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Finite Difference Solution Of The Burgers–Fisher Equation Based On The Crank–Nicolson Scheme

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***Abstract*—Nonlinear partial differential equations arise in many areas of science and engineering. The generalized Burgers–Fisher equation plays a significant role in modeling various physical and applied phenomena. This equation appears in fields such as financial mathematics, gas dynamics, traffic flow, applied mathematics, and physics. It serves as a prototypical model for describing the interaction among reaction mechanisms, convective effects, and diffusive transport. In this study, two implicit numerical methods are employed to solve the Burgers–Fisher equation**

Keywords—Burgers–Fisher equation; difference method; matlab implementation; numerical results

I. INTRODUCTION

We consider the following Burgers′-Fisher equation of the form

$$\frac{\partial v}{\partial t}-\frac{\partial^2 v}{\partial x^2}+\alpha v\frac{\partial v}{\partial x}+\beta v(1-v)=0 \quad a\le x\le b, t>0 \quad (1)$$

Where α and β are advection and source/sink constants.

Initial and boundary conditions are as follow

$$v(x,0)=f(x) \quad for\ a\le x\le b$$

$$v(a,t)=g(t) \quad for\ t\ge 0$$

$$v(b,t)=h(t) \quad for\ t\ge 0$$

This paper focuses on the numerical solution of the Burgers–Fisher equation, a nonlinear parabolic partial differential equation. It represents a mathematical model for numerous physical phenomena encountered in various fields of science and engineering, including heat conduction, gas dynamics, chemical physics, and nonlinear optics. For instance, the equation is used to describe the velocity profile of a viscous fluid in fluid dynamics and gas flow behavior in exhaust pipes. The Burgers–Fisher equation serves as a prototypical model for capturing the interplay among convective effects, reaction mechanisms, and diffusive transport. Due to its fundamental role in nonlinear physics, the equation possesses considerable theoretical significance and practical relevance.

Fisher proposed in 1937 that this equation models population dynamics explaining the spatial spread of an advantageous allele and discussing its travelling wave solutions. This equation was

$$\frac{\partial v}{\partial t}-D\frac{\partial^2 v}{\partial x^2}=Kv(1-v)$$

The Burgers' equation, which was proposed by Johannes Martinus Burgers in 1948 modeling various physical phenomena such as gas dynamics, fluid mechanics, traffic flow, nonlinear acoustics, is given as:

$$\frac{\partial v}{\partial t}-D\frac{\partial^2 v}{\partial x^2}+v\frac{\partial v}{\partial x}=0$$

The combination of these two equations is commonly known as the Burgers′-Fisher equation (1). Recently, various numerical and analytical methods have been used by various researchers to deal with the Burgers′-Fisher equation.

In this paper, we address the numerical solution of the Burgers–Fisher equation using the well-known Crank–Nicolson method. This method is straightforward to implement and provides reliable numerical results. We consider a representative example with prescribed initial and boundary conditions to demonstrate the effectiveness of the proposed approach.

II. CRANK-NICOLSON METHOD FOR THE SOLUTION OF THE BURGERS–FISHER EQUATION

Consider the problem below:

$$\frac{\partial v}{\partial t}-\frac{\partial^2 v}{\partial x^2}+\alpha v\frac{\partial v}{\partial x}+\beta v(1-v)=0$$

With initial and boundary conditions

$$v(x,0)=f(x) \quad for\ a\le x\le b$$

$$v(a,t)=g(t) \quad for\ t\ge 0$$

$$v(b,t)=h(t) \quad for\ t\ge 0$$

we replace in this equation the derivatives with the following differences. A more rewarding method can be derived by averaging the forward-difference method written at its jth step in t and sep (j+1).

$$\frac{\partial v}{\partial t} = \frac{v(x_i, t_{j+1}) - v(x_i, t_j)}{k} + O(k)$$

(the forward difference method)

$$\frac{\partial^2 v}{\partial x^2} = \frac{v(x_{i+1}, t_j) - 2v(x_i, t_j) + v(x_{i-1}, t_j)}{h^2} + O(h^2)$$

(the central difference method)

$$\frac{\partial v}{\partial x} = \frac{v(x, t_j) - v(x_{i-1}, t_j)}{2h} + O(h^2)$$

(the central difference method)

And in step j+1 can be written

$$\frac{\partial^2 v}{\partial x^2} = \frac{v(x_{i+1}, t_{j+1}) - 2v(x_i, t_{j+1}) + v(x_{i-1}, t_{j+1})}{h^2} + O(h^2)$$

$$\frac{\partial v}{\partial x} = \frac{v(x_{i+1}, t_{j+1}) - v(x_{i-1}, t_{j+1})}{2h} + O(h^2)$$

A more rewarding method can be derived by averaging the forward-difference method written at its jth step and at the (j+1)st step in t. The averaged difference method has local truncation error of order $O(k^2 + h^2)$. This method is known as Crank-Nicolson method.

