New Concepts of Alpha-Chaotic Maps

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***Abstract*—In this work, the definitions of alpha-type chaos, alpha-type exact chaos, topologically alpha-type mixing chaos, and weak alpha -type mixing chaos are introduced and extended to topological spaces. This paper proves that these chaotic properties are all preserved under αr -conjugation. We have the following relationships: α -type exact chaos⇒ α -type mixing chaos ⇒ weak α -type mixing chaos ⇒ α -type chaos which implies chaos**

***Keywords*— alpha-type chaos, alpha-type exact chaos, topologically alpha-type mixing chaos, and weak alpha -type mixing chaos.**

I. INTRODUCTION

In this paper, new type of chaotic map is introduced and studied called alpha-chaotic. This is intended as a survey article on some chaos type of a discrete system given by α -irresolute self-map of a topological space without isolated point. On one hand this subject introduces postgraduate students to the study of new types of chaotic and exact maps, it gives an overview of results on the topic, but, on the other hand, this study covers some of the recent developments. I introduced and defined a new type of chaotic and investigate some of its properties. Relationships with some other type of exact and chaotic maps are given (see for more knowledge [5]). I list some relevant properties of the α -type chaotic map. Further, I introduced the notions of α -exact mapping. I have shown that every alpha chaotic map is a chaotic map but the converse not necessarily true, and that every α-mixing map is a mixing map, but the converse not necessarily true. Further, *I studied a new class of α -type exact chaos and weakly α -mixing chaos. The existence of chaotic behavior in deterministic systems has attracted researchers for many years. In engineering applications such as biological engineering, and chaos control, chaoticity of a topological system is an important subject for investigation. The definitions of α -type chaos, α -type exact chaos, topologically α -type mixing chaos, and weak α -type mixing chaos are extended to topological spaces. This paper proves that these chaotic properties are all preserved under αr -conjugation. We have the following relationships: α -type exact chaos⇒α -type mixing chaos ⇒ weak α -type mixing chaos ⇒α -type chaos which implies chaos*

II. PRELIMINARIES AND DEFINITIONS

Definition 2.1. [1] A map is called α-irresolute if for every α-open set H of Y, then the inverse image is α-open in X.

Definition 2.2.[2] Let (X, τ) be a topological space, $f: X \to X$ be α-irresolute map, then the map f is called α-transitive if for every pair of non-empty α-open sets U and V in X there is a positive integer n such that $f^n(U) \cap V$ is not empty.

Definition 2.3

(1) A point $x \in X$ is α-recurrent if, for every α-open set U containing x, infinitely many $n \in \mathbf{N}$ satisfy $f^n(x) \in U$. Thus, recurrence means that, under the iteration of f, the point x returns to each of its α-neighborhoods infinitely often.

(2) Two points $x, y \in X$ are α-proximal if, for every α-neighborhood F of the diagonal $X \times X$, infinitely many $n \in \mathbf{N}$ satisfy $(f^n(x), f^n(y) \in F$.

Note that every α-proximal is proximal but the convers is not true.

Definition 2.4.

(1) Let (X, τ) be a topological space, $f: X \to X$ be α-irresolute map, then the map f is called topologically α-mixing if, given any nonempty α-open subsets $U, V \subseteq X$ there exist $N \geq 1$ such that $f^m(U) \cap V$ is not empty for all m>N. Clearly if *f* is topologically α-mixing then it is also α-transitive but not conversely.

(2) The map$f$ is α-exact if, for every nonempty α-open set $U \subset X$, there exists some $m \in \mathbf{N}$ such that $f^m(U)$ is the whole space.

(3) The map f is (topological) α-type transitive (resp., α-mixing) if for any two nonempty α- open sets U, $V \subset X$, there exists some $n \in \mathbf{N}$ such that $f^n(U) \cap V$ is not empty set (resp., $f^m(U) \cap V \neq \phi$. for all $m \geq n$).

(4) The map f is weak α-mixing, if the product $f \times f$ is α–type transitive on $X \times X$.

(5) A α-mixing map $f : X \to X$ is pure α-mixing if and only if there exists α- open set $U \subset X$ such that $f^n(U)$ is not the whole space for all $n \geq 0$.

(6) The map $f : X \to X$ is α-type chaotic if f is α-transitive on X and the set of periodic points of f is α-dense in X. For more knowlage see [3] and [4]

(7) The map f is called α-type exact chaos (resp., α-type mixing chaos and weakly α-type mixing chaos) if f is α-exact (resp., α-mixing and weakly α-mixing) and α-type chaotic map on the space X.

