Applying Dirichlet Series Of Elementary Arithmetic Functions For Studying The Zeta Zeros

Pericles Papadopoulos, George Athanasiou,
Konstantinos Kalkanis and Constantinos Psomopoulos
Department of Electrical and Electronics Engineering,
University of West Attica
ppapadop@uniwa.gr, georgeathanasioy78@gmail.com,
k.kalkanis@uniwa.gr, cpsomop@uniwa.gr

***Abstract*—This paper explores the inversion of Dirichlet series associated with elementary arithmetic functions and their deep connections to the non-trivial zeros of the Riemann zeta function. By analyzing key functions such as the prime characteristic function, the Von Mangoldt function, the Möbius function, and Euler's totient function, we develop a framework that highlights their spectral decomposition in terms of zeta zeros. Through rigorous contour integration techniques and Dirichlet series manipulations, we derive new insights into the analytic structure of these functions. Our findings contribute to the broader understanding of prime number distributions and their link to the critical strip of the Riemann zeta function. Our findings contribute to the broader understanding of prime number distributions and their link to the critical strip of the Riemann zeta function.**

Keywords— Number theory, arithmetic functions, Dirichlet series.

I Introduction

The study of Dirichlet series and their inversions plays a fundamental role in analytic number theory, particularly in understanding the distribution of prime numbers and related arithmetic functions. Among the most significant connections in this field is the interplay between elementary arithmetic functions and the non-trivial zeros of the Riemann zeta function. This relationship has far-reaching implications for key problems in number theory, including the Riemann Hypothesis and prime number theorems.

In this work, we focus on the Dirichlet series representations of several elementary arithmetic functions, including the Von Mangoldt function, the Möbius function, Euler's totient function, and the prime characteristic function. By employing contour integration techniques and leveraging properties of the zeta function, we derive explicit expressions that highlight the role of zeta zeros in shaping these functions' behavior.

A key objective of this paper is to demonstrate how these functions can be expressed in terms of the non-trivial zeros of the Riemann zeta function. This spectral decomposition provides insight into the analytic structure of fundamental arithmetic functions and their asymptotic properties. Furthermore, our results align with classical findings in number theory, such as Landau's results on prime number distributions.

The paper is structured as follows: In Section 2, we review the Dirichlet series of the Von Mangoldt function and its connection to prime numbers. Section 3 examines the prime characteristic function, while Section 4 focuses on the Möbius function and its spectral decomposition. Section 5 discusses Euler's totient function in the context of Dirichlet series. Finally, we summarize our findings and discuss potential extensions of this approach to broader problems in number theory.

II The Technique for Inverting the Dirichlet Series

In this section, we present a method for inverting Dirichlet series, which are central to analytic number theory. The inversion of these series allows us to recover the underlying arithmetic function $f(n)$ from its associated Dirichlet series $D(f,s)$, defined as:

$$D(f,s)=\sum_{n=1}^{\infty}\frac{f(n)}{n^s}, \text{where } s\in\mathbb{C}\#(2.0.1)$$

The inversion problem plays a crucial role in deriving properties of arithmetic functions, particularly those connected to the distribution of prime numbers. We begin by introducing a conformal mapping, which provide a direct approach to inversion.

2.1 The Classical Inversion of the Dirichlet Series

The classical inversion of the Dirichlet series is a cornerstone result in analytic number theory. Given a Dirichlet series of the form

$$D(f,s)=\sum_{n=1}^{\infty}\frac{f(n)}{n^s}, \Re(s)>\sigma_0, \#(2.1.1)$$

where $f(n)$ is an arithmetic function and σ_0 is the abscissa of convergence (see [5]), the goal of the inversion is to recover $f(k)$ for any natural number k.

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The classical inversion formula, derived using the Mellin transform, is given by the limit:

$$f(k)=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}k^{\sigma+it}D(f,\sigma+it)dt\#(2.1.2)$$

This is actually, a special case of Mellin inversion, or more generally, a special case of Fourier inversion. For a proof, the reader might see [2].

