A Ideal Integral Bochner Type On Banach Space

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Abstract— In this paper we propose on type of Bochner integral integration in concept of ideal convergence.This wants to construct a new convergence of functions in Banach spase to definite the measurable functions.The main result is construction on type of Bochner as the Ideal integral, continuing the results of Boccuto and Balcerzak.

Keywords—Bohner type ideal integrals, I-convergence,I-measurable function

Introduction :

This paper was inspired by [7] and [5] where the concept of I-conergence of the sequences of real numbers and I-convergence of the function of real valued.We will often quote some results from [5] that can be transferred to function in Banah space.In [7] it is shown that our I-convergence is, in a sense, equivalent to μ-statistical convergence of J. Connor ([15]). The concept of statistical convergence is introduced in [9] and [13] and developed in [17] . The concept of I-convergence is a generalization of statistical convergence and it is based on the notion of the ideal I of subsets of the set N of positive integers.

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PRELIMINARIES

(A) LET Y BE A SET THAT IS NOT THE EMPTY SET , Y≠∅. FAMILY$\mathfrak{I}\subset\Pi$(Y) IS CALLED*IDEAL OF THE SET* YIF AND ONLY IF,THAT FOR A, B$\in\mathfrak{I}$ IT FOLLOWS THAT, A∪B$\in\mathfrak{I}$ AND FOR EVERY A$\in\mathfrak{I}$AND B ⊂ A WE WILL HAVE B $\in\mathfrak{I}$.

(B) THE IDEAL $\mathfrak{I}$ IS CALLED *NON-TRIVIAL*IFFAND ONLY IF,$\mathfrak{I}$≠∅AND Y$\notin\mathfrak{I}$. A NON-TRIVIAL IDEAL IS CALLED *ACCEPTABLE*WHEN IT CONTAINS THE SETS WITH ONLY ONE POINT ON IT.

LET(T, Σ, μ)BE A SPACE WITH PROBABILISTIC MEASURE μ, WHERE T IS AN RANDOM SET ON A LINE, Σ-BOREL'S ALGEBRA AND μ IS A DEFINED MEASURE.

Throughout the paper N will denote the set of positive integers. Let be An a subset of ordered set N. It said to have density δ(A) ,if

$$\delta(A) = lim_{n\to\infty} \frac{|A_n|}{n} \text{ where } A_n = \{k < n ; k\in A\}.$$

Definition 1:

The vectorial sequence x is statistically convergent to the vector(element) L of a vectorial normed space if for each ε >0

$$lim_{n\to\infty} \frac{1}{n} |\{k \leq n : ||x_k - L|| \geq \varepsilon\}| = 0$$

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I-Convergence of Sequences of Elements in Banach Space.

Definition 2. A sequence $x = (x_n)$,n∈N of elements of X is said to be I-convergent to L∈ X if and only if for each $\varepsilon > 0$ the set $A(\varepsilon) = \{n\epsilon N : \|x_n - L\| \geq \varepsilon\}$ belongs to I. The element L is called the I-limit of the sequence x = {xn},n∈N. I-lim x_n =L.

Definition 3. . A sequence $x = (x_n)$,n∈N of elements of X is said to be I-Caushy if for each $\varepsilon > 0$ there exists q ∈ N such that $\{n\epsilon N : \|x_n - x_q\| \geq \varepsilon\} \in I$.

Definition 4. A sequence $x = (x_n)$,n∈N is called weackly I-covergent if the sequence $x^*(x_n)$ is I-convergent for every $x^* \in X^*$.

Now, we deals with generalization of Ideal convergence of functions on normed space.
The sequence of functions {fk} contains the functions with value in vectorial space.

Definition 4: The function $f: T \to X$, where X is a vector space is called *a simple function* according to μ, if for every family of measurable sets $\{E_i\}$that have no common point ,so$E_i \subset T$ and $E_i \cap E_j = \emptyset$, for$i \neq j$,where$T = \bigcup_{i=1}^{n} E_i$and $f(t) = x_i$, for $t \in E_i$, i=1, 2,..., n.
As we know before,the simple function is defined $f(t) = \sum_{i=1}^{n} x_i \chi_{E_i}$, where $\chi_{E_i}$is an characteristic function of E_i.

