Quaternionic Quadratic Formulas

Niki Lina Matiadou And Perikles Papadopoulos
Department of Electronics Engineering,
Technological Education Institute of Piraeus, Greece
lmatiadou@yahoo.gr and ppapadop@teipir.gr

***Abstract*— In this paper, we use the vector representation of quaternions in order to obtain simplified forms of the solutions of the quaternionic equation $x^2+bx+c=0$ with b,c ∈ $\mathcal{H}$. Similarly, we conclude that the equation $x^2+xab-bax=0$ with a,b∈$\mathcal{H}$ and $a=a_0+\underline{a}, b=b_0+\underline{b}$ has the unique solution x=0, except the case $(a_0,b_0)=(0,0)$ and $\underline{a}\times\underline{b}\neq 0$, where it has an infinite number of solutions $x=-(\underline{a}\times\underline{b})+q(\underline{a}\times\underline{b})q^{-1}, q\in H^*$.**

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I. Introduction

The set *H* of quaternions is the vector space R^4 with component-wise addition and scalar multiplication. We denote the canonical basis elements by

$$1=(1,0,0,0), i=(0,0,1,0), k=(0,0,0,1).$$

Thus, the general quaternion has the form

$$q=(\alpha,\beta,\gamma,\delta)=\alpha 1+\beta i+\gamma j+\delta k\equiv$$
$$a+\beta i+\gamma j+\delta\kappa, \alpha,\beta,\gamma,\delta\in R.$$

Hamilton in [1] was the first who defined multiplication on *H* by using the relations

$$i^2=j^2=k^2=ijk=-1.$$

The set *H* of quaternions is a skew-field and an associative division algebra with unit. The general quaternion $q=(\alpha,\beta,\gamma,\delta)=\alpha+\beta i+\gamma j+\delta k$, can be represented as a pair $q=(a,\underline{u})$, where $\alpha\in R$, $\underline{u}=(\beta,\gamma,\delta)\in R^3$. This representation simplifies the formula of multiplication. In fact, by using vector products for the arbitrary quaternions

$$q_1=a_0+a_1 i+a_2 j+a_3 k=(a_0,\underline{u}),$$
$$q_2=\beta_0+\beta_1 i+\beta_2 j+\beta_3 k=(\beta_0,\underline{v}),$$

we have the following relation:

$$q_1 q_2=(a_0,\underline{u})(\beta_0,\underline{v})=(a_0\beta_0-\underline{u}\cdot\underline{v}, \beta_0\underline{u}+a_0\underline{u}+\underline{u}\times\underline{v}).$$

For $q_1=a_0+a_1 i+a_2 j+a_3 k=(a_0,\underline{u})$ we define its *conjugate* by

$$\overline{q_1}=a_0-a_1 i-a_2 j-a_3 k=(a_0,-\underline{u}),$$

its *real part* by $\operatorname{Re} q_1=\frac{1}{2}(q_1+q_2)=a_0$,

its *imaginary part* by $Imq_1=q_1-Req_1=\underline{u}$ and its *norm* or *modulus* by

$$\|q_1\|=\sqrt{q_1\overline{q_1}}=\sqrt{a_0^2+a_1^2+a_2^2+a_3^2}.$$

The *inverse* of q_1 exists when $\|q_1\|\neq 0$ and it is equal to $q^{-1}=\frac{1}{\|q_1\|^2}\overline{q_1}$. We say that $q_1,q_2\in H$ are *similar* if there exists an invertible $q\in\mathcal{H}$ such that $q_1=q^{-1}q_2 q$ or equivalently, if $Req_1=Req_2$ and $\|q_1\|=\|q_2\|$, [2].

We consider the monic quadratic polynomial equation

$$x^2+bx+c=0, \tag{1.1}$$

where *b*, *c* $\in\mathcal{H}$. It is equivalent to a real system of four nonlinear equations.

Zang and Mu [3] worked on quadratic equation (1.1) and tried to compute some roots by solving a real linear system. They did not give a complete solution of (1.1). Poter [4] reduced the problem of solving equation (1.1) to the problem of solving a linear polynomial equation of the form *px*+*xq*+*r*=0, provided that a root of (1.1) is given. Niven [5] determined the number of roots of equation (1.1), but he did not give any formula for the computation of the roots. Finally, Huang and So [6] used an algebraic approach to solve equation (1.1) or equivalently the nonlinear system.

In this paper we use the vector representation of quaternions in order to give simplified explicit formulas for the roots of the quadratic equation (1.1).

