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On Semi–Normed Spaces, Via Semi–Pre–Inner Product And Orthogonality According To An Index

[1]Artur Stringa
[1]Department of Mathematics, Faculty of Natural Science, University of Tirana, Tirana 1057 Albania
artur.stringa@fshn.edu.al

[2]Arbër Qoshja
[2]Department of Applied Mathematics, Faculty of Natural Science, University of Tirana, Tirana 1057 Albania
arber.qoshja@fshn.edu.al

Abstract- In this paper, using the concept of semi – pre – inner product, some results on linear dependence and independence in semi-normed spaces are given. These results are characterized in terms of semi – inner products. We set up the concept of orthogonality and transversality by an index, obtaining similar results on linear dependence and independence in semi – normed spaces.

Keywords: semi – norm, semi – pre – inner product, linearly independent, linearly dependent, orthogonality according to an index, transversality according to an index.

I. Introduction

Definition 1.1 [3]
Let X be a real vector space. We shall say that a real semi – inner product (in short s.i.p.) is defined on X, if to any $x, y \in X$ there corresponds a real number $\langle x, y\rangle$ and the following properties holds:

(1) **(i)** $\langle x+y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$

(ii) $\langle \lambda x, y\rangle = \lambda \langle x, y\rangle$, for $x, y, z \in X$ and $\lambda \in R$,

(2) $\langle x, x\rangle > 0$, for $x \neq 0$,

(3) $\langle x, y\rangle^2 \leq \langle x, x\rangle \langle y, y\rangle$.

The pair $(X, \langle \cdot, \cdot \rangle)$ is called a semi – inner product space (in short s.i.p.s.).

An s.i.p.s. is a normed vector space with $\|x\| = \langle x, x\rangle^{1/2}$ **[3].**

It is further prove in **[3]** that every normed vector space can be made into s.i.p.s.

Aiming to generalize the condition (2) in the definition of s.i.p., we have introduction in **[1]** the concept of the semi – pre – inner product function, which is a generalization of the s.i.p. function's concept:

Definition 1.2 [1]
Let X be a real vector space. Consider a functional defined on $X \times X$ as follows: $X \times X \to R$
$(x, y) \mapsto [x, y]$.

If $[x, y]$ satisfies the postulates:

(1) $[x, x] \geq 0,\ x \in X$,

(2) $[\lambda x, y] = \lambda [x, y]$, $\lambda \in R$ and $x, y \in X$,

(3) $[x+y, z] = [x, z] + [y, z]$ $x, y \in X$,

(4) $[x, y]^2 \leq [x, x][y, y]$.

then we say that is a semi – pre – inner product on X (in short s.p.i.p.).

The pair $(X, [\cdot, \cdot])$ is called a semi – pre – inner product space (in short s.p.i.p.s.).

The following theorem proves the existence of the s.p.i.p.

Theorem 1.1 [1]
Let X be a real vector space. For every semi – norm function **p** in X, there is a s.p.i.p. $[\cdot, \cdot]$ in X, such that $p^2(x) = [x, x],\ x \in X$.

II. Main Results

Let X be a real vector space and $(X, \{p_\alpha\}_{\alpha \in \mathbf{A}})$ be a semi – normed space, where $\{p_\alpha\}_{\alpha \in \mathbf{A}}$ is a family of semi – norms on X and $\mathbf{A}$ is an index set. For every $\alpha \in \mathbf{A}$, let us denote by $[\cdot, \cdot]_\alpha$ the s.p.i.p., corresponding to the semi – norm p_α.

Definition 2.1

Let be $x_1, x_2, \ldots, x_n \in X$ and $\alpha \in \mathbf{A}$.

The real number, which is denoted by the symbol $D_\alpha(x_1, x_2, ..., x_n)$ and which is equal to:

$$D_\alpha(x_1, x_2, ..., x_n) = \det \begin{bmatrix} [x_1, x_1]_\alpha & [x_1, x_2]_\alpha & .... & [x_1, x_n]_\alpha \\ [x_2, x_1]_\alpha & [x_2, x_2]_\alpha & .... & [x_2, x_n]_\alpha \\ .... & .... & .... & .... \\ [x_n, x_1]_\alpha & [x_n, x_2]_\alpha & .... & [x_n, x_n]_\alpha \end{bmatrix},$$

is called Gram's determinant by index α of vectors $x_1, x_2, ..., x_n$.