Definition 5 : The function$f: T \to X$ is called $\mathfrak{I}$-*measurable on* T , if for every t∈T, $\varepsilon > 0$ and $A \subset \mathfrak{I}$ there is a sequence of simple functions $f_n: T \to X$ for which we have

$$\|f_n(t) - f(t)\| < \varepsilon \text{ for } n \in \mathbb{N}\backslash A.$$

Proposition1:The linear combination of functions $\mathfrak{I}$-measurable (measurable ideals) is an $\mathfrak{I}$-measurable function.

Proof. Let f and g be two functions I-measurable .For the function f there exists the sequence of functions $f_n(t)$ suth that I-lim $f_n(t) = f(t)$. This means that for each ε>0 and for $\frac{\varepsilon}{2|\alpha|} > 0$ and

exists $A_1 \in \mathfrak{I}$ such that ||fn(t) - f(t) || < $\frac{\varepsilon}{2|\alpha|}$ for $n \in$

$\mathbb{N}\backslash A_1$,tϵT.Similarly for the function I-measurable g(x),

there exists the sequence of functions $g_n(t)$ suth that

I-lim $g_n(t) = g(t)$.This means that for each ε>0 and for

$\frac{\varepsilon}{2|\alpha|} > 0$,exists $A_2 \in \mathfrak{I}$ such that ||gn(t) - g(t) || < $\frac{\varepsilon}{2|\alpha|}$ for

$n \in \mathbb{N}\backslash A_2$,tϵT , $\mathbb{N}\backslash A_1 \cup \mathbb{N}\backslash A_2 \subset \mathbb{N}\backslash(A_1 \cup A_2)$ for

$n \in \mathbb{N}\backslash(A_1 \cup A_2)$ and t ∊T we have $\|(\alpha f_n(t) +$

$\beta g_n(t)) - (\alpha f + \beta g)\| \leq |\alpha|\|f_n(t) - f(t)\| +$

$|\beta|\|g_n(t) - g(t)\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$

Hence, we get I – lim (αfn(t) + βgn(t)) = αf(t) + βg(t).

Definition 6. The subsequence $(f_{n_k})_{k\in\mathbb{N}}$of the sequence$(f_n)_{n\in\mathbb{N}} \overset{\mathfrak{I}}{\to} f$ is called fundamental if,for $A' = \{n_1 < n_2 < \cdots < n_k < \cdots\}; f_{n_k} \overset{\mathfrak{I}}{\to} f$ for $n \in \mathbb{N}\backslash A'$ where$A' \subset A$.

Definition 7:.Let(I, Σ, μ)be a measurable complete space with a non-negative measure. The sequence of measured functions $(f_n)_n$in I is ***$\mathfrak{I}$-convergent according to the measure μ*** to the function f, if for each ε> 0 and σ> 0 there is an essential subsequence$(f_{n_k})_k$of the sequence$(f_n)_n$ such that: $\mu\{t: \|f_{n_k}(t) - f(t)\| \geq \sigma\} < \varepsilon$ for $n_k \in \mathbb{N}\backslash A'$ and t ∊I. We denote $f_n(t) \overset{\mathfrak{I}-\mu}{\longrightarrow} f(t)$.

Definition 8. The sequence of measured functions$(f_n)_n$with values in $Banah\ space$ is called $\mathfrak{I}$-*fundamental according to the measure* μ ,$S \subset \mathfrak{I}$,if there is a natural number $(\sigma, S) \subset \mathbb{N}\backslash A$ and there is a subsequence $(f_{n_k})_k$of$(f_n)_n, if\ \forall\varepsilon > 0$ and $\sigma > 0$, $\mu\{t: \|f_{n_k}(t) - f(t)\| \geq \sigma\} < \varepsilon.$

Proposition 2. If the sequence $(f_n)_n$is $\mathfrak{I}$-convergent to f in $\mathfrak{I}$ it is $\mathfrak{I}$-fundamental.

