Determining a kink angle of a crack in mixed mode fracture using maximum energy release rate, SED and MTS criteria

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***Abstract*—This paper deals with the applicability of the maximum energy release rate criterion, of the minimum strain energy density (SED) criterion and of the maximum tangential stress (MTS) criterion, the purpose of which is to define the direction in which the crack kinks at different load modes, which leads to different opening of the crack tip. In addition to this, the suitability of the energy release rate method, using the J integral, is dealt with, used to determine stress intensity factors K_I and K_{II} for an initial crack. The suitability of the method is determined by comparing the results acquired with this method with the results obtained with the crack opening method. Based on FEM analysis, it is shown that the results obtained with any of the three methods used for determining the direction in which the crack kinks are comparable when tensile stresses at the crack tip are greater than shear stresses. If shear stresses prevail, the results differ most significantly when using maximum energy release rate criterion and the energy release rate method using the J integral. The differences increase with increased stress intensity factor K_{II}. When using the minimum strain energy density criterion, special attention must be paid to the correct use of its local minimum. The results when using energy release rate method with the J integral to determine stress intensity factors K_I in K_{II} are similar to the results obtained with the crack opening method; however, special attention should be paid to determining the results when the impact of shear stresses prevails near the crack tip.**

Keywords—Kink angle, Fracture mechanics, Stress intensity factor, Numerical simulations

I. INTRODUCTION

Fracture mechanics is widely used for analyzing crack propagation under given load. For this purpose it is necessary to know stress distribution around the crack tip. Due to stress concentration caused by crack tip, stresses can exceed yield stress. When a plastic zone is small compared to the crack length, a linear fracture mechanics can be used, where stresses can be described with William's equation. William's equation includes first order constants and higher order constants. First order constants describe magnitude of crack tip opening mode. Under general loading a crack tip can open in mixed-mode with tensile mode (described with stress intensity factor K_I), in-plane shear mode (described with stress intensity factor K_{II}) and out-of-plane shear mode (described with stress intensity factor K_{III}). Under mixed-mode loading, a pre-crack kinks and further propagates under mode I. Usually stress intensity factors are determined before the kink angle (crack propagation angle) is determined. Many substitution and energy methods exist for determination of these factors [1, 2, 3]. Normally, a kink angle under mixed-mode opening can be determined with the maximum energy release rate (G) [4] criterion, the minimum strain energy density (SED) criterion [5] and the maximum tangential stress (MTS) criterion [6] when stress intensity factors are known. In this paper stress intensity factors are determined using displacement correlation method and energy release rate by using the complex J integral, while the kink angle is determined using maximum energy release rate, SED and MTS criteria.

II. MIXED-MODE FRACTURE CRITERIA

A. The maximum energy release rate (G) criterion

The Virtual Crack Extension method (VCE), as proposed by Hellen [7], is based on the criteria of released strain energy dV per crack extension da

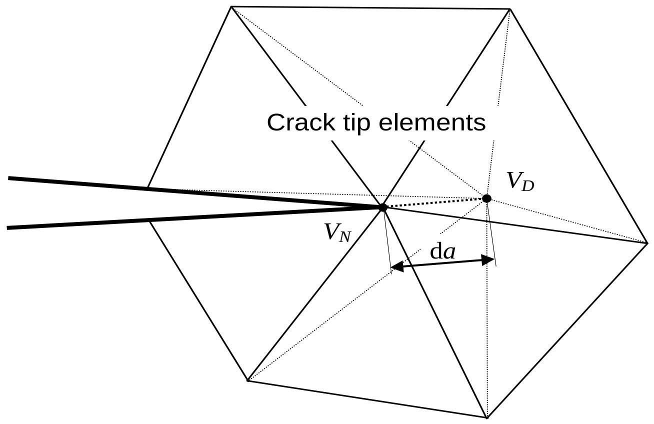
$$G=-\frac{\mathrm{d}V}{\mathrm{d}a} \tag{1}$$

which serves as a basis for determining the combined stress intensity factor around the crack tip

$$K=\begin{cases}\sqrt{E\cdot G} & \text{plane stress;}\\ \sqrt{\dfrac{E\cdot G}{(1-\nu^2)}} & \text{plane strain.}\end{cases} \tag{2}$$

with K further being equal to

$$K=\begin{cases}\sqrt{\left(K_I^2+K_{II}^2\right)+(1+\nu)\cdot K_{III}^2} & \text{plane stress;}\\ \sqrt{\left(K_I^2+K_{II}^2\right)\cdot(1-\nu^2)+(1+\nu)\cdot K_{III}^2} & \text{plane strain.}\end{cases} \tag{3}$$