II. The equation

$x^2+bx+c=0$, $b,c\in$ *H*. We use the transformation $x=X-\frac{b}{2}$ in order to obtain the equivalent equation

$$X^2+aX-Xa+b=0, \tag{2.1}$$

where $a = \frac{b}{2}$ and $\beta = c - \frac{b^2}{4} = \frac{4c-b^2}{4}$. Since, $a = \mathrm{Re}\,a + \mathrm{Im}\,a$ and $(\mathrm{Re}\,a)X = X(\mathrm{Re}\,a)$ equation (2.1) is equivalent to the equation

$$X^2 + aX - Xa + \beta = 0, \qquad (2.2)$$

where $a = \mathrm{Im}\frac{b}{2}$ with $\mathrm{Re}\,a = 0$ and $\beta = \frac{4c-b^2}{4}$.

Let $X = X_0 + \underline{w}$ where $X_0 \in \mathcal{R}$ and

$$\underline{w} = (X_1, X_2, X_3) = X_1 i + X_2 j + X_3 k,$$

$a = \underline{u} = (\frac{b_1}{2}, \frac{b_2}{2}, \frac{b_3}{2})$ and $\beta = \beta_0 + \underline{v}$. Then we have the following:

$$X^2 = X_0^2 - \underline{w}^2 + 2X_0\underline{w},$$

$$aX = -\underline{u}\underline{w} + X_0\underline{u} + \underline{u}\times\underline{w},$$

$$Xa = -\underline{u}\underline{w} + X_0\underline{u} - \underline{u}\times\underline{w}$$

and equation (2.2) becomes

$$(X_0^2 - \underline{w}^2 + \beta_0) + 2X_0\underline{w} + 2(\underline{u}\times\underline{w}) + \underline{v} = 0. \qquad (2.3)$$

Equivalently, we have the equations

$$X_0^2 - \underline{w}^2 + \beta_0 = 0, \qquad (2.4)$$

$$2X_0\underline{w} + 2(\underline{u}\times\underline{w}) + \underline{v} = 0. \qquad (2.5)$$

We now distinguish the following cases:

1. $\underline{u}, \underline{v}$ are collinear, i.e. $\underline{u}\times\underline{v} = \underline{0} \Leftrightarrow a\beta = \beta a$. In this case, we consider two different subcases.

(a) $\underline{v} = \underline{0}$ and $\underline{u}$ arbitrary. Then the equation

(2.5) becomes $X_0\underline{w} + \underline{u}\times\underline{w} = \underline{0}$ and gives

$X_0\underline{w}^2 + (\underline{u}\times\underline{w})\cdot\underline{w} = 0$ or $X_0\underline{w}^2 = 0 \Leftrightarrow X_0 = 0$ or $\underline{w} = \underline{0}$. If $X_0 = 0$, then from equation (2.4) we get $\underline{w}^2 = \beta_0 \Leftrightarrow \underline{w} = (X_1, X_2, X_3)$, with

$X_1^2 + X_2^2 + X_3^2 = \beta_0$, provided that $\beta_0 > 0$. If $\underline{w} = \underline{0}$, then equation (2.5) becomes

$$X_0^2 = -\beta_0 \Leftrightarrow X_0 = \pm\sqrt{-\beta_0}, \beta_0 \le 0.$$

(b) $\underline{u} = \lambda\underline{v}, \lambda \in R, \underline{v} \ne \underline{0}$.

We observe that it is impossible to have a solution with $X_0 = 0$. In fact, if $X_0 = 0$, then the equation (2.5) becomes $2\lambda(\underline{v}\times\underline{w}) + \underline{v} = \underline{0}$, which gives $2\lambda(\underline{v}\times\underline{w})\cdot\underline{v} + \underline{v}^2 = 0$ or $\underline{v} = \underline{0}$, (absurd). Thus, we look for solutions $X = X_0 + \underline{w}$ with $X_0 \ne 0$. From equation (2.5), we derive a pair of equations

$$2X_0\underline{w}^2 + \underline{v}\cdot\underline{w} = 0, \qquad (2.6)$$

$$2X_0\underline{w}\cdot\underline{v} + \underline{v}^2 = 0, \qquad (2.7)$$

from which, using equation (2.4), we conclude that X_0 is a root of the biquadratic equation

$$4X_0^4 + \beta_0 X_0^2 - \underline{v}^2 = 0. \qquad (2.8)$$

Hence, we have the following solutions

$$X_0 = \pm\sqrt{\frac{-\beta_0 + \sqrt{\beta_0^2 + \underline{v}^2}}{2}}.$$

Moreover, from equation (2.5), we get

$$2X_0(\underline{w}\times\underline{v}) + 2\lambda[\underline{v}^2\underline{w} - (\underline{w}\cdot\underline{v})\underline{v}] = \underline{0},$$

from which by using equations

$2X_0\underline{w} + 2\lambda(\underline{v}\times\underline{w}) + \underline{v} = \underline{0}$ and (2.7), in either case $\lambda \ne 0$ or $\lambda = 0$, we obtain $\underline{w} = -\frac{1}{2X_0}\underline{v}$.

