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# Theoretical Modeling Of A Cantilever Beam Design Of A Piezoelectric Energy Harvester To Be Attached To A Bridge Structure

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**Abstract- A significant amount of kinetic energy from road traffic is wasted in transportation networks. Piezoelectric energy harvesters (PEH) can convert vibrations into electrical energy. This technology is considered for usage in smart highways for collecting the wasted kinetic energy from passing traffic and turning it into useable energy. In this paper, a piezoelectric energy harvesting design using a cantilever beam set up is considered with the vibrations from a bridge structure. The nonlinear system of coupled electromechanical equations is implemented using symbolic and numerical tools to derive expressions for the complex voltage amplitude in both series and parallel circuit configurations. Although no quantitative results for energy are presented at this stage, the modeling framework establishes a foundation for further analysis and simulation of piezoelectric energy harvesters. This approach enables future optimization and design exploration for enhanced energy harvesting performance. Future research to be done will continue this optimization and design analysis.**

***Keywords—Piezoelectric, Smart Highway, Cantilever Beam, Bridge Structure, Modeling Framework, Energy Harvesting***

## 1. Introduction

With the world searching for new and effective renewable energy sources, piezoelectric energy harvesting is an interesting topic of research. Previously studied smart highways use the integration of piezoelectric technology in roadways to collect kinetic energy from moving traffic. Findings from this study showed that factors affecting the amount of energy harvested depend on the speed and volume of traffic [5]. This is due to the vibrations produced by vehicles traveling on the road. Based on this information, vibrations on piezoelectric integrated bridge structures became the next research point. Bridges inherently have more vibrations due to the structure and relationship between car and bridge [3]. This would have a differing effect on the energy harvesting abilities of the piezoelectric, which is vibration, or kinetic energy, based.

### Literature Review

Piezoelectric energy harvesting (PEH) offers a way to convert mechanical vibrations from bridge traffic into usable electrical energy. Cantilever-based PEH systems are especially effective for this purpose due to their simplicity and sensitivity to dynamic loads. This literature review explores recent research on bridge vibration energy harvesting, focusing on design models and optimization strategies.

### 1.1 Smart Infrastructure and Energy Harvesting Applications

Smart cities use technologies for information and communication that provide helpful development for a sustainable infrastructure [6]. Examples of such smart cities include Singapore that uses energy and water use, as well as waste management, monitoring in real time; Sandiego has incorporated sensors to optimize parking and traffic flow as well as aiding in safety of both people and the environment; Barcelona has sensors that can measure pollution, noise, and temperature [6]. Cities like these are using sensor technology to promote information important to the city's infrastructure and wellbeing. Integration of PEH's, similarly, would contribute to roadway infrastructure such as using the energy to power the road lights and signs [5].

### 1.2 Bridge Vibrations

Constant and consistent dynamic loading from vehicles cause vibrations throughout the bridge structure [3]. This form of mechanical energy, otherwise known as kinetic energy, is often considered a useless byproduct of the flow of traffic. Recently however, the need for clean and renewable energy has inspired scientists to attempt capturing the wasted energy. This energy is often put right back into the infrastructure and energy systems [5], [7].

### 1.3 How do Piezoelectric Materials Work?

The piezoelectric effect was discovered in 1880 by two French physicists named Jacques and Pierre Curie. They observed that when certain crystalline materials were hit with a mechanical force, the material became electrically polarized. Due to the opposite polarities from tension and compression, when the material would bend, one surface would be charged positively and the other negatively, as well as in proportion to the amount of force that was applied. They also discovered that the phenomenon could be reversed. If the material would be subjected to an energy field, the crystal shortened or lengthened according to the polarity. It is important to note that these effects are small and often require some sort of amplification to have noticeable effects [4].

### 1.4 Piezoelectric Designs

Many methods of energy harvesting have made an appearance in the energy world in the last century. Piezoelectric materials have gained notability for their ability to convert mechanical energy into useable, electrical energy [2]. To efficiently exploit vibrations, the optimal design for the harvester is important. The cantilever beam design is one of these efficient structures for bridge vibrations [9], [8]. A cantilever beam has a high sensitivity to excitations in the base and it has a simple behavior, making it a design that many are studying in regard to vibration based energy [2], [5].

