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On one Application of Newton's Method to Stability Problem

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***Abstract*— In this paper we consider stability problem for switched linear systems. This problem can be formulated as a convex minimization problem. By modifying the cost functions we apply the vector-valued Newton's method.**

***Keywords*— Switched system; Hurwitz stability; common quadratic Lyapunov function; Newton's method**

I. Introduction

Let A be $n \times n$ real matrix. If all eigenvalues of A lie in the open left half plane then A is said to be Hurwitz stable. Hurwitz stability of A is equivalent to the following: There exists positive definite symmetric matrix P such that

$$A^T P + PA < 0 \qquad (1)$$

where the symbol "T" stands for the transpose, and the symbol "<" for negative definiteness. Hurwitz stability of A implies the asymptotic stability of the zero solution of the linear system

$$\dot{x} = Ax \qquad (2)$$

where $x = x(t) \in \mathbb{R}^n$. If the matrix A switches between N matrices $A_1, A_2, \ldots, A_N$, i.e. $A \in \{A_1, A_2, \ldots, A_N\}$ then the obtained system

$$\dot{x} = Ax \qquad (3)$$

is called a switched system. Sufficient condition for the asymptotic stability of the zero solution of (3) is the existence of quadratic Lyapunov function of the form

$$V(x) = x^T P x$$

where $P > 0$ and

$$A_i^T P + PA_i < 0 \; (i = 1, 2, \ldots, N). \qquad (4)$$

The matrix P is called a common solution to the Lyapunov inequalities (4).

The stability problem of linear switched systems has been investigated in a lot of works (see [1-11] and references therein).

The papers [3-11] study theoretical results for the existence of a common solution to (4).

The papers [12-14] consider numerical algorithms for a common positive definite solution in the case of existence.

In this paper we apply Newton's root finding method for the numerical generating of a common positive definite solution to (4).

II. Lyapunov Equations

In this section, we consider the Lyapunov inequality (1) which is equivalent to the following equation.

$$A^T P + PA = -Q \qquad (5)$$

where $Q > 0$. We are looking for a positive definite solution P of (5). In the iteration steps, the obtained P is guaranteed to be symmetric. The following theorem shows that in the case of Hurwitz stability of A this implies the positive definiteness of P.

Theorem 1. Assume that A is Hurwitz stable. If there exists a symmetric solution P to (5) then $P > 0$.

Proof: Define

$$\tilde{P} = \int_0^\infty e^{A^T t} Q e^{At} dt \qquad (6)$$

where e^{At} stands for the matrix exponential.

Since A is Hurwitz stable, the matrix A^T is also Hurwitz stable. Therefore e^{At} and $e^{A^T t}$ define exponential functions with exponents $\mathrm{Re}(\lambda_i) \cdot t < 0$ where λ_i are the eigenvalues of A. This implies that the integral in (6) is well defined. The matrix $\tilde{P}$ is symmetric, positive definite and satisfies the following relation

$$A^T \tilde{P} + \tilde{P} A = -Q$$

(see [15]). Then

$$A^T (P - \tilde{P}) + (P - \tilde{P}) A = 0.$$

Multiplying by $e^{A^T t}$ and e^{At} give

$$\begin{aligned} 0 &= e^{A^T t}\left[A^T (P - \tilde{P}) + (P - \tilde{P}) A\right] e^{At} \\ &= \frac{d}{dt}\left[e^{A^T t}(P - \tilde{P}) e^{At}\right]. \end{aligned}$$

The integration from 0 to ∞ yields

$$\left[e^{A^Tt}(P-\tilde{P})e^{At}\right]_0^\infty = 0.$$

Using the fact that $e^{At} \to 0$, $e^{A^Tt} \to 0$ as $t \to \infty$ we obtain

$$0 - (P - \tilde{P}) = 0$$

and $P = \tilde{P} > 0$.

We are looking for a iterative procedure for a common P satisfying (4). Theorem 1 allows to guarantee positive definiteness of P obtained at each step of iteration.