Initial and extended crack tip configuration

If V_C is the strain energy obtained for all degrees of freedom not present in the crack tip elements, and V_N is the energy in the crack tip elements when the tip is not extended, while V_D is the energy in these elements when the tip is extended (see *Fig. 1*), then the total energies of the initial and altered bodies, V_N^T and V_D^T, respectively, are equal to

$$V_N^T = V_C + V_N \quad \text{and} \quad V_D^T = V_C + V_D \tag{4}$$

Thus, for a virtual crack extension δa it follows

$$\frac{\mathrm{d}V}{\mathrm{d}a} = \frac{V_D^T - V_N^T}{\delta a} = \frac{V_D - V_N}{\delta a} \tag{5}$$

which is clearly independent of VC. It follows that only strain energies V_N and V_D in the crack tip elements need to be calculated for every possible crack extension. This results in a very efficient method for determining the instantaneous energy release rate and, thus, the stress intensity factor for any given crack extension.

Following the same argument, the energy release rate G and the stress intensity factor K can be easily determined for several different possible crack extension directions for a cluster of points on an arc around the initial crack tip with radius da, see *Fig. 2a*

$$\left(\frac{\mathrm{d}V}{\mathrm{d}a}\right)^j = \frac{V_D^j - V_N}{\mathrm{d}a^j}. \tag{6}$$

Assuming the validity of the maximum energy release criterion, the crack will propagate in the direction corresponding to the maximum value of $(\mathrm{d}V/\mathrm{d}a)^j$, *i.e.* in the direction of the maximum stress intensity factor K^j. Computational procedure is based on incremental crack extensions, where the size of the crack increment is prescribed in advance. The virtual crack increment should not exceed 1/3 of the size of crack tip finite elements. For each crack extension increment the stress intensity factor is determined in several different possible crack propagation directions and the crack is actually extended in the direction of the maximum stress intensity factor, which requires local remeshing around the new crack tip. The incremental procedure is repeated until full fracture occurs or until the stress intensity factor reaches the critical value K_c, when full fracture is imminent. For improved numerical results, special fracture finite elements that exhibit $r^{-1/2}$ stress singularity are used in the first circle of elements around a crack tip, with ordinary elements elsewhere, *Fig. 2b*.

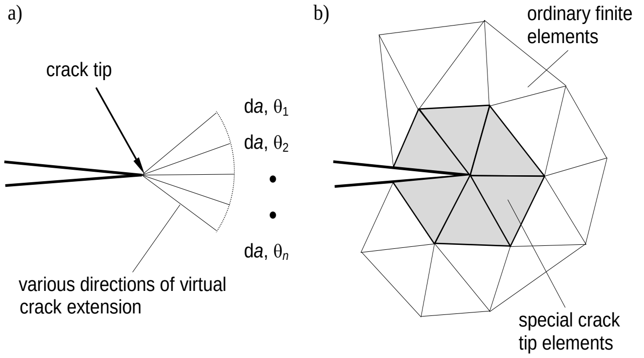


Fig. 2. *Virtual crack extensions of the crack tip*

Following the above procedure, one can numerically determine the functional relationship $K=f(a)$ and the critical crack length a_c at $K=K_c$ from the computed values of K at discrete crack extensions a.

Hellen and Blackburn [8] also showed that in case of maximum energy release rate the crack propagation angle can be determined using:

$$\theta_0 = \arctan\left(\frac{2K_I K_{II}}{K_I^2 + K_{II}^2}\right) \tag{7}$$

This equation can be used for determining crack propagation angle up to 45°. Therefore, special care must be taken when shear mode is dominant around the crack tip.

