2 \oplus S_3 \oplus \prod_{j=1}^n [T_0(\tau_{j4})]$ and $S_1 \oplus S_2 \oplus S_3$ generates all regularly solvable extensions of the product operator $\prod_{j=1}^n [T_0(\tau_j)]$. The other possibilities are entirely similar.

Example 5: Four singular end-points and $d = 4n^2$. We consider three cases.

- (i) There are two intervals, say I_1 and I_2 each one has two regular end-points, and each one of the others has two singular end-points. In this case $d_1 = d_2 = 2n^2$, $d_3 = d_4 = 0$, then $S = S_1 \oplus S_2 \oplus \prod_{j=1}^n [T_0(\tau_{j3})] \oplus \prod_{j=1}^n [T_0(\tau_{j4})]$, and $S_1 \oplus S_2$ generates all regularly solvable extensions of $\prod_{j=1}^n [T_0(\tau_j)]$.
- (ii) There are two intervals, say I_1 and I_2 each one has one regular and one singular end-points, and the others I_3 and I_4 has two regular and singular end-points respectively. In this case $d_1 = d_2 = n^2$, $d_3 = 2n^2$ and $d_4 = 0$, then $S = S_1 \oplus S_2 \oplus S_3 \oplus \prod_{j=1}^n [T_0(\tau_{j4})]$ as the case (ii) of Example 4.
- (iii) Each interval has one regular and one singular end-points. In this case, $d_p = n^2$, $p = 1, 2, 3, 4$. Then "mixing" can occur and we get **new regularly solvable extensions** of the product operator $\prod_{j=1}^n [T_0(\tau_j)]$. For the sake of definiteness assume that the end-points a_1, b_2, a_3 and b_4 are singular end-points and b_1, a_2, b_3 and a_4 are regular end-points. The other possibilities are entirely similar.

For $u \in D[\prod_{j=1}^n T(\tau_j)]$ and $\Phi_k \in D[\prod_{j=1}^n T(\tau_j^+)]$ where $u = \{u_1, u_2, u_3, u_4\}$ and $\Phi_k = \{\Phi_{1j}, \Phi_{2j}, \Phi_{3j}, \Phi_{4j}\}$ the conditions (5.1) reads:

$$0 = [u, \Phi_k] = \sum_{j=1}^4 \{ [u, \Phi_{kj}](b) - [u, \Phi_{kj}](a) \}, \quad (k = 1, 2, 3, 4). \quad (6.4)$$

Also, for $v \in D[\prod_{j=1}^n T(\tau_j^+)]$ and $\psi_i \in D[\prod_{j=1}^n T(\tau_j)]$ where $v = \{v_1, v_2, v_3, v_4\}$ and $\psi_i = \{\psi_{1i}, \psi_{2i}, \psi_{3i}, \psi_{4i}\}$ the conditions (5.2) reads:

$$0 = [\psi_i, v] = \sum_{p=1}^4 \{[\psi_{ip}, v](b) - [\psi_{ip}, v](a)\}, \quad (6.5)$$

$i = 1, 2, 3, 4$ and condition (ii) in Theorem 4.7 reads

$$0 = [\psi_i, \Phi_k] = \sum_{i=1}^4 \{[\psi_i, \Phi_k](b) - [\psi_i, \Phi_k](a)\}.$$

By [3, Theorem III.10.13], the terms involving the singular end-points a_1, b_2, a_3 and b_4 are zero, such that (6.4), (6.5) and (6.6) reduces to

$$[u, \Phi_{k2}](b_2) - [u, \Phi_{k1}](a_1) - [u, \Phi_{k3}](a_3) - [u, \Phi_{k4}](b_4) = 0,$$

$$[\psi_{i2}, v](b_2) - [\psi_{i1}, v](a_1) - [\psi_{i3}, v](a_3) - [\psi_{i4}, v](b_4) = 0$$

and

$$[\psi_{i2}, \Phi_{k2}](b_2) - [\psi_{i1}, \Phi_{k1}](a_1) - [\psi_{i3}, \Phi_{k3}](a_3) - [\psi_{i4}, \Phi_{k4}](b_4) = 0,$$

$(i, k = 1, 2, 3, 4)$ respectively. Thus the boundary conditions are not separated for the four intervals and hence the regularly solvable operator can not be expressed as a direct sum of regularly solvable operators defined in the separate intervals $I_p, p = 1, 2, 3, 4$.

We refer to Everitt and Zettl's papers [6] and [8] for more examples and more details.

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