III. Modified Newton's Method

Consider a differentiable function $F: \mathbb{R}^n \to \mathbb{R}^n$ and the following equation

$$F(x) = 0. \tag{7}$$

Here

$x = (x_1, x_2, \dots, x_n)^T \in \mathbb{R}^n$, $F(x) = \left(f_1(x), f_2(x), \dots, f_n(x)\right)^T$.

Denote the Jacobian matrix by $J(x)$, i.e.

$$J(x) = \left(\frac{\partial f_i(x)}{\partial x_j}\right) \ (i,j = 1,2,\dots,n).$$

The Newton method is a method for an approximate solution of (7) and starting from a suitable initial point x^0 the iteration is defined by

$$x^k = x^{k-1} - J(x^{k-1})^{-1} F(x^{k-1}) \ (k = 1,2,\dots)$$

Define $r = \frac{n(n+1)}{2}$ and

$$P = P(x) = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ x_2 & x_{n+1} & \cdots & x_{2n-1} \\ \vdots & \vdots & \ddots & \vdots \\ x_n & x_{2n-1} & \cdots & x_r \end{pmatrix}.$$

The matrix inequalities (4) are equivalent to

$$f_i(x) = \lambda_{\max}(A_i^T P + P A_i) < 0 \ (i = 1,2,\dots,N) \tag{8}$$

where $x = (x_1, x_2, \dots, x_r)^T \in \mathbb{R}^r$, $\lambda_{\max}(\cdot)$ stands for the maximal eigenvalue.

In the case of simple maximum eigenvalue of $A^T P(x) + P(x)A$, the gradient of $f(x) = \lambda_{\max}(A^T P(x) + P(x)A)$ should be easily calculated. Indeed since the function $x \to A^T P(x) + P(x)A$ is linear then

$$A^T P(x) + P(x)A = \sum_{j=1}^{r} x_j Q_j.$$

Then $\nabla f(x) = (u^T Q_1 u, \dots, u^T Q_r u)$, where "∇" stands for the gradient, u is the unit eigenvector corresponding to the maximum eigenvalues of $A^T P(x) + P(x)A$ (see [12]).

Proposition 1. The function $f_i(x)$ is convex for each i.

Proof: The relation $P \to A_i^T P + P A_i$ is linear. On the other hand for symmetric C, the function $C \to \lambda_{\max}(C)$ is convex [16]. Therefore $f_i(x)$ is convex as a composition of linear and convex functions.

□

The system (4) has a common solution $P > 0$ if and only if there exists $x_* \in \mathbb{R}^r$ such that

$$f_i(x_*) < 0 \ (i = 1,2,\dots,N). \tag{9}$$

In order to apply Newton's method instead of the minimization of the functions $f_i(x)$, we consider the system of equations

$$f_i(x) = 0 \ (i = 1,2,\dots,N).$$

Without loss of generality we can set $r = N$. Indeed if $N > r$, we can combine some function by using the operation maximum. For example if $N = r + 1$ then define

$$g_1(x) = \max\{f_1(x), f_2(x)\},$$

$$g_i(x) = f_{i+1}(x) \ (i = 2,3,\dots,N).$$

This operation preserves convexity. If $N < r$ we use the operation of duplication. Thus from the now we assume that $r = N$.

Define $F = (f_1, f_2, \dots, f_r)^T$ and consider the equation

$$F(x) = 0 \tag{10}$$

where $x \in \mathbb{R}^r$.

If we apply the classical Newton's method to (10) we obtain the trivial sequence $P_k \to 0$, since the functions $f_i(x)$ are positive homogenous. To avoid this we impose the condition $\mathrm{trace}(P) = 1$. The following proposition shows that this does not violate the generality.

Proposition 2. Assume that $P > 0$ and $A^T P + PA < 0$. Then $A^T P_* + P_* A < 0$ where $P_* = \frac{1}{\mathrm{trace}(P)} \cdot P$.