B. The minimum Strain Energy Density (SED) criterion

The SED fracture criterion locally focuses on the continuum element ahead of the crack and is based on the notion of weakness or severity experienced by the local material. Failure occurs when a critical amount of strain energy dW is accumulated within the element volume dV and the crack is then advanced incrementally in the corresponding direction [5, 9]. The strain energy density function dW/dV is assumed to have the form

$$\frac{dW}{dV} = \frac{S}{r} \tag{8}$$

where S is the strain energy density factor and r is the distance from the crack tip. Therefore, the minimum of the strain energy density factor S_{min} around the crack tip determines the direction of crack propagation.

The strain energy density can be determined directly from the relationship

$$\frac{dW}{dV} = \frac{1}{2}\sigma^T \varepsilon \tag{9}$$

which results in the following expression for the coordinate stresses evaluated at integration points of finite elements around the crack tip

$$\frac{dW}{dV}=\frac{1}{4\mu}\left[\frac{\kappa+1}{4}\left(\sigma_x+\sigma_y\right)^2-2\left(\sigma_x\sigma_y-\tau_{xy}^2\right)\right] \quad 10)$$

where μ is the shear modulus such that $E = 2\mu(1+\nu)$ with E being the Young's modulus.

The position of integration points actually defines the corresponding angle of calculated strain energy density and strain energy density factor S around the crack tip as can be observed from *Fig. 3*.

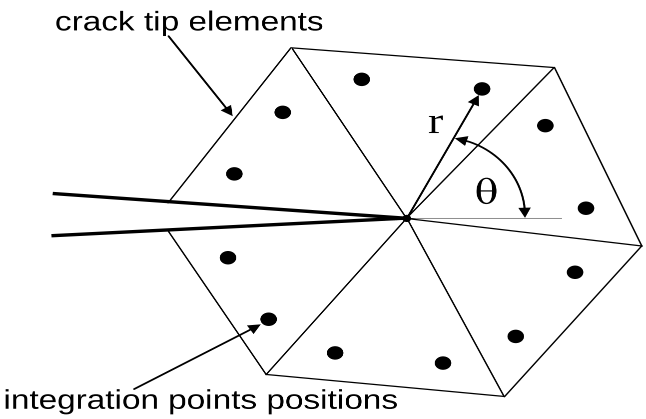


Fig. 3. *Layer of integration points around crack tip*

Discrete values for S are then fitted with the approximation function, which enables a simple algorithm for determining the local minimum.

The strain density function has several minimums around the crack tip, where the global minimum is not necessarily the true solution of the problem as can be observed from *Fig. 4*.

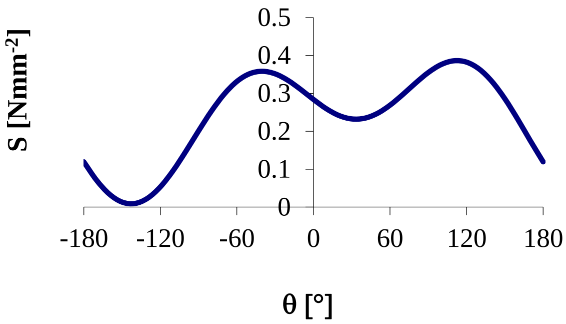


Fig. 4. *Distribution of the strain density function*

The minimum of strain density function S_{min} can be found numerically by incremental search for a local minimum of function S for different possible crack extension directions θ_i in the range ±π around the crack tip angle.

Crack propagation direction determined from coordinate stresses show very good correlation with experimental data. The accuracy of the method depends only on the accuracy of finite element method, which can be improved by using finer mesh around the crack tip.

Alternatively, the strain energy density factor S can be evaluated for 2-dimensional linear elastic crack problems from the stress intensity factors in the following manner

$$S=a_{11}K_{\mathrm{I}}^2+2a_{12}K_{\mathrm{I}}K_{\mathrm{II}}+a_{22}K_{\mathrm{II}}^2 \quad (11)$$

where the coefficients a_{ij} (i, j = 1, 2) are for 2-dimensional case equal to

$$\begin{aligned} a_{11}&=\frac{1}{16\mu}\left(1+\cos\theta\right)\left(\kappa-\cos\theta\right)\\ a_{12}&=\frac{1}{16\mu}\sin\theta\left[2\cos\theta-\left(\kappa-1\right)\right]\\ a_{22}&=\frac{1}{16\mu}\left[\left(\kappa+1\right)\left(1-\cos\theta\right)+\left(1+\cos\theta\right)\left(3\cos\theta-1\right)\right]\end{aligned} \quad (12)$$

