Projective Hilbert Space Instead Of Adiabatic Process

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***Abstract*—Berry's phase uses adiabatic approximation which only dependsrapidity of system's evolving. In Aharonov-Anandan's phase,cyclicity of evolution is single determinant in the development phase. With using a mathematical trick Geometrical Berry's phase indicates property of non-integrable topological Aharonov-Anandan's phase which has more universal form than the others. This universality is caused by the difference from using instead of projective Hilbert space to adiabatic approximation.**

Keywords—adiabatic process, Berry's phase, geometrical phase , topological phase, projective Hilbert space, Aharonov-Anandan's phase

I. INTRODUCTION

Phases are one of the most popular resarch area in physics. First of all, I have to say, this study for illustrate the similarities between geometrical Berry's phase and topological Aharonov-Anandan's phase. Geometrical phases in quantum or classical mechanics are acceptable under adiabatic approximation. Let we choose any quantum system like atom, electron or molecule etc. whose has discrete energy states with Hamiltonian $H(q,p;R(t))$ depends on a set of quite slowly changing parameters R as well as dynamical variables or operators and its environment e.g. electric or magnetic forces round a cycle which is on the initial point at time $= 0$, on final point namely same with initial point at time $t = T$. Afterwards, as I will explain later,altough environment conditions havechanged, system is still in state n according to adiabatic theorem. But its phase has changed, by $\gamma = \gamma(T) - \gamma(0)$. If we define the environment which does not change, then the phase is equal to

$$\gamma = -\frac{E_n(R)}{\hbar}T \qquad (1)$$

as Planck's says. When external parameter R begins changing, we have thesedue to Schrödinger equation not only dynamical but also geometrical phases. While dynamical phases are increasing with time T geometrical phases independent to that magnitude. In summary quantal or classical system with N freedom which under quite slowly changing external parameter has a different phase with the dynamical one and it calls Berry's phase [1] in physics literature. Unlike all of these definitions, Aharonov-Anandan's phase completely connected to arbitrary vectors on projective Hilbert space $P(\mathcal{H})$ [2]. Therewithal, its truely independent from time T. Let we will explain to this distinction with two special instance after describe adiabatic process and projective Hilbert space rapidly.

II. PROJECTIVE HILBERT SPACE

Even if about projective Hilbert space can be quite lengthy explanation [1], [2], [3], [4], [5], I think so far to explain the relationship between the phase of the Aharonov–Anandan and projective Hilbert spaceis enough. In Hilbert space,system could gainphase factor due to evolutionin the cyclic tarajectory. Thereforethat occur between the initial and final points singularity have shown in the Figure 1.

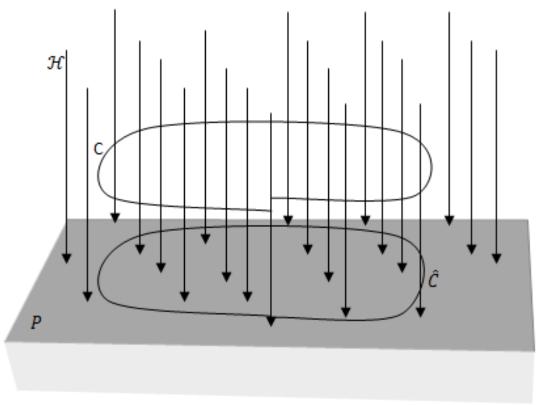

Figure 1. Projective Hilbert Space $P(\mathcal{H})$ of a complex Hilbert space $\mathcal{H}$ is the set of equivalence classes of vectors.

Whatever the rate of evolution of the system, state will not be affected and will be remained the same value. These results are associated with differential geometry and based on a mathematical convenience rather than we will be discussing in section 3.

III. ADIABATIC PROCESS

Let we choose a system, initially at the n th state with Hamiltonian $H(q,p;R(t))$. If the Hamiltonian is evolving during the time quite slowly, afterfor a while system is still at the same state but it has different Hamiltonian than the initial one. According to time dependent perturbation theory, $\hat{V}(t)$ is rather small but we neglected to variation of perturbation. Now we want to use adiabatic approximation to prove state invariance. If we demonstrate the transition probability nearly zero, there is no transition for the state of system. $p_{if}(t)$ is including the probability of transition and w_{fi} represents transition frequency between the states i and f ;

$$w_{fi} = \frac{E_f - E_i}{\hbar} \quad (2)$$

$$-\frac{1}{\hbar}\int_0^t d\,t' < \psi_f|\hat{V}(t)|\psi_i > e^{iw_{fi}t'} = \quad (3)$$

for the simplicity about integration by parts, we need to this transformation

$$e^{iw_{fi}t'} = \frac{1}{iw_{fi}}\frac{\partial}{\partial t}\left(e^{iw_{fi}t'}\right) \quad (4)$$

after the integration by parts , and before we accept $\hat{V}(t)$ change slowly , it vanishes at limits [3]. Then we have

$$= -\frac{1}{\hbar w_{fi}}\int_0^t d\,t'\left(\frac{\partial}{\partial t'} < \psi_f|\hat{V}(t)|\psi_i >\right)e^{iw_{fi}t'} \quad (5)$$