Proof: From $P = (p_{ij}) > 0$ it follows that for all $x \in \mathbb{R}^n$, $x \neq 0$, $x^T P x > 0$. Taking

$$x^i = (0, \dots, 0, 1, 0, \dots, 0)^T$$

we obtain $p_{ii} > 0$. Therefore $\mathrm{trace}(P) > 0$ and

$$A^T P_* + P_* A = \frac{1}{\mathrm{trace}(P)}[A^T P + PA] < 0.$$

The condition $\mathrm{trace}(P) = 1$ reduces the number of variables from r to $r - 1$. To solve (10) the following algorithm is suggested.

Algorithm 1.

1) Consider the equation (10). Take initial matrix $Q = \mathrm{diag}(1,2,2,\dots,2)$ and consider $A_1^T P + P A_1 = -Q$.

The solution of this matrix equation let be P_0. Dividing P_0 by trace(P_0) gives the initial iteration x^0.

2) Replace the functions $f_i(x)$ by $\tilde{f}_i(x) + \frac{1}{2\text{trace}(P_0)}$ $(i = 1,2,\dots,r-1)$. Apply Newton's iteration

$$x^k = x^{k-1} - J(x^{k-1})^{-1}\tilde{F}(x^{k-1})$$

where $\tilde{F} = (\tilde{f}_1, \tilde{f}_2, \dots, \tilde{f}_{r-1})$.

3) If $\tilde{f}_i(x^k) < 0$ $(i = 1,2,\dots,r-1)$ for some k then stop. Otherwise continue.

Example 1. Consider the Hurwitz stable matrices

$A_1 = \begin{pmatrix} -1 & -4 \\ -1 & -8 \end{pmatrix}$ and $A_2 = \begin{pmatrix} -3 & 5 \\ -2 & 1 \end{pmatrix}$.

The corresponding functions are:

$$f_1(x) = \lambda_{\max}(A_1^T P(x) + P(x)A_1),$$
$$f_2(x) = \lambda_{\max}(A_2^T P(x) + P(x)A_2)$$

where

$$P(x) = \begin{pmatrix} x_1 & x_2 \\ x_2 & 1 - x_1 \end{pmatrix}.$$

For the matrix

$$Q = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

the unique solutions of

$$A_1^T P + PA_1 = -Q \ (i = 1,2)$$

is

$$P_0 = \begin{pmatrix} 0.972 & -0.472 \\ -0.472 & 0.361 \end{pmatrix}.$$

Hence

$$\frac{1}{\text{trace}(P_0)} \cdot P_0 = \begin{pmatrix} 0.729 & -0.354 \\ -0.354 & 0.271 \end{pmatrix}$$

and take the initial point $x^0 = (0.729, -0.354)^T$. For this point calculations give the following maximum eigenvalues and its corresponding unit eigenvectors:

$$\lambda_{\max}(A_1^T P(x^0) + P(x^0)A_1) = -0.75$$

the maximum eigenvector: $u^1 = (1,0)^T$,

$$\lambda_{\max}(A_2^T P(x^0) + P(x^0)A_2) = 0.8333$$

the maximum eigenvector: $u^2 = (-0.7090, -0.7051)^T$.

Therefore

$$f_1(x^0) = -0.75,$$
$$f_2(x^0) = 0.833.$$

and

$$\nabla f_1(x)|_{x=x^0} = (-2, -2),$$
$$\nabla f_2(x)|_{x=x^0} = (2.988, 0.961).$$

Therefore the Jacobian matrix of $F(x) = \left(f_1(x), f_2(x)\right)^T$ at x^0 is

$$J(x^0) = \begin{pmatrix} -2 & -2 \\ 2.988 & 0.961 \end{pmatrix}$$

and

$$x^1 = \begin{pmatrix} 0.729 \\ -0.354 \end{pmatrix} + \begin{pmatrix} 0.237 & 0.493 \\ -0.737 & -0.493 \end{pmatrix} \begin{pmatrix} -0.75 + 0.375 \\ 0.833 + 0.375 \end{pmatrix}$$
$$= \begin{pmatrix} 0.130 \\ 0.119 \end{pmatrix}.$$