III. IMPLEMENTATION OF THE METHOD

Consider the problem below:

$$\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} + \alpha v \frac{\partial v}{\partial x} + \beta v(1 - v) = 0$$

With initial and boundary conditions

$$v(x, 0) = f(x) \quad for\ a \le x \le b$$

$$v(a, t) = g(t) \quad for\ t \ge 0$$

$$v(b, t) = h(t) \quad for\ t \ge 0$$

As a first step, the computational domain is discretized with respect to the independent variables x,t.

$$v_{i,j} = v(ih, jk,) \quad \text{where } h = \Delta x \text{ and } k = \Delta t$$

The backward-difference method written at the (j+1)st step in t

$$\frac{v_{i,j+1} - v_{i,j}}{k} - \frac{v_{i+1,j} - 2v_{i,j} + v_{i-1,j}}{h^2} + \alpha v_{i,j} \frac{v_{i+1,j} - v_{i-1,j}}{2h} + \beta v_{i,j}(1 - v_{i,j}) = 0$$

and the backward-difference method written at the (j+1)st step in t.

$$\frac{v_{i,j+1} - v_{i,j}}{k} - \frac{v_{i+1,j+1} - 2v_{i,j+1} + v_{i-1,j+1}}{h^2} + \alpha v_{i,j+1} \frac{v_{i+1,j+1} - v_{i-1,j+1}}{2h} + \beta v_{i,j+1}(1 - v_{i,j+1}) = 0$$

The Crank-Nicolson method can be written;

$$\frac{v_{i,j+1} - v_{i,j}}{k} - \frac{1}{2h^2}[v_{i+1,j} - 2v_{i,j} + v_{i-1,j} + v_{i+1,j+1} - 2v_{i,j+1} + v_{i-1,j+1}] + \frac{\alpha}{4h}[v_{i,j}(v_{i+1,j} - v_{i-1,j}) + v_{i,j+1}(v_{i+1,j+1} - v_{i-1,j+1})] + \frac{\beta}{2}[(v_{i,j} - {v_{i,j}}^2) + (v_{i,j+1} - {v_{i,j+1}}^2)] = 0$$

If we keep the terms $v^{(j+1)}$ on the left-hand side and $v^{(j)}$ on the right-hand side, the equation takes the form:

$$\frac{1}{k}v_{i,j+1} - \frac{1}{2h^2}(v_{i+1,j+1} - 2v_{i,j+1} + v_{i-1,j+1}) + \frac{\alpha}{4h}[v_{i,j+1}(v_{i+1,j+1} - v_{i-1,j+1})] + \frac{\beta}{2}(v_{i,j+1} - {v_{i,j+1}}^2) = \frac{1}{k}v_{i,j} + \frac{1}{2h^2}(v_{i+1,j} - 2v_{i,j} + v_{i-1,j}) - \frac{\alpha}{4h}[v_{i,j}(v_{i+1,j} - v_{i-i,j})] - \frac{\beta}{2}(v_{i,j} - {v_{i,j}}^2)$$

$$v_{i,j+1}\left(\frac{1}{k} + \frac{1}{h^2} + \frac{\beta}{2}(1 - v_{i,j+1})\right) + v_{i+1,j+1}\left(-\frac{1}{2h^2} + \frac{\alpha}{4h}v_{i,j+1}\right) + v_{i-1,j+1}\left(-\frac{1}{2h^2} - \frac{\alpha}{4h}v_{i,j+1}\right) = v_{i,j}\left(\frac{1}{k} - \frac{1}{h^2} - \frac{\beta}{2}(1 - v_{i,j})\right) + v_{i+1,j}\left(\frac{1}{2h^2} - \frac{\alpha}{4h}v_{i,j}\right) + v_{i-i,j}\left(\frac{1}{2h^2} + \frac{\alpha}{4h}v_{i,j}\right)$$

Define coefficients

$$a_i = \frac{1}{k} + \frac{1}{h^2} + \frac{\beta}{2}(1 - v_{i,j+1})$$

$$b_i = -\frac{1}{2h^2} + \frac{\alpha}{4h}v_{i,j+1}$$

$$c_i = -\frac{1}{2h^2} - \frac{\alpha}{4h}v_{i,j+1}$$

$$d_i = \frac{1}{k} - \frac{1}{h^2} - \frac{\beta}{2}(1 - v_{i,j})$$

$$f_i = \frac{1}{2h^2} - \frac{\alpha}{4h}v_{i,j}$$

$$g_i = \frac{1}{2h^2} + \frac{\alpha}{4h}v_{i,j}$$

The matrix form is

$$Av^{(j+1)} = Bv^{(j)}$$

Where A and B are tridiagonal matrix.