## 2. Methodology

In this study, mathematical modeling displaying the output complex voltage amplitude of a cantilever beam PEH design was developed using symbolic and numerical tools in MATLAB coding software. The mathematical model developed in this study is based on the coupled electromechanical equations formulated by Erturk and Inman (2009) [2], with modifications to suit the dynamic characteristics of bridge vibrations under traffic loading.

The cantilever design consists of two piezoelectric plates attached to either side of a substructure at the base and a mass at the free end.

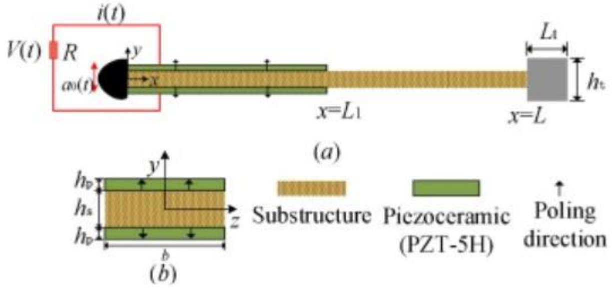


**Figure 1 Schematic of a piezoelectric energy harvester [9]**

The following table is a list of the needed information to proceed with the electromechanical equations. The numerical values are pulled from Zhang, Xiang, Shi, and Zhan (2018) paper titled *Experimental investigation on piezoelectric energy harvesting from vehicle-bridge coupling vibration.*

**Table 1 Parameters of the PEH**

| Parameters | Symbol | Value |
|---|---|---|
| Transducer Length | $L$ | 100 mm |
| PZT Length | $L_1$ | 50 mm |
| Transducer Width | $b$ | 30 mm |
| PZT Width | $h_p$ | 0.2 mm |
| Substrate Layer Thickness | $h_s$ | 1 mm |
| Density of PZT | $\rho_p$ | 7450 kg/m$^3$ |
| Density of substrate | $\rho_s$ | 8820 kg/m$^3$ |
| Elastic stiffness of PZT | $c_{11}$ | 66 GPa |
| Young's modulus of substructure | $E_s$ | 108 GPa |
| Permittivity constant | $\varepsilon_{33}$ | 24.10 nF/m |
| Piezoelectric strain constant | $e_{31}$ | -10.4 C/m$^2$ |
| Tip mass | $M_t$ | 27 g |
| Length of the tip mass | $L_t$ | 10 mm |
| Thickness of the tip mass | $h_t$ | 12 mm |
| Mechanical damping ratio | $\zeta$ | 0.0122 |

### 2.2 Systematic Use of Equations

To find the complex voltage amplitude, there are several variables to compute using the given numerical information. All equations in this paper are given from Erturk and Inman (2009) [2]. The first of these intermediate equations are ones that only use the information from Table 1.

$$m = b(\rho_s h_s + 2\rho_p h_p) \quad (1)$$

*m* refers to the mass per unit length, and the other variables can be found in the table above. *YI* is the bending stiffness term, which can be obtained from the following equation.

$$YI = \frac{2b}{3}\left[E_s \frac{h_s^3}{8} + c_{11}\left(\left(h_p + \frac{h_s}{2}\right)^3 - \frac{h_s^3}{8}\right)\right] \quad (2)$$

The next intermediate variable that will be needed is $I_t$. This is the rotary inertia of the tip mass $M_t$ which can be found using the parallel axis theorem in equation (3).

$$I_t = \left[\frac{1}{12} M_t\left(h_s^2 + L_t^2\right)\right] M_t L^2 \quad (3)$$

Using the values from the table and the previously found variables, λ can be found by solving the non-linear equation (4).

$1 + cos\lambda cosh\lambda + \lambda \frac{M_t}{mL}(cos\lambda sinh\lambda - sin\lambda cosh\lambda) - \frac{\lambda^4 I_t}{mL^3}(cosh\lambda sin\lambda + sinh\lambda cos\lambda) + \frac{\lambda^4 M_t I_t}{m^2 L^4}(1 - cos\lambda cosh\lambda) = 0$ (4)

Using λ and the previous variables, $\omega_r$ , the undamped natural frequency, and ςr, can then be found using equations (5) and (6).