III. Topological Alpha-Type Transitive Maps and Topological αr-Conjugacy

Topologically α-type transitive maps are defined and introduced [2] and α –type minimal maps [2]. I will study some of their properties and prove some results associated with these new definitions. I investigate some properties and characterizations of such maps. Let (X, f) be a topological dynamical system. A map $f : X \to X$ is called alpha- chaotic, if it is topological α- transitive and, its periodic points are α-dense in X [3], i.e. every non-empty α-open subset of X contains a periodic point. (A point $p \in X$ is called periodic if there exists $n \geq 1$ *with* $f^n(p) = p$. The set of all periodic points of f denoted by $Per(f)$.

Definition 3.1 Recall that a subset A of a space X is called α-dense in X if $Cl_\alpha(A) = X$, we can define equivalent definition that a subset A is said to be α-dense if for any x in X either x in A or it is a α-limit point for A.

Remark 3.2 any α-dense subset in X intersects any α-open set in X.

Definition 3.3 Recall that a subset A of a topological space (X, τ) is said to be nowhere α-dense, if its α-closure has an empty α-interior, that is, $\operatorname{int}_\alpha(Cl_\alpha(A)) = \phi$.

Definition 3.4 if for $x \in X$ the set $\{f^n(x):\ n \in \mathbf{N}\}$ is α-dense in X then x is said to have α-dense orbit. If there exists such an $x \in X$, then f is said to have α-dense orbit.

Definition 3.5. A function $f : X \to X$ is called αr-homeomorphism if f is α-irresolute bijective and $f^{-1} : X \to X$ is α-irresolute.

Definition 3.6 Two topological systems $f : X \to X$, $x_{n+1} = f(x_n)$ and $g : Y \to Y$, $y_{n+1} = g(y_n)$ are said to be topologically αr -conjugate if there is αr -homeomorphism $h : X \to Y$ such that $h \circ f = g \circ h$ *(i.e. h(f(x)) = g(h(x)))*. We will call h a topological αr -conjugacy.

Then I have proved some of the following statements:

1. $h^{-1} : Y \to X$ is a topological αr-conjugacy.
2. $h \circ f^n = g^n \circ h \quad \forall n \in \mathbf{N}$.
3. $p \in X$ is a periodic point of f if and only if h(p) is a periodic point of g .
4. If p is a periodic point of the map f with stable set $W_f(p)$, then the stable set of h(p) is $h(W_f(p))$.
5. The set of all periodic points of f are dense in X if and only if the set of all periodic points of g are dense in Y.
6. The map f is α-type chaotic if and only if g is α-type chaotic
7. *The map f is weakly α-mixing if and only if g is α-weakly mixing.*

Remark 3.7

If $\{x_0, x_1, x_2, \ldots\}$ denotes an orbit of $x_{n+1} = f(x_n)$ then $\{ y_0 = h(x_0),\ y_1 = h(x_1),\ y_2 = h(x_2), .. \}$ yields an orbit of $y_{n+1} = g(y_n)$. In particular, h maps periodic orbits of f onto periodic orbits of g . orbit of g since $y_{n+1} = h(x_{n+1}) = h(f(x_n)) = g(h(x_n) = g(y_n)$, i.e. f and g have the same kind of dynamics.

Proposition 3.8 if the two maps $f : X \to X$ *and* $g : Y \to Y$ are topologically αr -conjugate. Then

(1) The map f is α-exact if and only if g is α-exact

(2) f is weakly α-mixing if and only if g is weakly α-mixing.

(3) f is α-type chaotic on X if and only if g is α-type chaotic in Y.

Proof (1) $\Rightarrow$) given any nonempty α-open set V in Y, take $U = h^{-1}(V)$. clearly, U is a nonempty α-open set in X. Since f is α-exact, there exists an $n \in \mathbf{N}$ such that $f^n(U) = X$. Noting that f and g are αr-conjugate maps and that h is a αr-homeomorphism, it follows that

$g^n(V) = g^n(h(U)) = h(f^n(U)) = h(X) = Y$ .Since V is an arbitrary α-open set, this implies that g is α-exact.

It can be proved similarly.