For explained reason $\frac{\partial}{\partial t'} < \psi_f|\hat{V}(t)|\psi_i >$ term is almost constant so we can get this term out of the integral.Then $p_{if}(t)$ equals:

$$= \frac{1}{\hbar^2 w_{fi}^2}\left|\frac{\partial}{\partial t} < \psi_f|\hat{V}(t)|\psi_i >\right|^2\left|\int_0^t d\,t' e^{iw_{fi}t'}\right|^2 \quad (6)$$

$$\simeq \frac{1}{\hbar^2 w_{fi}^2}\left|\frac{\partial}{\partial t} < \psi_f|\hat{V}(t)|\psi_i >\right|^2\left|\frac{1}{iw}\left(e^{iw_{fi}t} - 1\right)\right|^2 \quad (7)$$

$$\simeq \frac{1}{\hbar^2 w_{fi}^4}\left|\frac{\partial}{\partial t} < \psi_f|\hat{V}(t)|\psi_i >\right|^2 \times \left|e^{\frac{iw_{fi}t}{2}}\left(\frac{e^{\frac{iw_{fi}t}{2}} - e^{-\frac{iw_{fi}t}{2}}}{2i}\right)2i\right|^2 \quad (8)$$

$$p_{if}(t) \simeq \frac{1}{\hbar^2 w_{fi}^4}\left|\frac{\partial}{\partial t} < \psi_f|\hat{V}(t)|\psi_i >\right|^2 sin^2\left(\frac{w_{fi}t}{2}\right) \quad (9)$$

Because of $sin^2(\frac{w_{fi}t}{2}) \ll 1$ we can say easily, $p_{if}(t) \ll 1$.

Actually, as we accepted before $\hat{V}(t)$ is changing quite slowly ; $\frac{\partial}{\partial t} < \psi_f|\hat{V}(t)|\psi_i > \to 0$. Now we can say that, if the perturbation is adiabatic, the state still holds same. System at initial time $t = 0$ and at nth state has $|\psi_n(0) >$ket, after time t it still placed nth state but it has $|\psi_n(t) >$ket with Hamiltonian $\hat{H} = \widehat{H_0} + \hat{V}(t)$ and energy $E_n(t)$.

IV. Berry's Phase

With time dependent parameter R(t), Hamiltonian is defined like $H(R(t))$. The ket $|n(R(t)) >$ is adapting with external time dependent parameter R(t)and we will use it for representing nth energy eigenstate.

$$H(R(t))n(R(t)) >= E_n(R(t))|n(R(t)) > \quad (10)$$

$|n(R(t)) >$ is normalized and it evolves the parameter which is including $R(0) = R_0$, this ket provide the Schrödinger equation $|n(R_0, t_0 = 0; t >$.

$$H(x(t))|\psi(t) >= i\hbar\frac{\partial}{\partial t}|\psi(t) > \quad (11)$$

$$H(x(t))|n(x(t)) >= E_n|n(x(t)) > \quad (12)$$

At this point, we assume that Hamiltonian has discrete energy spectrum whose eigenvalues are not degenerate and there is no level crossing during the evolution. If $R(t)$ parameter is adiabatic variable, system at initial time $t = 0$ at nth state after for a while it still placed nth state like we argued at chapter 2. Now

$$|\psi(0) >= |n(R(0)) > \quad (13)$$

But generally, system gains a phase factor at time t.

$$|\psi(t) >= e^{i\phi_n}|n(R(t)) > \quad (14)$$

from the point of first view, we can easily expect to the phase is dynamical as

$$\theta_n(t) = -\frac{1}{\hbar}\int_0^t H_n(\tau)d\tau \quad (15)$$

However, if we put the likewise $\phi_n = \theta_n + \gamma_n$phase factor in the Schrödinger equation,

$$i\hbar\frac{d}{dt}|\psi(t) >= H|\psi(t) > \quad (16)$$

$$i\hbar\frac{d}{dt}\left(e^{i(\theta_n+\gamma_n)}\right)|n(R(t)) > + i\hbar e^{i\phi_n}\frac{d}{dt}|n(R(t)) >= \quad (17)$$