In order for the limit to exist, it must obey the following convergence condition:

$$\left|\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}k^{\sigma+it}D(f,\sigma+it)dt\right| \leq k^{\sigma}D(|f|,\sigma)\#(2.1.3)$$

thus, the integral has the asymptotic behavior:

$$\lim_{T\to\infty}\int_{-T}^{T}k^{\sigma+it}D(f,\sigma+it)dt=O(T)\#(2.1.4)$$

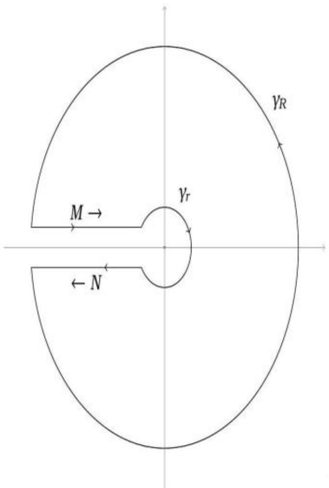
for $k^{\sigma}D(|f|,\sigma)$ convergent. using the exponential conformal mapping $z=\exp(it\pi/T)$ we deduce to the integral below:

$$f(k)=\lim_{T\to\infty}\frac{1}{2\pi i}\oint_{|z|=1}k^{\sigma+\left(\frac{T}{\pi}\right)\log z}D(f,\sigma+ (T/\pi)\log z)\frac{dz}{z}\ (2.1.5)$$

which it can be extended, using analytic continuation to any closed curve C :

$$f(k)=\lim_{T\to\infty}\frac{1}{2\pi i}\oint_{C}k^{\sigma+\left(\frac{T}{\pi}\right)\log z}D(f,\sigma+ (T/\pi)\log z)\frac{dz}{z}\ (2.1.6)$$

The curve C, that it will be used in all the length of this paper is as follows:



Where it is centered to zero with argument $-\pi\leq\arg(z)\leq\pi$, with the small radius $\epsilon>0$ really small and the big radius arbitrary large, with lower bound $R>e^{2}$.

2.2 The Von Mangoldt function

The result of the conformal mapping will be applied on the von Mangoldt function, $\Lambda(n)$, which is defined as (p prime and r integer):

$$\Lambda(n)=\begin{cases}\log(p), & \text{if } n=p^{r}, r\geq 1,\\ 0, & \text{otherwise}.\end{cases}\#\ (2.2.1)$$

this gives the following upper bounds:

$$\Lambda(n)\leq\log(n)\leq\sqrt{n}$$

Thus, this satisfies the convergence condition:

$$|D(\Lambda,s)|\leq\sum_{n\geq 1}\frac{\Lambda(n)}{n^{\sigma}}\leq\sum_{n\geq 1}\frac{\sqrt{n}}{n^{\sigma}}=\zeta(\sigma-1/2)$$

which converges for $\sigma>3/2$. To derive the Von Mangoldt function the integral that follows is used:

$$\Lambda(k)=\lim_{T\to\infty}\frac{1}{2\pi i}\oint_{C}k^{\sigma+\left(\frac{T}{\pi}\right)\log z}D(\Lambda,\sigma+ (T/\pi)\log z)\frac{dz}{z}.\#(2.2.2)$$

The Dirichlet series of the Von Mangoldt function is:

$$D(\Lambda,s)=-\frac{\zeta'(s)}{\zeta(s)}.\#(2.2.3)$$

According to (2.2.3) this Dirichlet series can be expressed as this sum:

$$-\frac{\zeta'(s)}{\zeta(s)}=\frac{1}{s-1}-\sum_{\zeta(\rho)=0}\frac{1}{s-\rho}\#(2.2.4)$$

Where ρ is the roots of the zeta function on the critical strip and the trivial zeroes of the zeta function, that is $\rho=-2m, m\in\mathbb{N}$. To calculate the integral, we use the branch cut contour at $[-\pi,\pi)$ by the keyhole contour C as shown before: To calculate the integral we use the residue theorem [4] at $=e^{(\pi/T)(w-\sigma)}$:

$$\Lambda(k)=\frac{1}{2\pi i}\lim_{T\to\infty}\oint_{C}k^{\sigma+\left(\frac{T}{\pi}\right)\log z}\frac{1}{\sigma+\left(\frac{T}{\pi}\right)\log z-1}-\sum_{\zeta(\rho)=0}\frac{1}{\sigma+(T/\pi)\log z-\rho}\frac{dz}{z}\#(2.2.5)$$

or:

$$\Lambda(k) = \lim_{T\to\infty} \lim_{z=e^{\left(\frac{\pi}{T}\right)(1-\sigma)}} \frac{k^{\sigma+\left(\frac{T}{\pi}\right)\log z}}{z} \frac{z - e^{\left(\frac{\pi}{T}\right)(1-\sigma)}}{\sigma + \left(\frac{T}{\pi}\right)\log z - 1} - \sum_{\zeta(\rho)=0} \lim_{z=e^{\left(\frac{\pi}{T}\right)(\rho-\sigma)}} \frac{k^{\sigma+\left(\frac{T}{\pi}\right)\log z}}{z} \frac{z - e^{(\pi/T)(\rho-\sigma)}}{\sigma + (T/\pi)\log z - \rho} \#(2.2.6)$$

or, by the definition of the derivative:

$$\Lambda(k) = \lim_{T\to\infty} \frac{\pi}{T}k - \frac{\pi}{T}\sum_{m=1}^{\infty} k^{-2m} - \frac{\pi}{T} \sum_{\substack{\zeta(\rho)=0 \\ 0<\Re(\rho)<1 \\ -T\leq\Im(\rho)\leq T}} k^{\rho} \#(2.2.7)$$

but, since $T \to +\infty$ and $k \ll T$, we deduce that:

$$\Lambda(k) = \lim_{T\to\infty} O\left(\frac{1}{T}\right) - \frac{\pi}{T} \sum_{\substack{\zeta(\rho)=0 \\ 0<\Re(\rho)<1 \\ -T\leq\Im(\rho)\leq T}} k^{\rho} \#(2.2.8)$$

And since $\lim_{T\to\infty} O\left(\frac{1}{T}\right) \to 0$, the result is as follows:

$$\Lambda(k) = \lim_{T\to\infty} -\frac{\pi}{T} \sum_{\substack{\zeta(\rho)=0 \\ 0<\Re(\rho)<1 \\ -T\leq\Im(\rho)\leq T}} k^{\rho} \#(2.2.9)$$

which agrees with Landau's findings in paper [1]. We must note, that since the argument is $-\pi \leq \arg(z) \leq \pi$, the argument of the transformation is $-T \leq \frac{T}{\pi}\arg(z) \leq T$ which restricts the imaginary parts of the zeta zeros on the critical strip to $-T \leq \Im(\rho) \leq T$.

2.3 Prime Characteristic Function

The prime characteristic function is a fundamental arithmetic function that identifies whether a given natural number n is prime. It is defined as:

$$\chi_{\mathbb{P}}(n) = \begin{cases} 1, & \text{if } n \in \mathbb{P} \\ 0, & \text{otherwise} \end{cases} \#(2.3.1)$$

where $\mathbb{P}$ denotes the set of prime numbers. According to [6], it can be expressed as:

$$\chi_{\mathbb{P}}(n) = \frac{1}{\log(n)} \sum_{\substack{m\geq 1 \\ n^{1/m}\in\mathbb{N}}} \Lambda(n^{1/m})\mu(m) \#(2.3.2)$$

where the summation is taken over all $m \geq 1$ such that $n^{1/m}$ is a natural number. This compact form emphasizes the role of the Möbius function $\mu(m)$ in eliminating contributions from composite powers of primes.

By changing the order of summation and applying the Taylor expansion for the exponential function, we obtain the following result:

$$F(n; s) = \sum_{m=1}^{\infty} \mu(m) n^{\frac{s}{m}} = \sum_{k=1}^{\infty} \frac{\log(n)^k}{k!\,\zeta(k)} s^k \#(2.3.3)$$

Furthermore, since $\inf_{n\in\mathbb{Z}^+} \zeta(n) = 1$, we obtain:

$$|F(n; s)| < |n^s|.$$

According to Landau [1], the prime characteristic function can also be expressed as a summation involving the nontrivial zeros ρ of the Riemann zeta function $\zeta(s)$. Specifically, we have:

$$\chi_{\mathbb{P}}(n) = \lim_{T\to\infty} -\frac{\pi}{\log(n)\,T} \sum_{\substack{\zeta(\rho)=0 \\ -T\leq\Im(\rho)\leq T \\ 0<\Re(\rho)<1}} F(n; \rho) \#(2.3.4)$$

where the summation is taken over all nontrivial zeros $\rho = \beta + i\gamma$ of $\zeta(s)$, with $0 < \Re(\rho) = \beta < 1$ and $\Im(\rho) = \gamma$.