Let be $\lambda_1, \lambda_2, ..., \lambda_n \in R$, such that holds equation:

$$\lambda_1 x_1 + \lambda_2 x_2 + ... + \lambda_n x_n = 0.$$

From here, using the property (3) of s.p.i.p., it follows that for every $\alpha \in \mathbf{A}$ hold system of equalizations:

$$(1_\alpha) \begin{cases} \lambda_1 [x_1, x_1]_\alpha + \lambda_2 [x_2, x_1]_\alpha + ... + \lambda_n [x_n, x_1]_\alpha = 0 \\ \lambda_1 [x_1, x_2]_\alpha + \lambda_2 [x_2, x_2]_\alpha + ... + \lambda_n [x_n, x_2]_\alpha = 0 \\ .... \\ \lambda_1 [x_1, x_n]_\alpha + \lambda_2 [x_2, x_n]_\alpha + ... + \lambda_n [x_n, x_n]_\alpha = 0 \end{cases}.$$

For all $\alpha \in \mathbf{A}$, system of equations (1_α) can be related to a homogeneous system of n equations with unknowns $\lambda_1^\alpha, \lambda_2^\alpha, ..., \lambda_n^\alpha$, by determinant $D_\alpha(x_1, x_2, ..., x_n)$:

$$(2_\alpha) \begin{cases} \lambda_1^\alpha [x_1, x_1]_\alpha + \lambda_2^\alpha [x_2, x_1]_\alpha + ... + \lambda_n^\alpha [x_n, x_1]_\alpha = 0 \\ \lambda_1^\alpha [x_1, x_2]_\alpha + \lambda_2^\alpha [x_2, x_2]_\alpha + ... + \lambda_n^\alpha [x_n, x_2]_\alpha = 0 \\ .... \\ \lambda_1^\alpha [x_1, x_n]_\alpha + \lambda_2^\alpha [x_2, x_n]_\alpha + ... + \lambda_n^\alpha [x_n, x_n]_\alpha = 0 \end{cases}.$$

For all $\alpha \in \mathbf{A}$, it is clear that if $D_\alpha(x_1, x_2, ..., x_n) \neq 0$, then system (2_α) has only one solution, which is the trivial solution and if $D_\alpha(x_1, x_2, ..., x_n) = 0$, then system (2_α) has infinitely solutions. If $\lambda_1, \lambda_2, ..., \lambda_n$ are n real numbers, which prove the equation:

$$\lambda_1 x_1 + \lambda_2 x_2 + ... + \lambda_n x_n = 0.$$

Then for all $\alpha \in \mathbf{A}$ system (2_α) has as a solution of the ordered system $(\lambda_1, \lambda_2, ..., \lambda_n)$.

This helps to prove the following theorem:

Theorem 2.1

If the system of vectors $x_1, x_2, ..., x_n$ of X is a linearly dependent system, then for each index $\alpha \in \mathbf{A}$, $D_\alpha(x_1, x_2, ..., x_n) = 0.$

Proof:

We assume that the system of vectors $x_1, x_2, ..., x_n$ from X is a linearly dependent system. Then, there are real constants $\lambda_1^0, \lambda_2^0, ..., \lambda_n^0$, where at least one of them is different from 0, such that there hold an equation:

$$\lambda_1^0 x_1 + \lambda_2^0 x_2 + ... + \lambda_n^0 x_n = 0.$$

This shows that there is true the equation:

$$\lambda_1 x_1 + \lambda_2 x_2 + ... + \lambda_n x_n = 0,$$

for $\lambda_1 = \lambda_1^0, \lambda_2 = \lambda_2^0, ..., \lambda_n = \lambda_n^0$.

From here it follows that for every $\alpha \in \mathbf{A}$ the system (2_α) there is a ordered system as a solution $(\lambda_1^0, \lambda_2^0, ..., \lambda_n^0)$. Since the system (2_α) is a homogeneous system and the ordered system $(\lambda_1^0, \lambda_2^0, ..., \lambda_n^0)$ is nontrivial, it follows that for every $\alpha \in \mathbf{A}$ we have that $D_\alpha(x_1, x_2, ..., x_n) = 0.$

Corollary 2.1

Let suppose that $x_1, x_2, ..., x_n$ is a vectors system in X. If exists the index $\alpha \in \mathbf{A}$, such that $D_\alpha(x_1, x_2, ..., x_n) \neq 0,$ then the system $x_1, x_2, ..., x_n$ is linearly independent.