The angle θ, which is measured from the local crack direction, corresponding to the minimum of S can be determined from the first and the second derivative of Eq. (11) in respect to θ.

$$\left[2\cos\theta-\left(\kappa-1\right)\right]\sin\theta K_I^2+\left[2\cos2\theta-\left(\kappa-1\right)\cos\theta\right]K_IK_{II}+\left[\left(\kappa-1-6\cos\theta\right)\sin\theta\right]K_{II}^2=0 \quad (13)$$

$$\left[2\cos2\theta-\left(\kappa-1\right)\cos\theta\right]K_I^2+2\left[\left(\kappa-1\right)\sin\theta-4\sin2\theta\right]K_IK_{II}+\left[\left(\kappa-1\right)\cos\theta-6\cos2\theta\right]K_{II}^2>0 \quad (14)$$

This approach strongly depends on correct evaluation of the stress intensity factors. To predict the crack growth with the use of equations (11), (13) and (14), the stress intensity factors K_{I} and K_{II} have to be known.

Stress intensity factors were determined using the displacement correlation method and energy release rate using the J integral. In case of plane strain case, stress intensity factors can be determined using the displacement correlation method, with the following equations

$$K_{\mathrm{I}}=\frac{E}{8(1-\nu^2)}\cdot\sqrt{\frac{2\pi}{r}}\cdot\left(v_1-v_2\right) \quad (15)$$

$$K_{\mathrm{II}}=\frac{E}{8(1-\nu^2)}\cdot\sqrt{\frac{2\pi}{r}}\cdot\left(u_1-u_2\right) \quad (16)$$

where u_i and v_i (i = 1,2) are displacements at nodes near the crack tip in x and y directions, respectively, and r is a distance of the node on the crack surface from the crack tip, see *Fig. 5*.

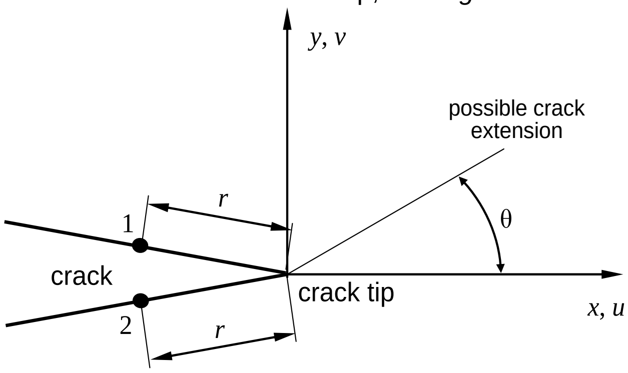


Fig. 5. *Displacement correlation method*

The corner nodes of special crack tip finite elements were chosen as points 1 and 2 on opposite crack faces.

Hellen and Blackburn [8] proposed a method for determining stress intensity factors using the *J* integral values in parallel (J_1) and perpendicular (J_2) direction of the crack, as shown in *Fig. 6*.

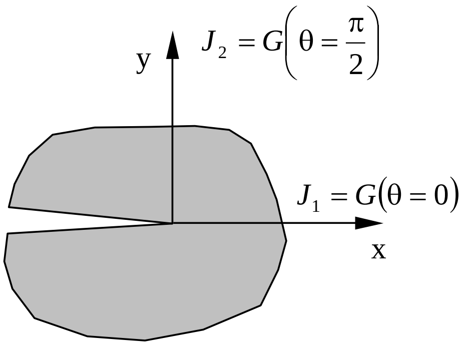


Fig. 6. *Definition of J_1 and J_2*

In linear elastic fracture mechanics, the *J* integral is equal to the energy release rate *G*, which can be calculated, using the VCE method. In this case stress intensity factors can be calculated from

$$K_{\mathrm{I}}^2 = \frac{2E}{(1+\nu)(1+\kappa)}\left(J_1 + \sqrt{J_1^2 - J_2^2}\right) \quad (17)$$

$$K_{\mathrm{II}}^2 = \frac{2E}{(1+\nu)(1+\kappa)}\left(J_1 - \sqrt{J_1^2 - J_2^2}\right) \quad (18)$$

When this method is used, special care should be taken when shear mode is dominant around the crack tip. In such a case a sign before the square root must be changed to get $K_{\mathrm{II}} > K_{\mathrm{I}}$. A physical intuition is needed when a sign should be changed to get correct results for K_{I} and K_{II}.