After 3 steps, we get $f_1(x^3) < 0$ and $f_2(x^3) < 0$ (see Table I). Hence for the matrix

$$P = P(x^3) = \begin{pmatrix} 0.390 & -0.247 \\ -0.247 & 0.609 \end{pmatrix},$$

$A_i^T P + PA_i < 0$ $(i = 1,2)$ are satisfied.

Table I

k	x^k	$f_1(x^k)$	$f_2(x^k)$
1	$(0.130, 0.119)^T$	-0.221	1.268
2	$(0.367, -0.135)^T$	-0.373	0.278
3	$(0.390, -0.247)^T$	-0.285	-0.074

Example 2. Consider the Hurwitz stable matrices

$A_1 = \begin{pmatrix} -1 & 2 \\ -1 & -2 \end{pmatrix}, A_2 = \begin{pmatrix} -2 & -3 \\ 2 & 1 \end{pmatrix}$ and $A_3 = \begin{pmatrix} 1 & 2 \\ -2 & -2 \end{pmatrix}$.

The corresponding functions are:

$$f_1(x) = \max(\lambda_{\max}(A_1^T P(x) + P(x)A_1), \lambda_{\max}(A_2^T P(x) + P(x)A_2)),$$
$$f_2(x) = \lambda_{\max}(A_3^T P(x) + P(x)A_3).$$

where

$$P(x) = \begin{pmatrix} x_1 & x_2 \\ x_2 & 1 - x_1 \end{pmatrix}.$$

For the matrix

$$Q = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

the unique solutions of

$$A_1^T P + PA_1 = -Q \ (i = 1,2)$$

is

$$P_0 = \begin{pmatrix} 0.41\bar{6} & 0.083 \\ 0.083 & 0.58\bar{3} \end{pmatrix}$$

and trace$(P_0) = 1$. Take the initial point $x^0 = (0.416, 0.083)^T$. For this point calculations give the following maximum eigenvalues and its corresponding unit eigenvectors:

$$\lambda_{\max}(A_1^T P(x^0) + P(x^0)A_1) = -1,$$
$$u^1 = (1,0)^T,$$
$$\lambda_{\max}(A_2^T P(x^0) + P(x^0)A_2) = 0.680,$$
$$u^2 = (-0.082, 0.996)^T,$$
$$\lambda_{\max}(A_3^T P(x^0) + P(x^0)A_3) = 0.567,$$
$$u^3 = (-0.987, 0.160)^T.$$

Therefore

$$f_1(x^0) = \max(-2.105, 0.074) = 0.680,$$
$$f_2(x^0) = 0.567.$$

and

$$\nabla f_1(x)|_{x=x^0} = (-1.191, -5.767),$$
$$\nabla f_2(x)|_{x=x^0} = (0.786, -3.478).$$

The Jacobian matrix of $F(x) = \left(f_1(x), f_2(x)\right)^T$ at x^0 is

$$J(x^0) = \begin{pmatrix} -1.191 & -5.767 \\ 0.786 & -3.478 \end{pmatrix}.$$

Therefore

$$\begin{aligned} x^1 &= \begin{pmatrix} 0.416 \\ 0.083 \end{pmatrix} + \begin{pmatrix} -0.400 & 0.664 \\ -0.090 & -0.137 \end{pmatrix} \begin{pmatrix} 0.680 + 0.5 \\ 0.567 + 0.5 \end{pmatrix} \\ &= \begin{pmatrix} 0.180 \\ 0.336 \end{pmatrix}. \end{aligned}$$

After 16 steps, we get $f_1(x^{16}) < 0$ and $f_2(x^{16}) < 0$ (see Table II). Hence for the matrix

$$P = P(x^{16}) = \begin{pmatrix} 0.461 & 0.318 \\ 0.318 & 0.538 \end{pmatrix},$$

$A_i^T P + P A_i < 0$ $(i = 1,2,3)$ are satisfied.