Proof: Let be $(f_{n_k}(t))_k$ subsequence fundamental of the sequence $(f_n(t))$. We choose a number $N \in \mathbb{N}, N \in \mathbb{N}\backslash A$ and consider the inequality: $\|f_{n_k}(t) - f_N(t)\| \leq \|f_{n_k}(t) - f(t)\| + \|f(t) - f_N(t)\|$ from here we can write $\{t: \|f_{n_k}(t) - f_N(t)\| \geq \sigma\} \subset \left\{t: \|f_{n_k}(t) - f(t)\| \geq \frac{\sigma}{2}\right\} \cup \left\{t: \|f(t) - f_N(t)\| \geq \frac{\sigma}{2}\right\}.$
We get for $n_k \in \mathbb{N}\backslash A'$, $A' \subset A$.
$\mu\{t: \|f_{n_k}(t) - f_N(t)\| \geq \sigma\} \leq \mu\left\{t: \|f_{n_k}(t) - f(t)\| \geq \frac{\sigma}{2}\right\} + \mu\left\{t: \|f(t) - f_N(t)\| \geq \frac{\sigma}{2}\right\}.$

Proposition 3. $\mathfrak{I}$-limit of the sequence$(f_n(t))_n$according to the measure μ is unique with the proximity of equivalence.

Proof: Let us assume that the statement is not true.This means that sequence $(f_n)_n$ is I-convergent in two different limits $f_1(t)$ and $f_2(t)$. For every $\varepsilon > 0$ and $\sigma > 0$ there exists the fundamental subsequence such that $\mu\left\{t: \|f_{n_k}(t) - f_1(t)\| \geq \frac{\sigma}{2}\right\} < \frac{\varepsilon}{2}$ for $n_k \in \mathbb{N}\backslash A_1'$ where $A_1' \subset A$ and $\mu\left\{t: \|f_{n_k}(t) - f_2(t)\| \geq \frac{\sigma}{2}\right\} < \frac{\varepsilon}{2}$ for $n_k \in \mathbb{N}\backslash A_2'$ where $A_2' \subset A$ and t ∊ T.

We have
$\{t: \|f_1(t) - f_2(t)\| \geq \sigma\} \subset \left\{t: \|f_{n_k}(t) - f_1(t)\| \geq \frac{\sigma}{2}\right\} \cup \left\{t: \|f_{n_k}(t) - f_2(t)\| \geq \frac{\sigma}{2}\right\}.$

For $n_k \in \mathbb{N}\backslash(A_1' \cup A_2'$ where $A_1' \cup A_2' \subset A$ we have
$\mu\{t: \|f_1(t) - f_2(t)\| \geq \sigma\} \leq \mu\left\{t: \|f_1(t) - f_{n_k}(t)\| \geq \frac{\sigma}{2}\right\} + \mu\left\{t: \|f_{n_k}(t) - f_2(t)\| \geq \frac{\sigma}{2}\right\} < \varepsilon.$

The above inequality shows that $f_1(t)$ and $f_2(t)$ can be different only in a set of zero measure.

Proposition 4.
If the sequence$(f_n)_n$is a $\mathfrak{I}$ –fundamental-sequence on Banah space , then there exists an $\mathfrak{I} - \lim_k \int f_k(t)d\mu$.
Proof: $(f_n)_n$ is I-fundamental sequence , for every $\varepsilon > 0$ exists $k \in \mathbb{N}\backslash A$ and N fixed natural we have $\|f_k(t) - f_N(t)\| < \frac{\varepsilon}{\mu(T)}$ a.a.k $k \in \mathbb{N}\backslash A$.
We have

$\|\int f_k(t)d\mu - \int f_N(t)d\mu\| \leq \int\|f_k(t) - f_N(t)\|d\mu \leq \|f_k(t) - f_N(t)\|\mu(T) < \varepsilon.$

Definition 9.
The function $f: T \to X$ is called weakly $\mathfrak{I}$-measurable if for each $x^*\epsilon X^*$ the real function $x^*(f): T \to R$ is $\mathfrak{I}$-measurable.