$\omega_r = \lambda_r{}^2 \sqrt{\frac{YI}{mL^4}}$ (5)

$\varsigma_r = \frac{sin\lambda_r - sinh\lambda_r + \lambda_r \frac{M_t}{mL}(cos\lambda_r - cosh\lambda_r)}{cos\lambda_r + cosh\lambda_r - \lambda_r \frac{M_t}{mL}(sin\lambda_r - sinh\lambda_r)}$ (6)

Also using these found variables, $\phi_r(x)$ can be found. Note here that $C_r$is a modal amplitude constant which should be evaluated by normalizing the eigenfunctions [2]. For the purpose of this modeling, $C_r$ is given a random value to proceed with the coding.

$\phi_r(x) = C_r \left[cos\frac{\lambda_r}{L}x - cosh\frac{\lambda_r}{L}x + \varsigma_r\left(sin\frac{\lambda_r}{L}x - sinh\frac{\lambda_r}{L}x\right)\right]$ (7)

Taking the derivative of $\phi_r(x)$ at $x = L$, the expression for the forward coupling term, $k_r$, can be found.

$h_{pc} = \frac{h_p + h_s}{2}$ (8)

$k_r = -e_{31} h_{pc} b \left.\frac{d\phi_r(x)}{dx}\right|_{x=L}$ (9)

An important note moving forward is the use of input variables and unknown constants. For the purpose of completing a functioning code, variables like $Y_0$ and $\vartheta_0$ , the translational and small rotational displacement amplitudes, are given a random value in the code for this instance. ω is considered an input frequency along with $R_1$, the load resistance. Using this information and previous equations, a forcing function $f_r(t) = F_r e^{j\omega t}$ where the amplitude, $F_r$, can be found.

$F_r = \omega^2 \left[m\left(Y_0 \int_0^L \phi_r(x)dx + \theta_0 \int_0^L x\phi_r(x)dx\right) + M_t \phi_r(L)(Y_0 + L\theta_0)\right]$ (10)

Once again using the information from Table 1, the series piezoelectric coupling term $\vartheta_s$ and the parallel circuit piezoelectric coupling term $\vartheta_p$can be found using equations (11) and (12).

$\vartheta_s = \frac{e_{31}b}{2h_p}\left[\frac{h_s{}^2}{4} - \left(h_p + \frac{h_s}{2}\right)^2\right]$ (11)

$\vartheta_p = 2\vartheta_s$ (12)

Next, to find the modal electromechanical coupling term, $\chi_r^s$ for the series connection and $\chi_r^p$ for the parallel connection, equations (13) and (14) were used.

$\chi_r^s = \vartheta_s \left.\frac{d\phi_r(x)}{dx}\right|_{x=L}$ (13)

$\chi_r^p = \vartheta_p \left.\frac{d\phi_r(x)}{dx}\right|_{x=L}$ (14)

Polishing up with the intermediate variables, the internal capacitance of the piezoceramic layer, $C_p$, can be found using equation (15).

$C_p = \frac{\varepsilon_{33} bL}{h_p}$ (15)

The resulting complex voltage amplitude, $V_s$ for the series circuit and $V_p$ for the parallel circuit, can be found and used in $v_s = V_s e^{j\omega t}$ and $v_p = V_p e^{j\omega t}$ to express the steady state voltage response.

$v_s = \frac{\sum_{r=1}^{\infty} \frac{j\omega k_r F_r}{\omega_r{}^2 - \omega^2 + 2j\zeta_r \omega_r \omega}}{\frac{1}{R_1} + j\omega\frac{C_p}{2} + \sum_{r=1}^{\infty} \frac{j\omega k_r \chi_r^s}{\omega_r{}^2 - \omega^2 + 2j\zeta_r \omega_r \omega}} e^{j\omega t}$ (16)

$v_p = \frac{\sum_{r=1}^{\infty} \frac{j\omega k_r F_r}{\omega_r{}^2 - \omega^2 + 2j\zeta_r \omega_r \omega}}{\frac{1}{2R_1} + j\omega C_p + \sum_{r=1}^{\infty} \frac{j\omega k_r \chi_r^p}{\omega_r{}^2 - \omega^2 + 2j\zeta_r \omega_r \omega}} e^{j\omega t}$ (17)

Note that due to the system having multiple natural frequencies, this code looked at the first one. This was determined using $\lambda$. This is the reason that equations (16) and (17) use summations of the variables that change with the state of $\lambda$. Because only the first mode is being observed, we can instead use the simplified equations (18) and (19).

