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HYBRID OF POWER SERIES EXPANSION AND ONE-TIME SEEDED SECANT APPROXIMATION SOLUTION TO KEPLER'S EQUATION APPLICABLE TO SATELLITE WITH KEPLERIAN ORBITS

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Abstract— **In this paper, a hybrid of power series expansion and one-time seeded secant approximation solution to Kepler's equation applicable to satellite with keplerian orbits is presented. Specifically, the estimation error of an existing non-iterative power series expansion method for computing the eccentricity anomaly, E keplerian orbits is considered and the estimation error performance is enhanced by introducing a one-time seeded secant approximation. The results show that for eccentricity, e = 0.093, at mean anomaly, M=3 degree, the estimation error of E for the power series expansion solution is -6.43$x10^{-10}$ whereas when the result of the power series expansion is enhanced by the onetime seeded secant approximation the estimation error is -1.05 $x10^{-22}$. In all the various values of M considered, the enhanced version had significant reduction in the estimation error.**

Keywords— Power Series Expansion, Keplerian Orbits , Estimation Error Performance , Seeded Secant, Kepler's Equation, Eccentricity Anomaly

1. Introduction

In the telecommunication industry, determination of the spatio-temporal location of satellites in their orbits is very essential [1,2,3,4,5,6,7,8,9,10,11]. Accordingly, Kepler provided equation that can be used to determine some of the key parameters for defining the spatio-temporal location of the satellite in their orbits. Particularly, Kepler's equation is a transcendental expression for computing eccentricity anomaly (E) of a satellite when the mean motion (M) or the satellite and the eccentricity of the orbit are known [12,13,14,15,16,17]. Generally, iterative solution to Kepler's transcendental equation is more popular. However, researchers has over the years tried to develop non-iterative solution to the Kepler's equation [18,19,20,21,22,23].

Specifically, in this paper, an existing non-iterative solution approach to Kepler's equation is examined and modified to improve on the accuracy of its results. The original method used power series expansion [19] to determine the solution for E when M and e are given. However, given the limited accuracy level of the results obtained from such approach when compared with the results obtained from iterative solution, in this paper, a onetime seeded secant computation [24,25] is performed on the output of the power series expansion solution. By doing so, the accuracy of the results obtained is greatly improved. Some numerical examples are used to demonstrate the effectiveness of the proposed method.

2. Methodology

2.1 The power series expansion solution

Kepler's equation for Keplerian orbit is given as;

$$E = M + eSin(E) \quad (1)$$

Where E is the eccentricity anomaly, e is the eccentricity of the orbit and M is the mean anomaly. In order to solve the Kepler's equation without iterative approach, Mikkola [19] used a series expansion method along with other approaches to arrive at the value of E for any given values of M and e. Particularly, Mikkola [19] introduced the variable, s and presented the series expansion of sin(s) as follows;

$$s = \sin\left(\frac{E}{3}\right) \quad (2)$$

$$\sin^{-1}(s) = \frac{M\, e(3s-4s^3)}{3} \quad (3)$$

An 11[th] order power series approximation of $\sin^{-1}(s)$ by Mikkola [19] is given as;

$$\sin^{-1}(s) \approx s + \frac{s^3}{6} + \frac{3s^5}{40} + \frac{5s^7}{112} + \frac{35s^9}{1152} + \frac{63s^{11}}{2816} + \cdots \quad (4)$$

A third order truncation of the power series approximation is given by Mikkola [19] as;

$$M = 3(1-e)s_0 + \left(4e + \frac{1}{2}\right){s_0}^3 \quad (5)$$

$$\alpha = \frac{1-e}{4e+\frac{1}{2}} \quad (6)$$

$$\beta = \left(\frac{1}{2}\right)\frac{M}{4e+\frac{1}{2}} \quad (7)$$

$$z^2 = \left(\beta + \sqrt{\alpha^3 + \beta^2}\right)^{\frac{2}{3}} \quad (8)$$

$$s_0 = \frac{2\beta}{z^2 + \alpha + \frac{\alpha^2}{z^2}} \quad (9)$$

$$S = s_0\left[1 - \frac{0.07925({s_0}^5)}{1+e}\right] \quad (10)$$

$$S = sin\left(\frac{\acute{E}_0}{3}\right) \quad (11)$$

$$E = 3\left(sin^{-1}(S)\right) \quad (12)$$

2.2 The one time seeded secant approximation of E

In order to improve on the accuracy of E obtained from the power series expansion solution presented by Mikkola [19], a seeded secant iteration formula is applied once to E. In this case, the starting point is the value of E obtained from the power series expansion solution presented by Mikkola [19], as follows;