C. The Maximum Tangential Stress (MTS) criterion

Erdogan and Sih [6] proposed a criterion for determining crack propagation angle using a direction, which is perpendicular to the direction of maximum tangential stress. Shear stress can be expressed in the polar coordinate system as:

$$\tau_{r\theta} = \frac{K_{\mathrm{I}}}{\sqrt{2\pi r}}\left(\frac{1}{4}\sin\frac{\theta}{2} + \frac{1}{4}\sin\frac{3\theta}{2}\right) + \frac{K_{\mathrm{II}}}{\sqrt{2\pi r}}\left(\frac{1}{4}\cos\frac{\theta}{2} + \frac{3}{4}\cos\frac{3\theta}{2}\right) \quad (19)$$

In the direction of maximum tangential stress the shear stress is equal to zero. Therefore, the equation for the shear stress can be expressed as:

$$K_{\mathrm{I}}\left(\frac{1}{4}\sin\frac{\theta}{2} + \frac{1}{4}\sin\frac{3\theta}{2}\right) + K_{\mathrm{II}}\left(\frac{1}{4}\cos\frac{\theta}{2} + \frac{3}{4}\cos\frac{3\theta}{2}\right) = 0 \quad (20)$$

or

$$\cos\frac{\theta}{2}\left[K_{\mathrm{I}}\sin\theta + K_{\mathrm{II}}(\cos\theta - 1)\right] = 0 \quad (21)$$

Solutions for the above equation are:

1. $\theta_0 = \pm\pi$
2. $K_{\mathrm{I}}\sin\theta + K_{\mathrm{II}}(3\cdot\cos\theta - 1) = 0$

The first solution has no physical meaning, because a crack cannot kink in the direction ± π. The second solution can be written in the explicit form as:

$$\theta_0 = -\arccos\left[\frac{3K_{\mathrm{II}}^2 + K_{\mathrm{I}}\sqrt{K_{\mathrm{I}}^2 + 8K_{\mathrm{II}}^2}}{K_{\mathrm{I}}^2 + 9K_{\mathrm{II}}^2}\right] \quad (22)$$

For pure mode I loading of a crack tip ($K_{\mathrm{I}} \neq 0$, $K_{\mathrm{II}} = 0$), a crack propagation angle is equal to 0, while for mode II ($K_{\mathrm{I}} = 0$, $K_{\mathrm{II}} \neq 0$), it is equal to -70,6°.

III. PRACTICAL APPLICATION OF THE COMPUTATIONAL MODEL

The determination of stress intensity factor and of crack propagation angle was evaluated, using the CTS specimen [10, 11] shown in Figure 3, with a 55 mm long crack. The CTS specimen is loaded with a static load of *F*=15 kN, where $\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3$ (*Fig. 7*). Mixed mode crack tip opening was simulated using different load cases between pure mode I (load angle is equal to 0°) and pure mode II (load angle is equal to 90°). Material properties were defined for an aluminum alloy with Young´s modulus E = 72400 MPa and Poisson´s ratio v = 0,33.