$\hat{v}_s = \frac{j2\omega R_1 k_r F_r}{(2 + j\omega R_1 C_p)(\omega_r{}^2 - \omega^2 + j2\xi_r \omega_r \omega) + j2\omega R_1 k_r \chi_r^s}$ (18)

$\hat{v}_p = \frac{j2\omega R_1 k_r F_r}{(1 + j2\omega R_1 C_p)(\omega_r{}^2 - \omega^2 + j2\xi_r \omega_r \omega) + j2\omega R_1 k_r \chi_r^p}$ (19)

These equations are the final equations used in Figures 2-5. They are then compared to the results from the reference paper [9].

## 3. Results and Discussion

Within this research, the cantilever beam piezoelectric design seems to be simple and effective at utilizing minimal frequencies that would be present on a bridge structure [9]. By coding a mathematical modeling of the voltage produced by this design, further experimental tests and final hardware results will be able to be better understood and compared with theoretical numbers.

Due to a few variables needing a random value in their place, the resultant numbers are not entirely accurate to the journals studied, however, as can be seen in the graphs below, they bear an uncanny resemblance in curve pattern. This eludes that with all the correct information, the model produced should provide accurate results.

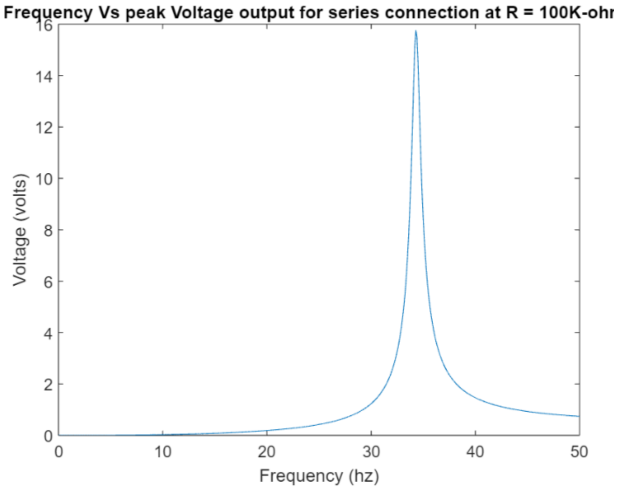


**Figure 2 Voltage vs Frequency Graph of the modeling results of the series connection at a load resistance of 100k-ohms**

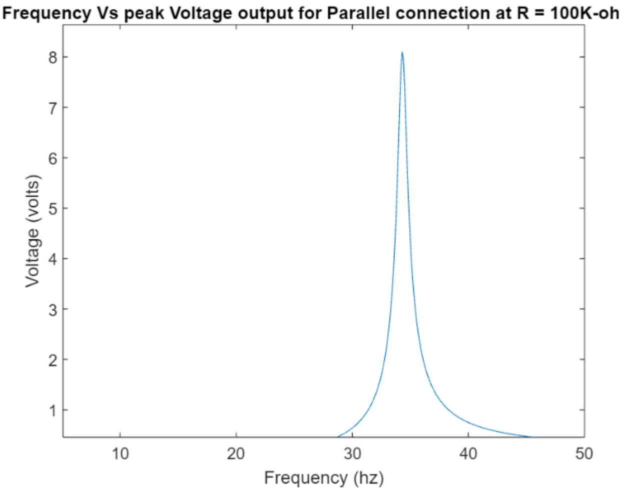


**Figure 3 Voltage vs Frequency Graph of the modeling results of the parallel connection at a load resistance of 100k-ohms**