Note 2.1

We assume that space $(X, \{p_\alpha\}_{\alpha \in \mathbf{A}})$ is separate, i.e. we assume that this space satisfies the condition: If $x \in X$ and $x \neq 0$, then exist index $\alpha \in \mathbf{A}$, such that

$$p_\alpha(x) \neq 0.$$

For all $\alpha \in \mathbf{A}$, since p_α is a semi – norm on X, we have that $p_\alpha^{-1}\{0\} = \{x \in X / p_\alpha(x) = 0\}$, is a closed subspace of the vector space X. We note that the relation:

$$x \sim y \Leftrightarrow (x - y) \in p_\alpha^{-1}\{0\},$$

is an equivalence relation on X.

Let us denote by X_α the quotient set $X /_{p_\alpha^{-1}\{0\}}$, regarding this equivalence relation, i.e. $X_\alpha = X /_{p_\alpha^{-1}\{0\}}$. Let be $\alpha \in \mathbf{A}$. For all $x \in X$, we denote by $\dot{x}$ the corresponding equivalence class from X_α of vector x. The function $p_\alpha : X_\alpha \to R^+$, such that for every $x \in X_\alpha$, we have $p_\alpha(x) = p_\alpha(x)$, for some x from the equivalence class x, is a norm on X_α **[1].**

Let be $x \in X$ and $x \neq 0$. Since the space $(X, \{p_\alpha\}_{\alpha \in \mathbf{A}})$ it is separable, there is $\alpha \in \mathbf{A}$, such that $p_\alpha(x) \neq 0$. From the truth of equality $p_\alpha(x) = p_\alpha(x)$, since the function p_α is a norm in X_α, we deduce that $x \neq 0$. Conversely, let be $\alpha \in \mathbf{A}$, $x \in X_\alpha$ and $x \neq 0$. From the truth of equality $p_\alpha(x) = p_\alpha(x)$, we deduce that $x \neq 0$.

Theorem 2.2

Let assume that the space $(X, \{p_\alpha\}_{\alpha \in \mathbf{A}})$ is separable.

The system of vectors $x_1, x_2, \ldots, x_n$ from X is a linearly independent system if and only if exists the index $\alpha \in \mathbf{A}$, such that the system of corresponding equivalence classes of these vectors, $x_1, x_2, \ldots, x_n$, from X_α is a linearly independent system.

Proof:

Necessary condition. Let be $x_1, x_2, \ldots, x_n$ a system linearly independent from X. We assume that for each index $\alpha \in \mathbf{A}$, the system of corresponding equivalence classes of these vectors, $x_1, x_2, \ldots, x_n$, from X_α is a linearly dependent system. Then, based on the assumption made, there are real constants $\lambda_1, \lambda_2, \ldots, \lambda_n$, where at least one of them is different from 0, such that holds $\lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_n x_n = 0$. Since the system of vectors $x_1, x_2, \ldots, x_n$ is a linearly independent system, there is true the equation $\lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_n x_n \neq 0$.

As the space $(X, \{p_\alpha\}_{\alpha \in \mathbf{A}})$ is separable, exists index $\alpha \in \mathbf{A}$, such that:

$$p_\alpha(\lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_n x_n) \neq 0.$$

We consider X_α corresponding. Based on Note 2.1., it follows that:

$$\lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_n x_n \neq 0,$$

where $\lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_n x_n$ is the corresponding equivalence class of the vector $\lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_n x_n$ from X_α. The equivalence is true:

$$\lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_n x_n \neq 0$$

$\Leftrightarrow \lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_n x_n \neq 0$. **Contradiction.**

Sufficient condition. Let be $x_1, x_2, \ldots, x_n$ a system of vectors from X, such that the equation holds:

$$\lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_n x_n = 0,$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are n real constants.

We assume that exists the index $\alpha \in \mathbf{A}$, such that the system of corresponding equivalence classes of these vectors, $x_1, x_2, \ldots, x_n$, from X_α is a linearly independent system.

If at least one of the numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ is different from 0, then the equation holds:

$$\lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_n x_n \neq 0,$$

which is equivalent to equation:

$$\lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_n x_n \neq 0.$$

Based on Note 2.1., we deduce that

$\lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_n x_n \neq 0$. **Contradiction**.

Theorem 2.3

Let assume that the space $(X, \{p_\alpha\}_{\alpha \in \mathbf{A}})$ is separable.

The system of vectors $x_1, x_2, \ldots, x_n$ from X is a linearly independent system if and only if there exists the index $\alpha \in \mathbf{A}$, such that $D_\alpha(x_1, x_2, \ldots, x_n) \neq 0$.