$$\acute{E}_0 = 3\left(sin^{-1}(S)\right) \quad (13)$$

$$\acute{E}_1 = M + e\left(Sin(\acute{E}_0)\right) \quad (14)$$

$$\acute{E}_{0f} = \acute{E}_0 - \acute{E}_1 \quad (15)$$

$$\acute{E}_{1f} = \acute{E}_1 - \left(M + e\left(Sin(\acute{E}_1)\right)\right) \quad (16)$$

$$E = \frac{\acute{E}_0(f(\acute{E}_1)) - \acute{E}_1(f(\acute{E}_0))}{\acute{E}_{1f} - \acute{E}_{0f}} \quad (17)$$

$$f(E) = E - M + e(Sin(E)) \quad (18)$$

3. Results and Discussion

The results of the computation S, E', E, Eact versus M in degree for e = 0.093 are presented in Table 1 , Table 2 and Figure 1, where Eact is the actual value of E obtained through fixed point iteration. The results show that for e = 0.093, at M=3 degree, the estimation error for the power series expansion solution is -6.43E-10 whereas when the result of the power series expansion is enhanced by the onetime seeded secant approximation the estimation error is -1.05E-22 . In all the various values of M considered in Table 1 , Table 2 and Figure 1, the enhanced version had significant reduction in the estimation error.

The results of the computation S, E', E, Eact versus M in degree for e = 0.53 are presented in Table 3 , Table 4 and Figure 2. The results show that for e = 0.53, at M=3 degree, the estimation error for the power series expansion solution is 6.7451E-04 and when the result of the power series expansion is enhanced by the onetime seeded secant approximation the estimation error is also 6.7451E-04 . However, as the values of M increases in Table 3, Table 4 and Figure 2, the enhanced version had significant reduction in the estimation error.

The results of the computation S, E', E, Eact versus M in degree for e = 0.993 are presented in Table 5 , Table 6 and Figure 3. The results show that for e = 0.993, at M=3 degree, the estimation error for the power series expansion solution is -4.9967E-04 and when the result of the power series expansion is enhanced by the onetime seeded secant approximation the estimation error is 3.3527E-05. Again, as the values of M increases in Table 5, Table 6 and Figure 3, the enhanced version had significant reduction in the estimation error.

Table 1 The results of the computation S, E', E, Eact versus M in degree for e = 0.093

e	M	M	S	E'	E	Eact	Error	
Unit	Degree	Radian	Radian	Radian	Radian	Radian	Eact-E'	Eact-E
0.093	3	0.05236	0.019240598	0.057725	5.77E-02	0.057725	-6.43E-10	-1.05E-22
0.093	13	0.226893	0.083201214	0.249893	2.50E-01	0.249892	-9.18E-07	-7.16E-16
0.093	23	0.401426	0.146519976	0.441148	4.41E-01	0.441133	-1.46E-05	-3.90E-13
0.093	33	0.575959	0.208750229	0.630891	6.31E-01	0.63081	-8.06E-05	-1.45E-11
0.093	43	0.750492	0.269511247	0.818656	8.18E-01	0.818386	-2.71E-04	-1.69E-10
0.093	53	0.925025	0.328496747	1.004135	1.00E+00	1.003454	-6.80E-04	-9.54E-10
0.093	63	1.099557	0.385472084	1.187158	1.19E+00	1.185748	-1.41E-03	-3.10E-09
0.093	73	1.27409	0.440264168	1.367679	1.37E+00	1.36513	-2.55E-03	-6.69E-09
0.093	83	1.448623	0.492748272	1.545736	1.54E+00	1.541584	-4.15E-03	-3.03E-08
0.093	93	1.623156	0.542834894	1.721427	1.72E+00	1.715188	-6.24E-03	5.31E-08

Table 2 The results of the computation for Log of Error versus M in degree for e = 0.093

	Log of Error	
M in degree	LOG($\lvert Eact - E' \rvert$)	LOG(|Eact − E |)
3	-9.1917	-21.9788
13	-6.03735	-15.1451
23	-4.83479	-12.4089
33	-4.09358	-10.8394
43	-3.56742	-9.77237
53	-3.16733	-9.02035
63	-2.85072	-8.50907
73	-2.59377	-8.17472
83	-2.38173	-7.51856
93	-2.20492	-7.27491