$$i\hbar(i\dot{\theta}_n + i\dot{\gamma}_n)e^{i\phi_n}|n(R(t)) > + i\hbar e^{i\phi_n}\frac{d}{dt}|n(R(t)) >= \quad (18)$$

$$i\hbar\left(i\dot{\theta}_n + i\dot{\gamma}_n\right)\Big|n(R(t)) > + i\hbar\frac{d}{dt}\Big|n(R(t)) > = H|n(R(t)) > \quad (19)$$

equation is multiplied by left with $< n(R(t))|$

$$i\hbar < n(R(t))|i\dot{\theta}_n + i\dot{\gamma}_n|n(R(t)) > + i\hbar < n(R(t))\left|\frac{d}{dt}\right|n(R(t)) > = < n(R(t))|H|n(R(t)) > \quad (20)$$

$$< n(R(t))|-\dot{\theta}_n - \dot{\gamma}_n|n(R(t)) > + i < n(R(t))\left|\frac{d}{dt}\right|n(R(t)) >= = \frac{1}{\hbar} < n(R(t))|H|n(R(t)) > \quad (21)$$

$$= -\int_0^t \frac{d}{dt}\theta_n dt - \int_0^t < n(R(t))|\dot{\gamma}_n|n(R(t)) > dt + i\int_0^t < n(R(t))|\frac{d}{dt}|n(R(t)) > dt \quad (22)$$

$$-\theta_n(\tau) = \int_0^t < n(R(t))|\dot{\gamma}_n|n(R(t)) > dt - i\int_0^t < n(R(t))\left|\frac{d}{dt}\right|n(R(t)) > + \frac{1}{\hbar}\int_0^t < n(R(t))|H|n(R(t)) > dt \quad (23)$$

we has known from(15) write instead of dynamical phase factor, last term on the RHS vanishes and its back on

$$\int_0^t < n(R(t))|\dot{\gamma}_n|n(R(t)) > dt = i\int_0^t < n(R(t))|\frac{d}{dt}|n(R(t)) > dt \quad (24)$$

equation. At this point it is easy to check (25) equation can derive easily

$$\frac{d}{dt}\gamma_n|n(R(t)) >= i\frac{d}{dt}|n(R(t)) > \quad (25)$$

$$i\frac{d\gamma_n}{dt}|n(R(t)) > + \frac{d}{dt}|n(R(t)) >= 0 \quad (26)$$

$$\frac{d\gamma_n}{dt} = i < n(R(t))|\frac{d}{dt}|n(R(t)) > \quad (27)$$

$$\frac{d\gamma_n}{dt} = i < n(R(t))|\,\vec{\nabla}.\boldsymbol{n}(R(t)) > \frac{dR(t)}{dt} \quad (28)$$

at the end of the story, if we remember the initial conditions our external parameter R is given under cyclic evolution namely $x(T) = x(0)$ and the evolution is adiabatically,

$$\gamma_n = i\oint_c < n(R(t))|\,\vec{\nabla}.\boldsymbol{n}(R(t)) > dR \quad (29)$$

we defined which is called Berry Phase easily.

V. AHARONOV-ANANDAN'S PHASE

When the system is evolving, it was defined by time dependent Schrödinger equation. we defined as previously the dynamical phase factor was given :

$$\theta(T) = -\frac{1}{\hbar}\int_0^T < \psi(t)|H(t)|\psi(t) > dt \quad (30)$$

and $\mathcal{H}$ defines Hilbert space, also P is formed by Hilbert space's all equivalance class "Projective Hilbert Space" , projection mapping will be

$$\Pi = \mathcal{H} \longrightarrow P \quad (31)$$

$$\Pi|\psi >= |\psi' > \quad (32)$$

and we assume

$$|\psi' >= c|\psi > \quad (33)$$

with all rays in the Hilbert space represents all possible state vectors in the Projective Hilbert space we could define "density-matrix operator" $\rho(t)$ as

$$\rho(t) = |\psi(t) >< \psi(t)| \quad (34)$$

this operator in compatible to Projective Hilbert Space because it has own phase information will vanishes. Now our state vector evolves with Schrödinger equation as

$$|\psi(T) >= e^{i\phi}|\psi(0) > \quad (35)$$

it seems final state bonding to initial state with a phase factor.$|\psi(t) >$state vector has T period in P . It is easy to understand with these definitions, evolution is drawing an arbitrary C which is cycle's trajectory in the Hilbert space. However, this cycle's projection gives$\hat{C}$ closed cycle and it is defined at :

$$\hat{C} = \Pi(C) \quad (36)$$

In the Projective Hilbert Space , $\hat{C} = \Pi(C)$ which has single value vector$|\xi(t) >$there will be