Proof:

Necessary condition. Let be $x_1, x_2, \ldots, x_n$ a linearly independent system from X.

Based on Theorem **2.2.,** exists the index $\alpha \in \mathbf{A}$, such that the system of corresponding equivalence classes of these vectors, $x_1, x_2, \ldots, x_n$ from X_α, is a linearly independent system. From **[1]** we have that function $p_\alpha : X_\alpha \to R^+$, such that for every $x \in X_\alpha$, $p_\alpha(x) = p_\alpha(x)$, for some x from the equivalence class x, is a norm on X_α. Then, by **[3],** there exists a semi – inner – product on $X_\alpha \times X_\alpha$:

$\langle \cdot,\cdot \rangle_\alpha : X_\alpha \times X_\alpha \to R,$ such that

$p_\alpha(x) = \left(\langle x, x \rangle_\alpha\right)^{1/2}$, for every $x \in X_\alpha$,

and by **[1]**, we have that $[x, y]_\alpha = \langle x, y \rangle_\alpha$.

Since the system $x_1, x_2, ..., x_n$ is a linearly independent system, from **[2]** we get that:

$$D_\alpha(x_1, x_2, ..., x_n) = \det \begin{bmatrix} \langle x_1, x_1 \rangle_\alpha & \langle x_1, x_2 \rangle_\alpha & .... & \langle x_1, x_n \rangle_\alpha \\ \langle x_2, x_1 \rangle_\alpha & \langle x_2, x_2 \rangle_\alpha & .... & \langle x_2, x_n \rangle_\alpha \\ .... & .... & .... & .... \\ \langle x_n, x_1 \rangle_\alpha & \langle x_n, x_2 \rangle_\alpha & .... & \langle x_n, x_n \rangle_\alpha \end{bmatrix} \neq 0.$$

On the other hand, since the equation holds $D_\alpha(x_1, x_2, ..., x_n) = D_\alpha(x_1, x_2, ..., x_n)$, we will deduce that $D_\alpha(x_1, x_2, ..., x_n) \neq 0.$

Sufficient condition.
It is evident from Corollary 2.1.

Corollary 2.2
Let assume that the space $\left(X, \{p_\alpha\}_{\alpha \in \mathbf{A}}\right)$ is separable.

The system of vectors $x_1, x_2, ..., x_n$ from X is a linearly dependent system if and only if for each index $\alpha \in \mathbf{A}$, holds the equation $D_\alpha(x_1, x_2, ..., x_n) = 0.$

Let us now give the concept of orthogonality and transversality by an index $\alpha \in \mathbf{A}$, to obtain some similar results on linear dependence and independence in semi-normed spaces.

Definition 2.2
Let be $x, y \in X$ and $\alpha \in \mathbf{A}$. The vector x is called orthogonal according to the index α over the vector y, if $[y, x]_\alpha = 0.$ In this case, the vector y is called transversal according to the index α on the vector x.

Definition 2.3
Let be $x, y \in X$. The vector x is called orthogonal over the vector y, is the vector x is orthogonal according to the any index $\alpha \in \mathbf{A}.$ over the vector y. In this case, the vector y is called transversal over the vector x.

Theorem 2.4
Let assume that the space $\left(X, \{p_\alpha\}_{\alpha \in \mathbf{A}}\right)$ is separable and the system of vectors $x_1, x_2, ..., x_n$ from X is a linearly independent system. If the vector $x \in X$ satisfies the conditions:

(i) For each index $\alpha \in \mathbf{A}$;, $p_\alpha(x) \neq 0$;

(ii) It is orthogonal or transversal to each of the vectors $x_1, x_2, ..., x_n$;

then the system of vectors $x, x_1, x_2, ..., x_n$ is a linearly independent system.