Hence, we have two solutions

$X = X_0 - \frac{1}{2X_0}\underline{v}$, where we have that

$$X_0 = \pm\sqrt{\frac{-\beta_0 + \sqrt{\beta_0^2 + \underline{v}^2}}{2}}.$$

2. $\underline{u}, \underline{v}$ are not collinear, i.e. $\underline{u}\times\underline{v} \ne \underline{0} \Leftrightarrow a\beta \ne \beta a$.

In this case, we consider two different subcases.

(a) If $X_0 = 0$, then the equations (2.4) and (2.5) give the system

$$\underline{w}^2 = \beta_0 \qquad (2.9)$$

$$\underline{v}\times\underline{w} = -\frac{1}{2}\underline{v}, \qquad (2.10)$$

which is compatible when $\beta_0 \ge 0$ and $\underline{u}\cdot\underline{v} = 0$. Since $\underline{u}, \underline{w}$ and $\underline{u}\times\underline{v}$ are coplanar, $\underline{w}$ can be written as

$$\underline{w} = \lambda\underline{u} + \mu(\underline{u}\times\underline{v}), \ \lambda, \mu \in R$$

and it must satisfy equations (2.9) and (2.10). Thus, from equation (2.10), we find that

$$\underline{w} = \lambda \underline{u} + \frac{1}{2\underline{u}^2}(\underline{u} \times \underline{v}), \lambda \in R$$

and from equation (2.9), we find that $\lambda = \pm \frac{\sqrt{4\beta_0 \underline{u}^2 - \underline{v}^2}}{2\underline{u}^2}$, provided that

$$4\beta_0 \underline{u}^2 - \underline{v}^2 \geq 0.$$

(b) If $X_0 \neq 0,$ then the equation (2.5), gives

$$2X_0(\underline{w} \cdot \underline{u}) + \underline{v} \cdot \underline{u} = 0 \Leftrightarrow \underline{w} \cdot \underline{u} = -\frac{\underline{v} \cdot \underline{u}}{2X_0}. \quad (2.11)$$

Moreover, from some equation, we get

$$2(X_0^2 + \underline{u}^2)\underline{w} = 2(\underline{w} \cdot \underline{u})\underline{u} - X_0 \underline{u} - \underline{v} \times \underline{u}.$$

By using (2.11), we finally find

$$\underline{w} = -\frac{1}{2X_0(X_0^2 + \underline{u}^2)}[(\underline{v} \cdot \underline{u})\underline{u} + X_0^2 \underline{v} + X_0(\underline{v} \times \underline{u})]. \quad (2.12)$$

It is to verify that $\underline{w}$ given by (2.12), is a solution of the equation (2.5). Moreover, this is the unique solution of equation (2.5). In fact, if we suppose that there exists a second solution $\underline{w}_1 \neq \underline{w},$ then from equations

$$2X_0\underline{w}_1 + 2(\underline{u} \times \underline{w}_1) + \underline{v} = \underline{0}$$

$$2X_0\underline{w} + 2(\underline{u} \times \underline{w}) + \underline{v} = \underline{0}$$

by subtraction, we obtain

$$2X_0(\underline{w}_1 - \underline{w})^2 = 0 \text{ or } \underline{w}_1 = \underline{w}.$$

Moreover, from equations (2.4) and (2.5), we obtain

$$\underline{v} \cdot \underline{w} = -2X_0 \underline{w}^2 = -2X_0(X_0^2 + \beta_0). \quad (2.13)$$

Also, from equation (2.12), we obtain

$$\underline{v} \cdot \underline{w} = -\frac{(\underline{u} \cdot \underline{v})^2 + X_0^2 \underline{v}^2}{2X_0(X_0^2 + \underline{u}^2)}. \quad (2.14)$$

Thus, from (2.13) and (2.14), we find that X_0^2 must satisfy the equation

$$4X_0^6 + 4(\underline{u}^2 + \beta_0)X_0^4 + (4\beta_0 \underline{u}^2 - \underline{v}^2)X_0^2 - (\underline{v} \cdot \underline{u})^2 = 0. \quad (2.15)$$