IV. NUMERICAL EXPERIMENTS AND DISCUSSIONS

The exact solution of the Burgers-Fisher equation over the domain [0,1]x[0,T] is:

$$v(x,t) = \frac{1}{2} + \frac{1}{2}\tanh\left(\frac{-\alpha}{4}\left[x - \left(\frac{\alpha}{2} - \frac{2\beta}{\alpha}\right)t\right]\right)$$

This solution is obtained from the existing literature.
Initial and boundary conditions are as follows

$$v(x,0) = \frac{1}{2} + \frac{1}{2}\tanh\left(\frac{-\alpha x}{4}\right)$$

$$v(0,t) = \frac{1}{2} + \frac{1}{2}\tanh\left(\frac{\alpha}{4}\left[\left(\frac{\alpha}{2} - \frac{2\beta}{\alpha}\right)t\right]\right)$$

$$v(1,t) = \frac{1}{2} + \frac{1}{2}\tanh\left(\frac{-\alpha}{4}\left[1 - \left(\frac{\alpha}{2} - \frac{2\beta}{\alpha}\right)t\right]\right)$$

In order to test the accuracy and efficiency of the proposed scheme, comparisons of the obtained results are made with the above exact solution.

Example 1. Numerical results are computed for $\alpha = -1$ and $\beta = -1$ at different time livels. Figure 4.1 presents the computed solutions in three-dimensional form for T=0.01. Figure 4.2 illustrates the comparison of absolute errors at grid points for the time levels T=0.001, 0.005 and 0.01.
It is observed that the absolute error decreases as the time step is reduced. Moreover, the proposed method exhibits high accuracy, particularly at the interior grid points. Figure 4.2 also shows the corresponding error profiles.

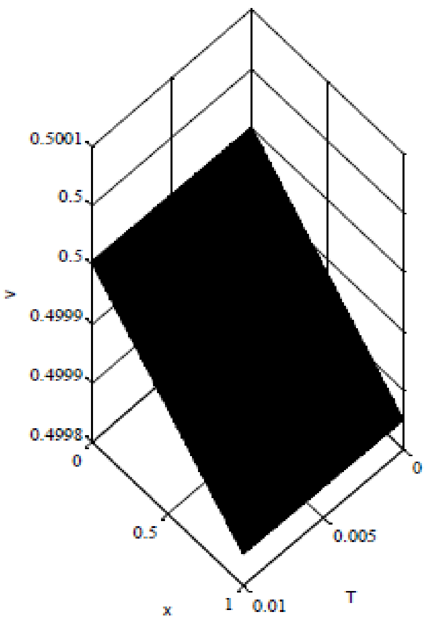

Fig. 1. *Solutions of* excemple 1 with space and time variables for T=0.01.

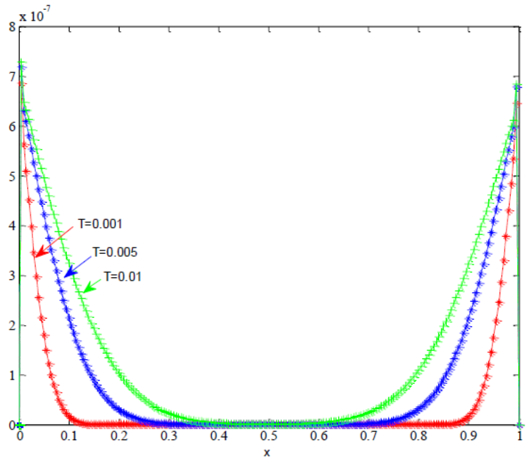

Fig. 2. *Absolute errors of excemple 1 for $\alpha = -1$ and $\beta = -1$ at different times T.*

V. CONCLUSION

The Crank–Nicolson method is employed for temporal discretization, while the quasi-linearization technique is used to handle the nonlinear nature of the equation. Several examples with varying parameter values are considered to demonstrate the effectiveness of the proposed method.
The numerical results obtained are consistent with the inherent behavior of the Burgers–Fisher equation and show improved accuracy compared to results reported in the existing literature. Furthermore, the method is computationally efficient, relatively simple to implement, and can be readily extended to higher-dimensional partial differential equations.

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