Proof:

Since the system of vectors $x_1, x_2, ..., x_n$ is a linearly independent system, then based on Theorem **2.3.**, exists the index $\alpha_0 \in \mathbf{A}$, such that $D_{\alpha_0}(x_1, x_2, ..., x_n) \neq 0.$ We have that:

$$D_{\alpha_0}(x, x_1, ..., x_n) = \det \begin{bmatrix} [x, x]_{\alpha_0} & [x, x_1]_{\alpha_0} & .... & [x, x_n]_{\alpha_0} \\ [x_1, x]_{\alpha_0} & [x_1, x_1]_{\alpha_0} & .... & [x_1, x_n]_{\alpha_0} \\ .... & .... & .... & .... \\ [x_n, x]_{\alpha_0} & [x_n, x_1]_{\alpha_0} & .... & [x_n, x_n]_{\alpha_0} \end{bmatrix}.$$

If the vector x is orthogonal to each of the vectors $x_1, x_2, ..., x_n$, it follows that for any natural index $k \in \{1, 2, ..., n\}, [x_k, x]_{\alpha_0} = 0.$ In this case, if we expand the determinant $D_{\alpha_0}(x, x_1, x_2, ..., x_n)$ by the first column, we will draw that there holds:

$$D_{\alpha_0}(x, x_1, x_2, ..., x_n) = [x, x]_{\alpha_0} \cdot D_{\alpha_0}(x_1, x_2, ..., x_n).$$

If the vector x is transversal to each of the vectors $x_1, x_2, ..., x_n$, it follows that for any natural index $k \in \{1, 2, ..., n\}, [x, x_k]_{\alpha_0} = 0.$ In this case, if we expand the determinant $D_{\alpha_0}(x, x_1, x_2, ..., x_n)$ by the first row, we will draw that there holds:

$$D_{\alpha_0}(x, x_1, x_2, ..., x_n) = [x, x]_{\alpha_0} \cdot D_{\alpha_0}(x_1, x_2, ..., x_n).$$

As $[x, x]_{\alpha_0} = [p_\alpha(x)]^2 \neq 0,$ it implies that $D_{\alpha_0}(x, x_1, x_2, ..., x_n) \neq 0.$ Based on Theorem **2.3.**, we deduce that the system of vectors $x, x_1, x_2, ..., x_n$ is a linearly independent.

Theorem 2.5
Let assume that the space $\left(X, \{p_\alpha\}_{\alpha \in \mathbf{A}}\right)$ is separable. If the system of vectors $x_1, x_2, ..., x_n$ from X is linearly independent, then there is a linearly

independent system of n vectors, $y_1, y_2, ..., y_n$, from X, and an index $\alpha_0 \in \mathbf{A}$, such that for any two natural indices $i, j \in \{1, 2, ..., n\}$, the equivalence is true:

$$i \neq j \Leftrightarrow [y_i, x_j]_{\alpha_0} = 0.$$

Proof:

Since the system of vectors $x_1, x_2, ..., x_n$ from X is linearly independent, then by Theorem **2.3**. it follows that exists the index $\alpha_0 \in \mathbf{A}$, such that:

$$D_{\alpha_0}(x_1, ..., x_n) = \det \begin{bmatrix} [x_1, x_1]_{\alpha_0} & .... & [x_1, x_i]_{\alpha_0} & ... & [x_1, x_n]_{\alpha_0} \\ [x_2, x_1]_{\alpha_0} & ... & [x_2, x_i]_{\alpha_0} & ... & [x_2, x_n]_{\alpha_0} \\ .... & & .... & ... & .... \\ [x_n, x_1]_{\alpha_0} & ... & [x_n, x_i]_{\alpha_0} & ... & [x_n, x_n]_{\alpha_0} \end{bmatrix} \neq 0.$$

We denote by $A_{1i}, A_{2i}, ..., A_{ni}$, the algebraic complements of the corresponding elements of the column with the index i of the determinant $D_{\alpha_0}(x_1, x_2, ..., x_n)$. For every index $i \in \{1, 2, ..., n\}$, denote by y_i the vector:

$$y_i = A_{i1}x_1 + A_{i2}x_2 + ... + A_{in}x_n.$$

Thus we obtain the system of vectors $y_1, y_2, ..., y_n$ from X. Let be $i, j \in \{1, 2, ..., n\}$.

If $i \neq j$, then we can write that:

$$[y_i, x_j]_{\alpha_0} = A_{i1}[x_1, x_j]_{\alpha_0} + A_{i2}[x_2, x_j]_{\alpha_0} + ... + A_{in}[x_n, x_j]_{\alpha_0}.$$

The right side of the above equation is nothing else than the sum of the products of the column entries with index j of the determinant $D_{\alpha_0}(x_1, x_2, ..., x_n)$ with the corresponding algebraic complements of the column entries with index i of this determinant. Using a property of determinants, since $i \neq j$, we get that:

$$A_{i1}[x_1, x_j]_{\alpha_0} + A_{i2}[x_2, x_j]_{\alpha_0} + ... + A_{in}[x_n, x_j]_{\alpha_0} = 0,$$

Thus, $[y_i, x_j]_{\alpha_0} = 0$. Let assume that

$[y_i, x_j]_{\alpha_0} = 0$. If $i = j$, also using a property of determinants, we get that:

$$[y_i, x_i]_{\alpha_0} = D_{\alpha_0}(x_1, x_2, ..., x_n) \neq 0.$$

Contradiction!