Proof (2)

For any two nonempty α-open sets $U,V \subset Y\times Y$, according to the construction of product topology, it follows that there exist nonempty α-open sets $U_1,U_2,V_1,V_2 \subset Y$ such that $U_1\times U_2 \subset U$ and $V_1\times V_2 \subset V$. Since f is weakly α-mixing, there exists $n \in$ N such that $(f\times f)^n(h^{-1}(U_1)\times h^{-1}(U_2))\cap(h^{-1}(V_1)\times h^{-1}(V_2))\neq\phi$, This implies that

$$(g\times g)^n(U)\cap V \supset (g\times g)^n(U_1\times U_2)\cap(V_1\times V_2)$$
$$=(g\times g)^n[(h\times h)(h^{-1}(U_1)\times h^{-1}(U_2))]\cap[(h\times h)(h^{-1}(V_1)\times h^{-1}(V_2))]$$
$$=(h\times h)(f^n(h^{-1}(U_1))\times f^n(h^{-1}(U_2)))\cap[(h\times h)(h^{-1}(V_1)\times h^{-1}(V_2))]$$
$$\supset (h\times h)([f^n(h^{-1}(U_1))\times f^n(h^{-1}(U_2))]\cap(h^{-1}(V_1)\times h^{-1}(V_2)))\neq\phi$$

Therefore, g is weakly α-mixing map on Y.

It can be proved similarly.

Proof (3)

Necessity. Similarly to the proof of Proposition 3.11 part (1), it can be verified that g is α-type transitive on Y, as f is α-type transitive on X. According to the definition of periodic points, it is easy to check that $Per(g)\supset h(Per(f))$. Applying this, we have $Cl_\alpha(Per(g))\supset Cl_\alpha[h(Per(f))]\supset h[Cl_\alpha(Per(f)]=h(X)=Y$. This implies that $Cl_\alpha[Per(g)]=Y$. Therefore g is α-type chaotic map on Y. Sufficiency can be proved similarly.

IV. Alpha-Chaos in Product Topological Spaces

Given two maps $f:X\to X$ and $g:Y\to Y$ on topological spaces X and Y. resp., consider their product $f\times g: X\times Y\to X\times Y$, $(f\times g)(x,y)=(f(x),g(y))$, with product topology on X × Y

Lemma 4.1 Let (X,f), (Y,g) be topological systems. The set of periodic points of $f\times g$ is α-dense in the product space $X\times Y$ if and only if, for both of f and g, the sets of periodic points in X and Y are α-dense in X, respectively Y.

Proof: Assume that the set of periodic points of f is α-dense in X (i.e. $Cl_\alpha(Per(f))=X$) and the set of periodic points of g is α-dense in Y (i.e. $Cl_\alpha(Per(g))=Y$). We have to prove that the set of periodic points of $f\times g$ is α-dense in $X\times Y$. Let $W\subset X\times Y$ be any non-empty α-open set. Then there exist non-empty α-open sets $U\subset X$ and $V\subset Y$ with $U\times V\subset W$. By assumption, there exists a point $x\in U$ such that $f^n(x)=x$ with $n\geq 1$. Similarly, there exists $y\in V$ such that $g^m(y)=y$ with $m\geq 1$. For $p=(x,y)\in W$ and k = mn we get

$$(f\times g)^k(p)=(f\times g)^k(x,y)=((f^k(x),g^k(y))=(x,y)=p$$

Therefore W contains a periodic point and thus the set of periodic points of $f\times g$ is α-dense in $X\times Y$.

Conversely let $U\subset X$ and $V\subset Y$ be non-empty α-open subsets. Then $U\times V$ is a non-empty α-open subset of $X\times Y$. As the set of the periodic points of $f\times g$ is α-dense in $X\times Y$, there exists a point $p=(x,y)\in U\times V$ such that $(f\times g)^n(x,y)=((f^n(x),g^n(y))=(x,y)$ for some n. From the last equality we obtain $f^n(x)=x$ for $x\in U$ and $g^n(y)=y$ for $y\in V$.

The α-denseness of periodic points carries over from factors to products. But, topological α-type transitivity may not carry over to products. The converse of this situation is however true:

Lemma 4.2 Let $f:X\to X$ and $g:Y\to Y$ be maps and assume that the product $f\times g$ is α-transitive on $X\times Y$. Then the maps f and g are both topological α- transitive on X and Y respectively.