Putting $X_0^2 = Y$, we have that Y must be a positive root of the equation $f(Y) = 0$, where

$$f(Y) = 4Y^3 + 4(\underline{u}^2 + \beta_0)Y^2 + (4\beta_0 \underline{u}^2 - \underline{v}^2)Y - (\underline{v} \cdot \underline{u})^2 = 0.$$

Since $f(0) = -(\underline{u} \cdot \underline{v})^2 < 0$ and $\lim_{Y \to +\infty} f(Y) = +\infty$ we conclude that the polynomial $f(Y)$ has at least one positive root. Moreover, looking at the sequence of coefficients

$$-(\underline{u} \cdot \underline{v})^2 < 0, \; 4\beta_0 \underline{u}^2 - \underline{v}^2, \; 4(\underline{u}^2 + \beta_0), \; 4 > 0,$$

we observe that we can have only one change of sign of the coefficients. It is due to that the system $4\beta_0 \underline{u}^2 - \underline{v}^2 > 0, \; 4(\underline{u}^2 + \beta_0) < 0,$

is impossible. Therefore, according to the *Harriot-Descartes rule* [7], the polynomial $f(Y)$ has exactly one positive root. Thus, we have proved the following

Theorem 2.1 Let

$$X^2 + aX - Xa + \beta = 0, \quad (2.16)$$

be a quadratic equation with $\alpha, \beta \in$ H and $\mathrm{Re}\, a = 0.$ Let also $X = X_0 + \underline{w},$ $a = \underline{u}$ and $\beta = \beta_0 + \underline{v},$ where $X_0, \beta_0 \in R$ and $\underline{w}, \underline{u}, \underline{v} \in R^3.$ Then we distinguish the following cases:

1. Let $\underline{u}, \underline{v}$ are collinear (i.e. $\underline{u} \times \underline{v} = \underline{0} \Leftrightarrow a\beta = \beta a$).

(a) We have the subcases:

i. if $\underline{v} = \underline{0}$ and $\beta_0 < 0,$ then $X = \pm\sqrt{-\beta_0}$ (two solutions)

ii. if $\underline{v} = \underline{0}$ and $\beta_0 = 0,$ then $X = 0$ (one solution)

iii. if $\underline{v} = \underline{0}$ and $\beta_0 > 0,$ then $X = \underline{w},$ with $\underline{w}^2 = \beta_0$ (infinite solutions).

(b) If $\underline{u} = \lambda \underline{v},$ $\lambda \in R$ and $\underline{v} \neq \underline{0},$ then $X = X_0 - \frac{1}{2X_0}\underline{v},$ where X_0^2 is the positive root of the equation $4X_0^4 + 4\beta_0 X_0^2 - \underline{v}^2 = 0,$

i.e. $X_0 = \pm\sqrt{\frac{-\beta_0 + \sqrt{\beta_0^2 + \underline{v}^2}}{2}}$ (two solutions).

2. Let $\underline{u}, \underline{v}$ be non collinear (i.e. $\underline{u} \times \underline{v} \neq 0 \Leftrightarrow a\beta \neq \beta a$). Then the equation (2.16) has a unique solution $X = X_0 + \underline{w},$ with $X_0 \neq 0$ and

$$\underline{w} = -\frac{1}{2X_0(X_0^2 + \underline{u}^2)}[(\underline{u} \cdot \underline{v})\underline{u} + X_0^2 \underline{v} + X_0(\underline{v} \times \underline{u})],$$

where X_0^2 is the unique positive root of the polynomial equation

$4X_0^6+4(\underline{u}^2+\beta_0)X_0^4+(4\beta_0\underline{u}^2-\underline{v}^2)X_0^2-(\underline{u}\cdot\underline{v})^2=0.$

Moreover, when $\underline{u}\cdot\underline{v}=0$ and $4\beta_0\underline{u}^2-\underline{v}^2\geq 0$, then the equation (2.16) has one or two pure imaginary solutions:

$X=\lambda\underline{u}+\frac{1}{2\underline{u}^2}(\underline{u}\times\underline{v})$, where we have that

$$\lambda=\pm\frac{\sqrt{4\beta_0\underline{u}^2-\underline{v}^2}}{2\underline{u}^2}.$$

III. The equation

$x^2+xab-bax=0, a,b\in H$. We consider the quaternionic equation

$$x^2+xab-bax=0, a,b\in H, \tag{3.1}$$

arising in the study of the spectrum of 2×2–quaternionic matrices [8]. We use the vector representation of quaternions in order to determine the explicit solutions of the equation (3.1).