Table II

k	x^k	$f_1(x^k)$	$f_2(x^k)$
1	$(0.180, 0.336)^T$	1.035	0.224
2	$(0.581, 2.025)^T$	8.502	6.641
3	$(0.553, 0.313)^T$	0.108	−0.122
⋮	⋮	⋮	⋮
15	$(0.293, 0.919)^T$	2.527	1.514
16	$(0.461, 0.318)^T$	−0.063	−0.074

Example 3. Consider the Hurwitz stable matrices

$$A_1 = \begin{pmatrix} -32 & 5 & 12 \\ -10 & 1 & -2 \\ -9 & 7 & -17 \end{pmatrix}, A_2 = \begin{pmatrix} -4 & 5 & 2 \\ -6 & -11 & 3 \\ 1 & 0 & -10 \end{pmatrix},$$
$$A_3 = \begin{pmatrix} -5 & -3 & 1 \\ 2 & -4 & 2 \\ 4 & 1 & -5 \end{pmatrix}, A_4 = \begin{pmatrix} -6 & 1 & -2 \\ 3 & -3 & 4 \\ 1 & -2 & -4 \end{pmatrix},$$
$$\text{and } A_5 = \begin{pmatrix} -10 & -4 & -2 \\ -7 & -8 & 20 \\ 7 & -2 & -22 \end{pmatrix}.$$

The corresponding functions are:

$$f_i(x) = \lambda_{\max}(A_i^T P(x) + P(x) A_i)$$

where

$$P(x) = \begin{pmatrix} x_1 & x_2 & x_3 \\ x_2 & x_4 & x_5 \\ x_3 & x_5 & 1 - x_1 - x_4 \end{pmatrix}.$$

For the matrix

$$Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

the unique solutions of

$$A_1^T P + P A_1 = -Q \ (i = 1,2)$$

is

$$P_0 = \begin{pmatrix} 0.0512 & -0.138 & 0.027 \\ -0.138 & 0.588 & -0.128 \\ 0.027 & -0.128 & 0.093 \end{pmatrix}.$$

Hence

$$\frac{1}{\mathrm{trace}(P_0)} \cdot P_0 = \begin{pmatrix} 0.069 & -0.188 & 0.037 \\ -0.188 & 0.803 & -0.174 \\ 0.037 & -0.174 & 0.126 \end{pmatrix}$$

and the initial point is

$$x^0 = (0.069, -0.188, 0.037, 0.803, -0.174)^T.$$

For this point calculations give the following:

$$F(x^0) = (-1.363, 1.952, 0.408, 1.843, 11.909)^T,$$

$$J(x^0) = \begin{pmatrix} -64 & -20 & -17.999 & 0 & 0 \\ -8.937 & -9.033 & 4.011 & 1.273 & 0.357 \\ -5.916 & -4.210 & -4.211 & -0.128 & 0.677 \\ 0.047 & -3.062 & -6.494 & 5.374 & -2.547 \\ 21.888 & -8.146 & 15.925 & 35.579 & -11.025 \end{pmatrix}$$

and $x^1 = (-0.090, 0.252, 0.079, 0.617, -0.216)^T$.

After 36 steps, we get

$$x^{36} = (0.421, -0.138, -0.044, 0.281, -0.028)^T$$

and

$$F(x^{36}) = (-0.590, -0.360, -0.608, -0.447, -0.164)^T.$$

Hence $A_i^T P + P A_i < 0$ $(i = 1,2,3,4,5)$ where

$$P = P(x^{36}) = \begin{pmatrix} 0.421 & -0.138 & -0.044 \\ -0.138 & 0.281 & -0.028 \\ -0.044 & -0.028 & 0.297 \end{pmatrix}.$$

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