So, $i \neq j$. As conclusion, the statements $i \neq j$ dhe $[y_i, x_j]_{\alpha_0} = 0$ are equivalent statements.

Now, let's prove that the system of vectors $y_1, y_2, ..., y_n$ is linearly indipendent.

Let be $\lambda_1, \lambda_2, ..., \lambda_n \in R$, such that the equation is true:

$$\lambda_1 y_1 + \lambda_2 y_2 + ... + \lambda_n y_n = 0.$$

From here, using the property **(3)** of s.p.i.p., it follows that holds the system of equations:

$$(\mathbf{3}_{\alpha_0}) \begin{cases} \lambda_1[y_1, x_1]_{\alpha_0} + \lambda_2[y_2, x_1]_{\alpha_0} + ... + \lambda_n[y_n, x_1]_{\alpha_0} = 0 \\ \lambda_1[y_1, x_2]_{\alpha_0} + \lambda_2[y_2, x_2]_{\alpha_0} + ... + \lambda_n[y_n, x_2]_{\alpha_0} = 0 \\ .... \\ \lambda_1[y_1, x_n]_{\alpha_0} + \lambda_2[y_2, x_n]_{\alpha_0} + ... + \lambda_n[y_n, x_n]_{\alpha_0} = 0 \end{cases}.$$

The system $(\mathbf{3}_{\alpha_0})$, is a homogeneous system with unknowns $\lambda_1, \lambda_2, ..., \lambda_n$, determinant of which is the amount $\prod_{i=1}^{n}[y_i, x_i]_{\alpha_0} \neq 0$. From here we deduce that the system $(\mathbf{3}_{\alpha_0})$ there is only one solution, which is the trivial solution $\lambda_1 = \lambda_2 = ... = \lambda_n = 0$.

Finally, the system of vectors $y_1, y_2, ..., y_n$ is a linearly independent system.

Theorem 2.6

Let assume that the space $(X, \{p_\alpha\}_{\alpha \in \mathbf{A}})$ is separable. If for the system of vectors $x_1, x_2, ..., x_n$ from X there is a system of vectors, $y_1, y_2, ..., y_n$, from X and an index $\alpha_0 \in \mathbf{A}$, such that for any two natural indices $i, j \in \{1, 2, ..., n\}$, the equivalence is true:

$$i \neq j \Leftrightarrow [x_i, y_j]_{\alpha_0} = 0,$$

then the system of vectors $x_1, x_2, ..., x_n$ is linearly independent,

Proof:

Let be $\lambda_1, \lambda_2, ..., \lambda_n \in R$, such that holds the equation:

$$\lambda_1 x_1 + \lambda_2 x_2 + ... + \lambda_n x_n = 0.$$

From here, using the property **(3)** of s.p.i.p., it follows that holds the system of equations:

$$\left(\mathbf{4}_{\alpha_0}\right)\quad \begin{cases} \lambda_1[x_1,y_1]_{\alpha_0}+\lambda_2[x_2,y_1]_{\alpha_0}+\ldots+\lambda_n[x_n,y_1]_{\alpha_0}=0 \\ \lambda_1[x_1,y_2]_{\alpha_0}+\lambda_2[x_2,y_2]_{\alpha_0}+\ldots+\lambda_n[x_n,y_2]_{\alpha_0}=0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \lambda_1[x_1,y_n]_{\alpha_0}+\lambda_2[x_2,y_n]_{\alpha_0}+\ldots+\lambda_n[x_n,y_n]_{\alpha_0}=0 \end{cases}.$$

The system $\left(\mathbf{4}_{\alpha_0}\right)$, is a homogeneous system with unknowns $\lambda_1,\lambda_2,\ldots,\lambda_n$, determinant of which is the amount $\prod_{i=1}^{n}[x_i,y_i]_{\alpha_0}\neq 0$. From here we deduce that the system $\left(\mathbf{4}_{\alpha_0}\right)$ there is only one solution, which is the trivial solution $\lambda_1=\lambda_2=\ldots=\lambda_n=0.$ Thus, the system of vectors $x_1,x_2,\ldots,x_n$ is a linearly independent system.

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