Since we have that $\mathrm{Re}(ab)=\mathrm{Re}(ba)$ and $x\,\mathrm{Re}(ab)=\mathrm{Re}(ba)x$, without loss of generality we suppose that $\mathrm{Re}(ab)=\mathrm{Re}(ba)=0$.

Let $a=a_0+\underline{a}, b=b_0+\underline{b}, x=x_0+\underline{w}=x_0+(x_1,x_2,x_3)$. Then equation (3.1), is equivalent to the system

$$x_0^2-\underline{w}^2-2\underline{w}\cdot(\underline{a}\times\underline{b})=0, \tag{3.2}$$

$$x_0^2\underline{w}+x_0(\underline{a}\times\underline{b})+\underline{w}\cdot(a_0\underline{b}+b_0\underline{a})=\underline{0}. \tag{3.3}$$

We distinguish the following cases:

1. Let $(a_0,b_0)\neq(0,0)$. We have the subcases:

(a) If $x_0=0$, then from (3.3), we get that $\underline{w}=\lambda(a_0\underline{b}+b_0\underline{a}), \lambda\in R$ and from equation (3.2), we conclude that $\underline{w}=\underline{0}$.

(b) If $x_0\neq 0$, then considering the scalar product of both parts of (3.3) succesively by $\underline{w}$ and $\underline{a}\times\underline{b}$, we derive the equations

$$\underline{w}^2+\underline{w}\cdot(\underline{a}\times\underline{b})=0, \tag{3.4}$$

$$\underline{w}\cdot(\underline{a}\times\underline{b})+(\underline{a}\times\underline{b})^2=0, \tag{3.5}$$

from which by addition we find $(\underline{w}+\underline{a}\times\underline{b})^2=0$ or $\underline{w}=-\underline{a}\times\underline{b}$ It is easy to prove that $\underline{w}=-\underline{a}\times\underline{b}$ is the unique solution of (3.3). However, from (3.2), we have

$x_0^2-\underline{w}^2-2\underline{w}\cdot(\underline{a}\times\underline{b})=0$ or $x^2+(\underline{a}\times\underline{b})^2=0$,

which is impossible for $x_0\neq 0$. Therefore, for $(a_0,b_0)\neq(0,0)$ we have the unique solution $x=0$.

2. Let $(a_0,b_0)=(0,0)$. Then we have two subcases:

(a) If $x_0=0$, then equation (3.2) gives

$\underline{w}^2+2\underline{w}\cdot(\underline{a}\times\underline{b})=0$ or $(\underline{w}+\underline{a}\times\underline{b})^2=(\underline{a}\times\underline{b})^2$.

Thus, we have $\|\underline{w}+\underline{a}\times\underline{b}\|=\|\underline{a}\times\underline{b}\|$ and since $\mathrm{Re}(\underline{w}+\underline{a}\times\underline{b})=\mathrm{Re}(\underline{a}\times\underline{b})$ we conclude that $\underline{w}+\underline{a}\times\underline{b}$ is similar to $\underline{a}\times\underline{b}$, that is

$$\underline{w}=-\underline{a}\times\underline{b}+q(\underline{a}\times\underline{b})q^{-1}=0, q\in H^*. \tag{3.6}$$

Therefore, when $\underline{a}\times\underline{b}\neq 0$, equation (3.1) has an infinity of solutions given by (3.6). When $\underline{a}\times\underline{b}=0$, equation (3.1) has the unique solution $x=0$.

(b) If $x_0\neq 0$, then from equation (3.3), we find $\underline{w}=-\underline{a}\times\underline{b}$. However, from equation (3.2), we have $x_0^2+(\underline{a}\times\underline{b})^2=0$, which is impossible.

Thus, we have the following

Theorem 3.1. Let

$$x^2+xab-bax=0, \tag{3.7}$$

where $x=x_0+\underline{w}, a=a_0+\underline{a}, b=b_0+\underline{b}$, with $a_0b_0=\underline{a}\cdot\underline{b}$. Then we have the following cases:

1. When $(a_0,b_0)\neq(0,0)$, or $(a_0,b_0)=(0,0)$ and $\underline{a}\times\underline{b}=\underline{0}$, then equation (3.7) has the unique solution $x=0$.

2. When $(a_0,b_0)=(0,0)$, and $\underline{a}\times\underline{b}\neq 0$, then equation (3.7) has an infinite numbers of solutions $x=-\underline{a}\times\underline{b}+q(\underline{a}\times\underline{b})q^{-1}$, $q\in H^*$.

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