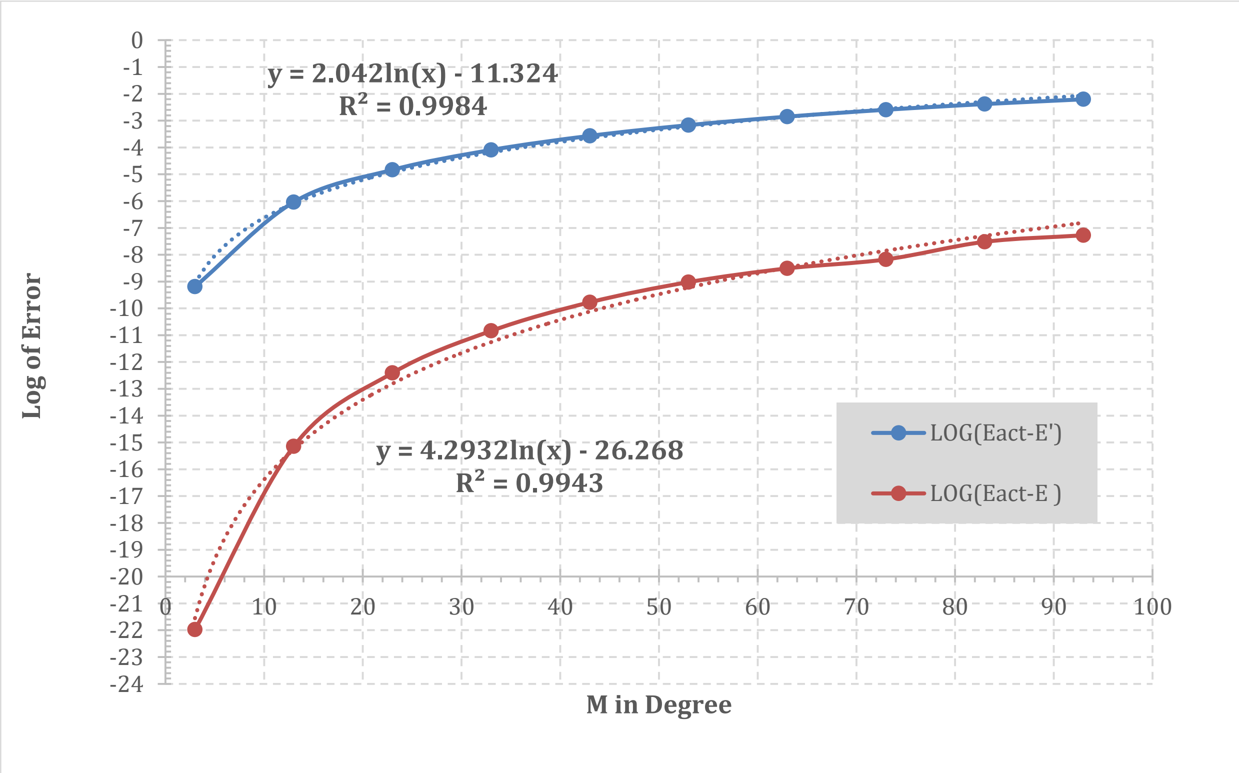

Figure 1 The graph of Log of Error versus M in degree for e = 0.093

Table 3 The results of the computation S, E', E, Eact versus M in degree for e = 0.53

e	M	M	S	E'	E	Eact	Error	
Unit	Degree	Radian	Radian	Radian	Radian	Radian	Eact-E'	Eact-E
0.53	3	0.05236	0.037040236	0.111146	1.11E-01	0.111821	6.7451E-04	6.7455E-04
0.53	13	0.226893	0.154114474	0.464193	4.64E-01	0.464169	-2.4179E-05	1.1433E-05
0.53	23	0.401426	0.254169408	0.770966	7.71E-01	0.770643	-3.2380E-04	3.3533E-05
0.53	33	0.575959	0.337147006	1.031654	1.03E+00	1.03046	-1.1944E-03	1.0122E-05
0.53	43	0.750492	0.406758472	1.256709	1.25E+00	1.254142	-2.5666E-03	5.9519E-07
0.53	53	0.925025	0.466371752	1.455554	1.45E+00	1.451241	-4.3128E-03	-3.1706E-07
0.53	63	1.099557	0.518406573	1.63496	1.63E+00	1.62867	-6.2895E-03	3.2747E-07
0.53	73	1.27409	0.564538956	1.799624	1.79E+00	1.791262	-8.3617E-03	1.8245E-06
0.53	83	1.448623	0.605943631	1.952855	1.94E+00	1.94244	-1.0414E-02	3.6437E-06
0.53	93	1.623156	0.643464176	2.097046	2.08E+00	2.084696	-1.2350E-02	5.2394E-06