$$|\xi(T) >= |\xi(0) > \quad (37)$$

$$|\psi(t) >= e^{if(t)}|\xi(t) > \quad (38)$$

Now we assume that vector obtained by multiplying with appropriate $f(t)$ complex factor in the Hilbert space. For the determined phase factor it is chosen as $f(t) - f(0) = \phi$

$$i\hbar \frac{d}{dt} |\psi(t) >= H|\psi(t) > \quad (39)$$

$$i\hbar \frac{d}{dt} \{e^{if(t)} |\xi(t) >\} = He^{if(t)} |\xi(t) > \quad (40)$$

$$i\hbar \{\frac{d}{dt} e^{if(t)}\} |\xi(t) > + i\hbar e^{if(t)} \frac{d}{dt} |\xi(t) > = He^{if(t)} |\xi(t) > \quad (41)$$

$$i\hbar (i\dot{f}(t) e^{if(t)} |\xi(t) >) + i\hbar e^{if(t)} \frac{d}{dt} |\xi(t) >= \quad (42)$$

$$-\hbar \dot{f}(t) |\xi(t) > + i\hbar e^{if(t)} \frac{d}{dt} |\xi(t) >= \quad (43)$$

$$i\hbar \frac{d}{dt} |\xi(t) >= (H + \hbar \dot{f}(t)) |\xi(t) > \quad (44)$$

derive the equation easily,

$$i\hbar \frac{d}{dt} |\xi(t) >= H|\xi(t) > + \hbar \dot{f}(t)) |\xi(t) > \quad (45)$$

$$i\hbar \frac{d}{dt} |\xi(t) >= H|\xi(t) > + \hbar \dot{f}(t)) |\xi(t) > \quad (46)$$

than equation is multiplied by left with$< \xi(t)|$;

$$i\hbar < \xi(t) \left| \frac{d}{dt} \right| \xi(t) >= < \xi(t)|H|\xi(t) > + \hbar < \xi(t)| \frac{df(t)}{dt} |\xi(t) > \quad (47)$$

and integrate by t ,

$$\int_0^T \hbar \frac{d}{dt} f(t) dt = = -\int_0^T \{ < \xi(t)|H|\xi(t) > - i\hbar < \xi(t)| \frac{d}{dt} |\xi(t) > \} dt \quad (48)$$

$$\int_0^T \frac{d}{dt} f(t) dt = -\frac{1}{\hbar} \int_0^T < \xi(t)|H|\xi(t) > dt + i \int < \xi(t)| \frac{d}{dt} |\xi(t) > dt \quad (49)$$

Now LHS of this equation becomes $f(t) - f(0)$ with fundamental theory of mathematics. Well we assumed before$f(t) - f(0) = \phi$;

$$f(t) - f(0) = -\frac{1}{\hbar} \int_0^T < \xi(t)|H|\xi(t) > dt + i \oint_c < \xi(t)| \frac{d}{dt} |\xi(t) > dt \quad (50)$$

If we look more carefully, first term of the RHS as we mentioned (30) , dynamical phase :

$$\theta(T) = -\frac{1}{\hbar} \int_0^T < \xi(t)|H|\xi(t) > dt = -\frac{1}{\hbar} \int_0^T < \psi(t)|H(t)|\psi(t) > dt \quad (51)$$

Finally, last term of the RHS, knows Aharonov-Anandan's phase and it forms with dynamical one to total phase of the system.

$$\beta = i \oint_c < \xi(t)| \frac{d}{dt} |\xi(t) > dt \quad (52)$$

VI. Discussion

The most important point for Aharonov-Anandan's phase belong to our description is opposite of Berry's phase. We did not use adiabatic approximation which phase only depends evolution's cyclicity. Furthermore, it is independent Hamiltonian's eigenstates. Projective Hilbert Space gives us the dependence of arbitrary state vectors. Topological phases as known as non-integrable phases. But it is easy to see original equation (27) and (52) say the same results, in fact these are structurally identical. Then the question arises: Are the geometric phases and topological phase same ? To be honest answer is no. Aharonov-Anandan's phase demonstrates more general features than Berry's phase. Projective Hilbert Space is a trick in differential geometry for phycisists. Adiabatic condition says system evolves slowly. Projective Hilbert Space already does not need this kind of approximation because of as we define in section 2 . System's motion along the cyclic path if we want to final and initial points are not coincident, while we projection the cycle of Hilbert space, it has already final and initial points overlapped on the projective Hilbert space. Namely states will be same like adiabatic approximation. To tell the truth this mathematical trick can be used for all kind of geometrical phases.

VII. Referances

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