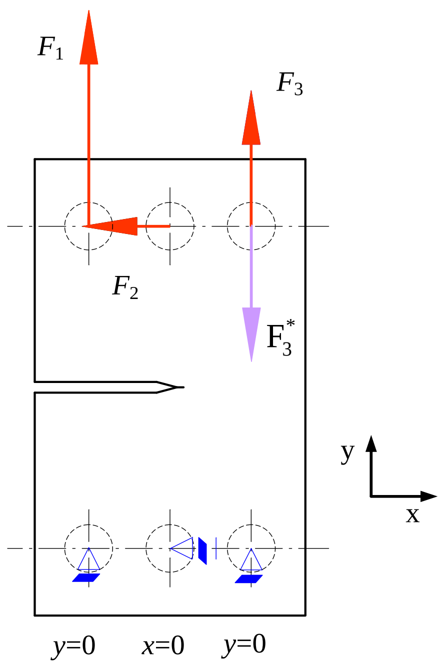


Fig. 7. *Boundary conditions for CTS specimen*

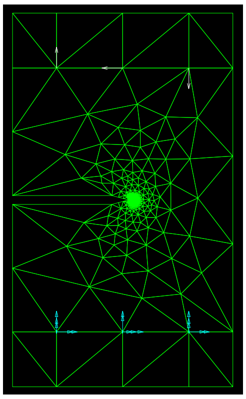


Fig. 8. *Discretization of CTS specimen using finite elements*

IV. Results

A. *Stress intensity factor*

When loading an initial fatigue crack, the crack kinks. Consequently, depending on the load angle α, various proportions of intensity factors K_I and K_{II} are obtained.

The *BERSAFE* [12] software package was used to calculate shift values; afterwards equations Eg.(17) and Eg.(18) were used to calculate the values of stress intensity factors K_I and K_{II}. In relation to the load angle of 75°, the stress intensity factor K_{II} prevails over the stress intensity factor K_I. In such a case it is required to change the sign that precedes the square root in equations Eg.(17) and Eg.(18).

With energy release rate method using the *J* integral, very good results for values K_I and K_{II} are obtained; however, for this method, physical intuition is required to determine K_I and K_{II}, which can be problematic when analysing crack propagation in very complex models.

TABLE I. Comparison of two methods for determination of stress intensity factor

α	DCM		J(G)	
	K_I	K_{II}	K_I	K_{II}
0°	542,7	-0,4	549,9	-0,2
15°	524,3	-57,0	531,3	-56,5
30°	470,0	-110,6	476,7	-109,2
45°	383,4	-156,2	389,8	-153,7
60°	271,3	-191,4	279,1	-186,2
75°	140,7	-213,7	138,3	-217,3
90°	0,1	-221,8	0,1	-222,6

B. *Kink angle*

In *Fig. 9*, a comparison of results for the kink angle θ_0 for various load angles α of an initial fatigue crack is presented.

The results match well up to the load angle α=60°. When the load angle is larger than 60°, the results obtained using the maximum energy release rate criterion deviate significantly from the results obtained on the basis of the remaining two criteria.

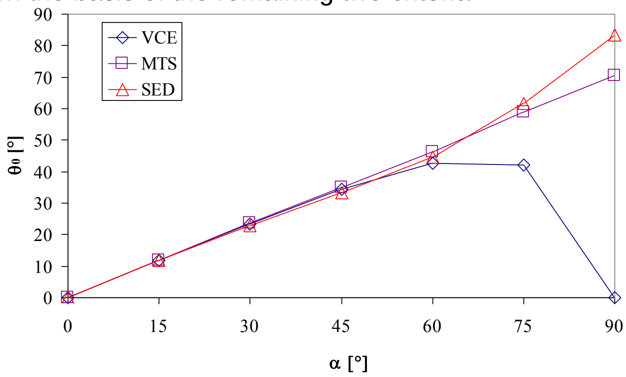


Fig. 9. *Comparison of θ_0 for different criteria*

The deviation of the results is largest in case of the pure loading mode II. Therefore, it can be concluded that the results obtained with the maximum energy release rate are not very accurate when shear stresses prevail over the tensile stresses near the crack tip.

If the results for the kink angle using the criteria of the minimum strain energy density and the test results [13, 14] are compared, it can be observed that the results match well for load angles α=30° and α=60°. When it comes to load angles α=75° and α=90°, the difference between the results for the kink angle is larger (approximately 5° and 11°). When it comes to further crack propagation, the test revealed a larger slope of the direction of crack propagation than numerical analysis.

On the basis of the comparison between numerical results and test results it can be concluded that, using the criteria of minimum strain energy density, the kink angle for load angles α up to 60° can be determined very reliably, whereas the results do not match so well when it comes to further crack propagation.

When using the criteria of minimum strain energy density, special attention must be paid when trying to find a solution for the kink angle θ_0. It is evident from *Fig. 10* that the global minimum for S_{min} is obtained at the angle of $\theta \approx -130°$ when the crack cannot propagate. Consequently, it is necessary to observe the local minimum for the value of the angle when the crack can physically propagate. In our analysis, that was the second minimum.