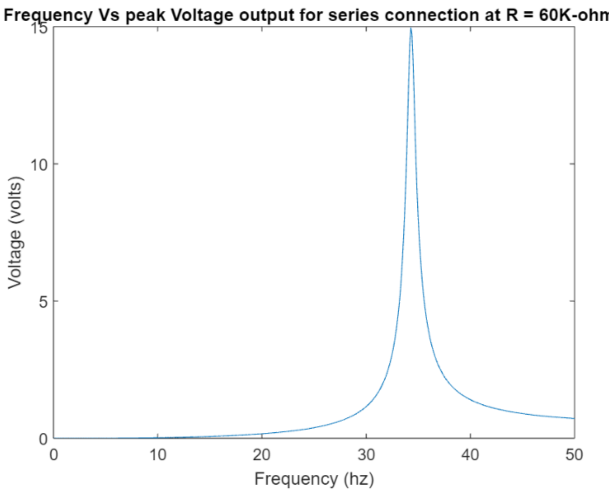


**Figure 4 Voltage vs Frequency Graph of the modeling results of the series connection at a load resistance of 60k-ohms**

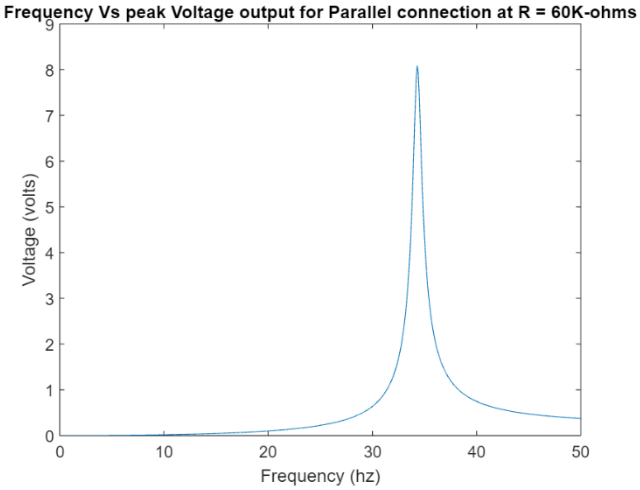


**Figure 5 Voltage vs Frequency Graph of the modeling results of the parallel connection at a load resistance of 60k-ohms**

These graphs can be compared to the results from the reference paper [9]:

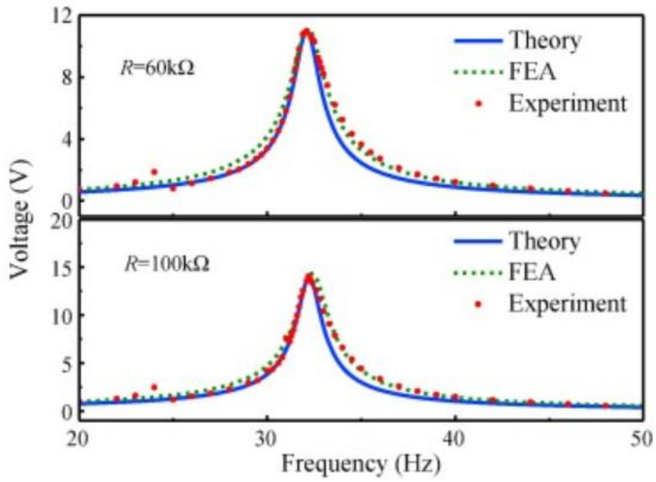


**Figure 6 Reference Voltage vs Frequency Data from paper [9]**

The modeling results observe the same pattern as the experimental and theoretical results that the authors of the reference paper discuss.

The similarities between the generated mathematical modeling graphs and the reference data, it is obvious that the modeling is accurate in replicating the frequency dependent voltage response. With all variables known, the model can be trusted to give a relatively accurate output voltage based on the properties of the designed piezoelectric energy harvester.

With this information on bridge vibrations, harvester design, and frequency-based voltage behavior, an experimental model will continue to be developed, and then the project will move into a final design stage in the future. The information from this study will be helpful as a base of knowledge about the piezoelectric energy harvesting design.

## 4. Conclusion

While the resultant quantitative outcomes of mathematical modeling do not display energy output, this study is an important step in the continued research of design optimization for PEH's. It serves as a fundamental model that can be utilized to predict the voltage output of a PEH in testing and provide important troubleshooting information. The voltage output of this modeling can be compared to a physical test by inputting the physical and electrical properties of the test materials to the coding, and the researcher will be able to know if their outputs are within the expected range. It is important to note that this study refers to only the software validation of the system described. Further research on piezoelectric cantilever beam designs will proceed, using this information as a reference and foundation. This research will then be used to develop both experimental validation and implementation.

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