2.4 Möbius Function

The Möbius function $\mu(n)$ is an important multiplicative function in number theory and is defined as follows:

$$\mu(n) = \begin{cases} (-1)^{\omega(n)}, & \text{if } p^2 \nmid n \forall p \in \mathbb{P}, \\ 1, & \text{if } n = 1, \\ 0, & \text{otherwise}, \end{cases} \#(2.4.1)$$

where $\omega(n)$ is the number of distinct prime factors of n. The Möbius function satisfies the following key properties:

- The Möbius inversion formula:

$$\delta_{n,1} = \sum_{d|n} \mu(d) \#(2.4.2)$$

where $\delta_{n,1}$ is the Kronecker delta function.

- The Dirichlet series representation of $\mu(n)$:

$$\frac{1}{\zeta(s)} = D(\mu, s) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}, \Re(s) > 1 \#(2.4.3)$$

In order to find a form concerning the zeta zeros of the Möbius function, it will be derived by the following integral:

$$\mu(k) = \lim_{T\to\infty} \frac{1}{2\pi i} \oint_C k^{\sigma+\left(\frac{T}{\pi}\right)\log z} D(\mu, \sigma + (T/\pi)\log z) \frac{dz}{z} \#(2.4.4)$$

Which satisfies the convergence condition:

$$|D(\mu, s)| \leq \sum_{n=1}^{\infty} \frac{|\mu(n)|}{n^{\sigma}} \leq \zeta(\sigma) \#(2.4.5)$$

since $|\mu(n)| \leq 1$ which converges for $\sigma > 1$. Working in the same way as the Von Mangoldt function was derived by using the residue theorem close to the non-trivial zeros by a Taylor expansion of the zeta function, we get the result:

$$\mu(k) = \lim_{T\to\infty} \frac{\pi}{T} \sum_{\substack{\zeta(\rho)=0 \\ 0<\mathcal{R}(\rho)<1 \\ -T<\Im(\rho)<T}} \frac{k^\rho}{\zeta'(\rho)} \#(2.4.6)$$

2.5 Euler's totient function

In number theory, Euler's totient function counts the positive integers up to a given integer n that are relatively prime to n. $\phi(n)$ is the Euler totient function, defined by:

$$\phi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right) \#(2.5.1)$$

The Dirichlet series for $\phi(n)$ may be written in terms of the Riemann zeta function as:

$$D(\phi, s) = \sum_{n=1}^{\infty} \frac{\phi(n)}{n^s} = \frac{\zeta(s-1)}{\zeta(s)}, \Re(s) > 2\#(2.5.2)$$

with the convergence condition:

$$|D(\phi, s)| \leq \sum_{n=1}^{\infty} \frac{\phi(n)}{n^\sigma} = \frac{\zeta(\sigma-1)}{\zeta(\sigma)}$$

which is convergent for $\sigma > 2$. To find another form of the Euler's totient function, the integral to be used is the following:

$$\phi(k) = \lim_{T\to\infty} \frac{1}{2\pi i} \oint{}_C k^{\sigma+\left(\frac{T}{\pi}\right)\log z}$$
$$D(\phi, \sigma + (T/\pi)\log z)\frac{dz}{z}\#(2.5.3)$$

Working in a same manner as for the Möbius function, the Euler's totient function according to the non-trivial zeros of the zeta function, gets the form:

$$\phi(k) = \lim_{T\to\infty} \frac{\pi}{T} \sum_{\substack{\zeta(\rho)=0 \\ 0<\Re(\rho)<1 \\ -T<\Im(\rho)<T}} k^\rho \frac{\zeta(\rho-1)}{\zeta'(\rho)} \#(2.5.4)$$

where we take the analytic continuation of the zeta function.

III Conclusions and Final Remarks

In this paper, we investigated the inversion of Dirichlet series associated with elementary arithmetic functions and their deep connections to the non-trivial zeros of the Riemann zeta function. Through rigorous contour integration techniques and spectral decomposition methods, we demonstrated how fundamental arithmetic functions such as the Von Mangoldt function, the Möbius function, Euler's totient function, and the prime characteristic function can be expressed in terms of zeta zeros.

A key outcome of our study is the explicit representation of these functions via the non-trivial zeros of the Riemann zeta function, reinforcing their connection to the critical strip. Our findings align with classical results in number theory, including Landau's work on prime number distributions, and provide further insight into the intricate relationship between arithmetic functions and the analytic structure of the zeta function.

The results presented here contribute to a broader understanding of the spectral nature of arithmetic functions. Future research could explore further generalizations of these techniques, including applications to other special functions in number theory, as well as deeper implications for open problems such as the Riemann Hypothesis.

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