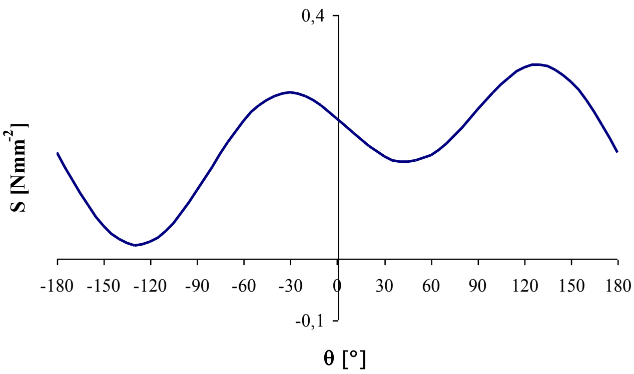
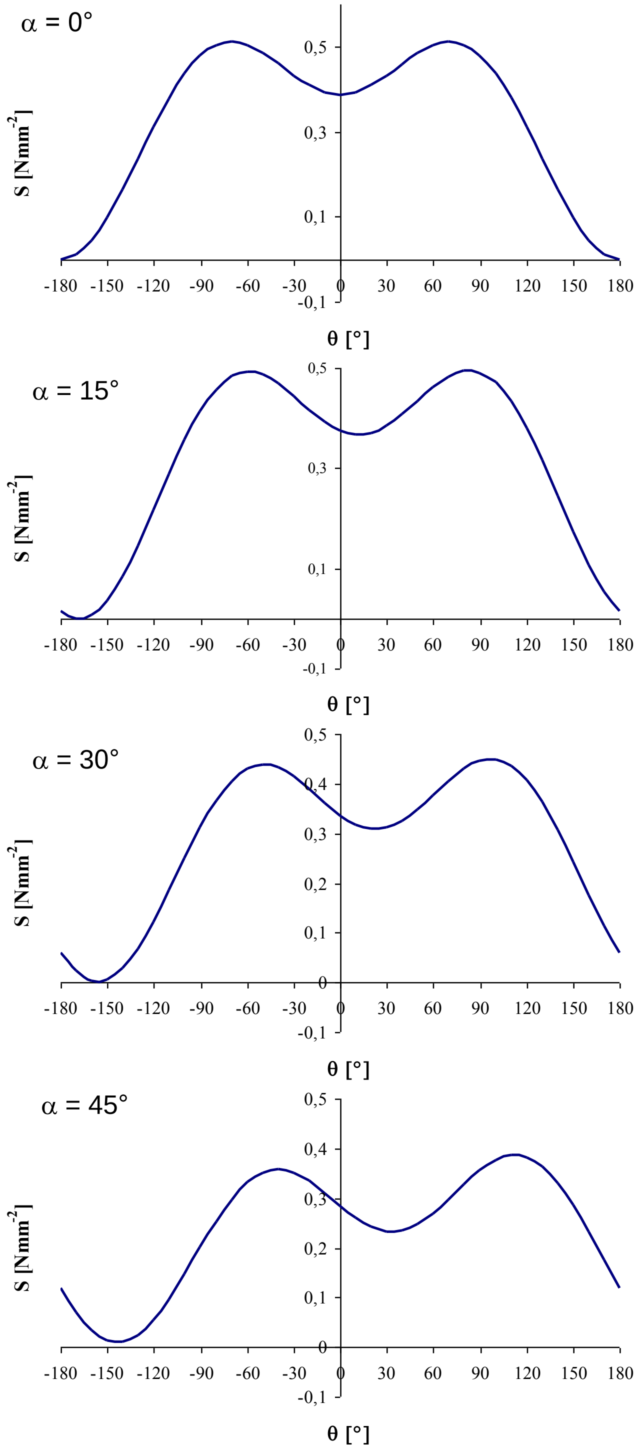


Fig. 10. *Distribution of factor S round the crack tip*

Fig. 10 presents the distribution of the strain energy density round the crack tip for various load angles α. From figures, the increase in the kink angle θ_0 related to the increase in load angle α is evident.