Definition 10. The function $f: T \to X$ is called $\mathfrak{I}$-*Bochner integrable,*if there is a fundamental sequence $\mathfrak{I}$-measurable such that,

a) $(f_k)_k$is $\mathfrak{I}$-convergent to f.

b) $\mathfrak{J} - \lim_k \int ||f_k(t) - f_N(t)||d\mu = 0$ almost everywhere $\mathfrak{J} - B - \int f(t)d\mu$ and is called $\mathfrak{J}$-Bochner integral.

The sequence function $(f_n)_n$ is called determines of the function f.

Proposition 5. If (f_n) and (g_n) as I-fundamental sequences are determinants of the same function f than

I-lim$\int f_n(t)d\mu$ = I-lim $\int g_n(t)d\mu$

Proof:

The inequality $||f_n(t)-g_n(t)|| \le ||f_n(t)-f(t)||+||f(t)-g_n(t)||$ shows that (fn) and (gn) are equivalent

I-lim||fn-gn|| =0 or for every ε>0, $||f_k(t) - g_k(t)|| < \varepsilon$ a.a.k.

By the definition of integral

$||\int f_n(t) - (I - lim \int f_n(t)d\mu))|| < \varepsilon$ and

$|| \int g_n(t) - (I - lim \int g_n(t)d\mu))|| < \varepsilon$ a.a.k and $t \in T$.

Consider the difference

$||(I - lim \int f_n(t)d\mu) - (I - lim \int g_n(t)d\mu)|| \le$
$||(I - lim \int f_n(t)d\mu) - \int g_n(t)d\mu)|| + ||\int f_n(t)d\mu - \int g_n(t)d\mu|| + ||\int g_n(t)d\mu) - (st - lim \int g_n(t)d\mu)|| < 3\varepsilon$

Proposition 11: If the function f is I-Bohner integrable then the function ||f|| is also I-Bohner integrable.

Proof. Following definition of I-integrability, there exists the sequence of fundamental functions fk I-convergent almost everywhere and a.a.k. to the function f and $\int ||f_k(t) - f_N(t)||d\mu < \varepsilon$ a.a.k We consider the inequality $|||f_k||-||f|| \le ||f_k - f||$,

Hence I-lim $f_k(t)=f(t)$ a.a.k I-lim$||f_k(t)||=||f(t)||$ a.a.k

Inequality $\int |||f_k|| - ||f_N|||d\mu \le \int ||f_k - f_N|| d\mu$,Shows that ||f|| is I-integrable.

The equality $I - \lim_n \int f_n d\mu = (IB) \int f d\mu$ where (fn) is sequence of functions determinant to f and the well known properties of classical integral allow us to formulate the following properties of **I- Bohner integral.**

- (IB) $\int (\alpha f(t) + \beta g(t))d\mu = (IB)\alpha \int f(t)d\mu + (IB)\beta \int g(t) d\mu$

- $(IB)|| \int_S f d\mu|| \le (IB) \int_S ||f|| d\mu$

This inequality we obtain from the same inequality for fundamental functions and isotonic property $|| \int f_k||d\mu \le \int ||f_k|| d\mu$.

(III) The inequality for the simple determinant functions $||fk|| \le ||gk||$ a.a.k implies

$\int ||f_k||d\mu \le \int ||g_k|| d\mu$ a.a.k

Isotonic property of integrals gives $(IB) \int ||f||d\mu \le (IB) \int ||g|| d\mu$

Theorem. Let f(x) be the function with value in separable Banach space and I-measurable by a probability measure. If for almost all t∈T holds the inequality $||f(t)|| \le g(t)$,

where g(t) is a function I-integrable then the function f(t) is I- integrable.

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