Proof. We have to prove the α-type transitivity of f; the α-type transitivity of g can be proved similarly. Let U_1,V_1 be non-empty α-open sets in X. Then the sets $U=U_1\times Y$ and $V=V_1\times Y$ are α-open in $X\times Y$. As $f\times g$ is α- transitive, there exists a positive integer n such that $(f\ x\ g)^n(U)\cap V\neq\phi$. From the equalities:

$$(f\times g)^n(U)\cap V=[f^n(U_1)\times g^n(Y)]\cap[V_1\times Y]$$
$$=[f^n(U_1)\cap V_1]\times[g^n(Y)\cap Y]\neq\phi,$$

so $f^n(U_1)\cap V_1\neq\phi$. Thus f is topological α-transitive.

Definition 4.3[7] Let $f:X\to X$ be a map on the topological space X. If for every nonempty α-open subsets $U,V\subset X$ there exists a positive integer n_0 such that for every $n\geq n_0$, $f^n(U,)\cap V\neq\phi$ then f is called topologically α- mixing.

It is clear that topological α-mixing implies topological α- transitive which implies topologically transitive. There is an even stronger notion that implies topological α-type mixing.

Definition 4.4 Let $f:X\to X$ be a map on the space X. If for every nonempty α-open subset $U\subset X$ there exists $n_0\in\mathbf{N}\setminus\{0\}$ such that for every

$n \geq n_0$, $f^n(U) = X$, then f is called locally α-type eventually onto.

Lemma 4.5 The product of two topologically α-mixing maps is topologically α-mixing.

Proof. Let (X, f) , (Y, g) be topological dynamical systems and *f*, *g* be topologically α- mixing maps. Given $W_1, W_2 \subset X \times Y,$ there exists α-open sets $U_1, U_2 \subset X$ *and* $V_1, V_2 \subset Y$, such that $U_1 \times V_1 \subset W_1$ and $U_2 \times V_2 \subset W_2$. By assumption there exist n_1 *and* n_2 such that

$$f^k(U_1) \cap U_2 \neq \phi \text{ for } n \geq n_1 \text{ and } g^k(V_1) \cap V_2 \neq \phi$$

$$\text{for } n \geq n_{12}.$$

$$\text{For } n \geq n_0 = \max\{n_1, n_2\}$$

we get

$$[(f\, x\, g)^k (U_1\ x\ V_1)] \cap (U_2\ x\ V_2) = [f^k(U_1)\ x\, g^k(V_1)] \cap (U_2\ x\ V_2)$$
$$= [f^k(U_1) \cap U_2]\ x\ [g^k(V_1) \cap V_2] \neq \phi$$

Which means that *f* × *g* is topologically α- mixing.

We give some sufficient conditions for a product map to be α-type chaotic.

Theorem 4.6 Let $f : X \to X$ *and* $g : Y \to Y$ be α-type chaotic and topologically α- mixing maps on topological spaces X and Y. Then $f \times g : X \times Y \to X \times Y$ is α-type chaotic.

Proof.: The map $f \times g$ has α-dense periodic points by Lemma 4.1 and it is topologically α- mixing by Lemma 4.5 and hence topologically α-transitive. Thus the two conditions of α-type chaos are satisfied.

V. Conclusions

There are the following results:

Proposition5.1

Topologically α-type exact chaos⇒ α- mixing chaos ⇒ weak α- mixing chaos ⇒ α-type chaos

Proposition5.2

If $f : X \to X$ *and* $g : Y \to Y$ are topologically αr -conjugate . Then

(1) The map f is α-exact if and only if g is α-exact

(2) f is weakly α-mixing if and only if g is weakly α-mixing.

(3) f is α-type chaotic on X if and only if g is α-type chaotic in Y.

Lemma 5.3 Let (X, f) , (Y, g) be topological systems. The set of periodic points of $f \times g$ is α-dense in $X \times Y$ if and only if, for both of f and g , the sets of periodic points in X and Y are α-dense in X, respectively Y.

Lemma 5.4 Let $f : X \to X$ *and* $g : Y \to Y$ be maps and assume that the product $f \times g$ is topological α- transitive on $X \times Y$. Then the maps f and g are both topological α-transitive on X and Y respectively.

Lemma 5.5 The product of two topologically α-mixing maps is topologically α- mixing.

Theorem 5.6 Let $f : X \to X$ *and* $g : Y \to Y$ be α-type chaotic and topologically α-mixing maps on a spaces X and Y. Then $f \times g : X \times Y \to X \times Y$ is α-type chaotic.

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