Fig. 11 presents the distribution of the strain energy density round the crack tip for various load angles α. From figures, the increase in the kink angle θ_0 related to the increase in load angle α is evident.



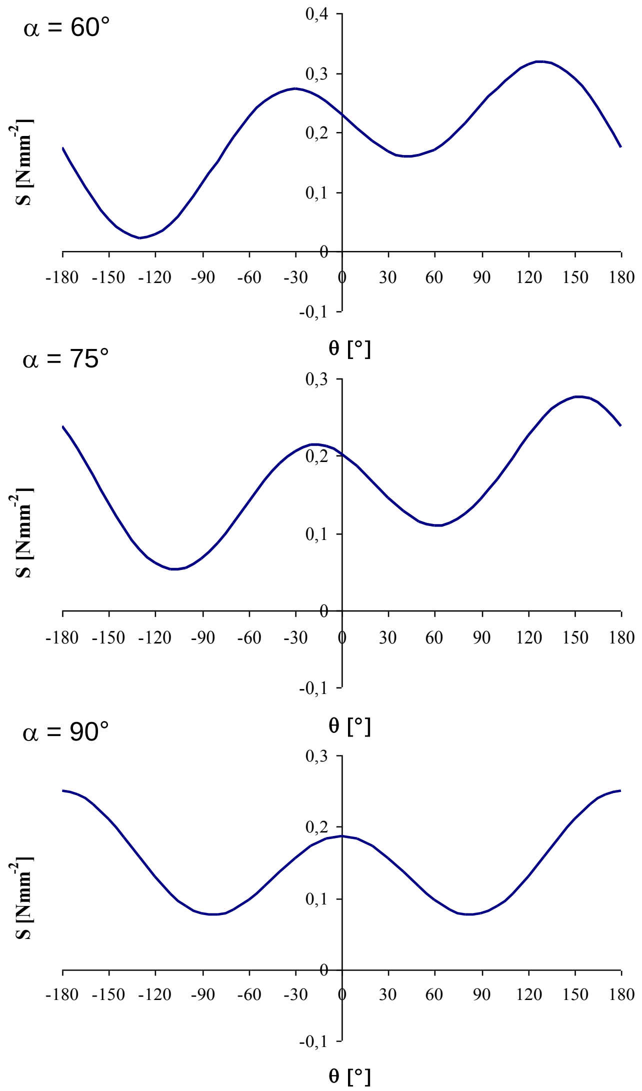


Fig. 11. *Distribution of S round the crack tip, depending on the load angle α*

For α=0° and α=90°, a symmetrical curve *S* round the crack tip is obtained, *Fig. 11*.

V. Conclusions

The crack tip of a CTS test piece is loaded with different proportions of tensile and shear loads; FEM is used to discretize the CTS test piece. Due to the impact of the shear stress near the initial crack tip, the crack kinks. The direction in which the crack kinks and, consequently, in which the crack propagates, is determined, using the maximum energy release rate criterion, the minimum strain energy density (SED) criterion and the maximum tangential stress (MTS) criterion. Stress intensity factors K_I in K_{II} for the initial crack (pre-crack) were determined with the energy release rate method using the J integral, and the results are compared with the results obtained using the crack opening method.

For the analysis, plane strain and material properties of the aluminum alloy are taken into account, which can be described with linear elastic fracture mechanics.

Generally speaking, the direction in which the crack kinks in case of opening modes I and II can be fairly reliably described when $K_I > K_{II}$.

With the energy release rate method using the J integral, special attention must be paid when shear stress prevails near the crack tip as in such a case physical intuition is required to define stress intensity factors K_I and K_{II}. Similarly, the analysis shows that, with the energy release rate method, on which the VCE method is based, the direction in which a crack kinks can be determined up to 45°. Consequently, it is impossible to define the direction in which a crack kinks from an initial crack in case of mode II of pure loading.

In case of the minimum strain energy density (SED) criterion, results show that, when this criterion is used to determine the crack kink, the second minimum of the strain energy density must be taken into consideration as at the global minimum, the crack cannot propagate physically.

With the maximum tangential stress (MTS) criterion, the direction in which a crack kinks can be uniformly determined.

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