Table 4 The results of the computation for Log of Error versus M in degree for e = 0.53

	Log of Error	
M in degree	LOG($\|Eact - E'\|$)	LOG($\|$Eact − E $\|$)
3	-3.17	-3.17
13	-4.62	-4.94
23	-3.49	-4.47
33	-2.92	-4.99
43	-2.59	-6.23
53	-2.37	-6.50
63	-2.20	-6.48
73	-2.08	-5.74
83	-1.98	-5.44
93	-1.91	-5.28

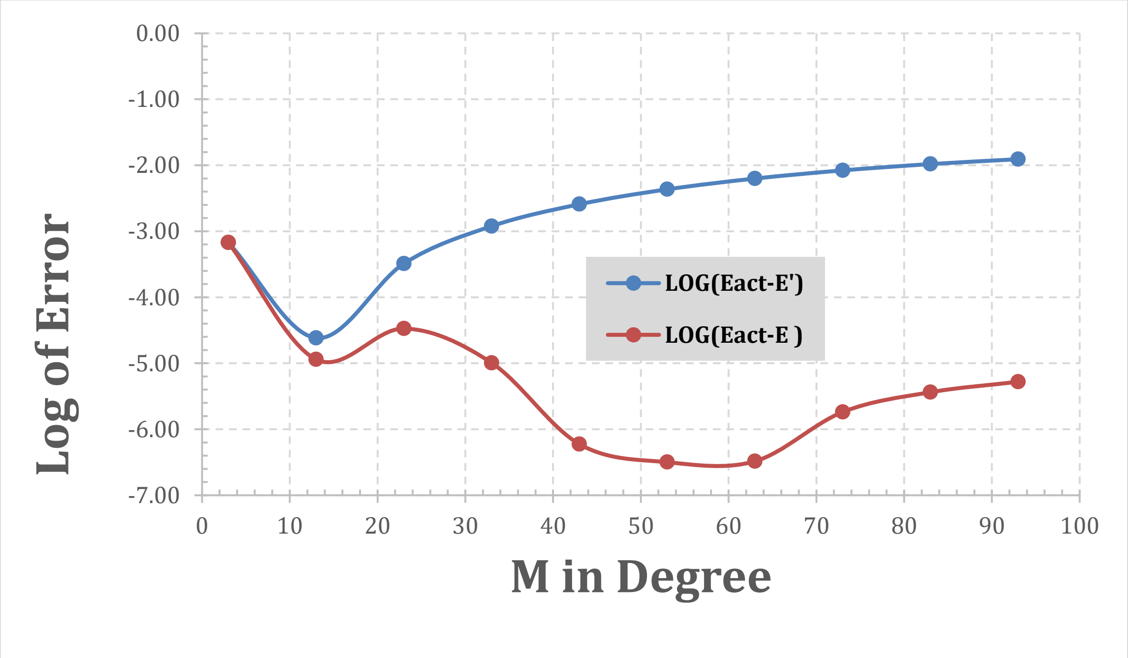

Figure 2 The graph of Log of Error versus M in degree for e = 0.53

Table 5 The results of the computation S, E', E, Eact versus M in degree for e = 0.993

e	M	M	S	E'	E	Eact	Error	
Unit	Degree	Radian	Radian	Radian	Radian	Radian	Eact-E'	Eact-E
0.993	3	0.05236	0.220177354	0.665989	6.65E-01	0.665489	-4.9967E-04	3.3527E-05
0.993	13	0.226893	0.365879183	1.123732	1.12E+00	1.121446	-2.2859E-03	2.2447E-04
0.993	23	0.401426	0.443947304	1.379997	1.38E+00	1.375605	-4.3918E-03	4.2517E-05
0.993	33	0.575959	0.501270777	1.5752	1.57E+00	1.568957	-6.2433E-03	2.4612E-08
0.993	43	0.750492	0.547664283	1.73871	1.73E+00	1.730802	-7.9084E-03	-4.6666E-07

0.993	53	0.925025	0.587100489	1.882417	1.87E+00	1.872958	-9.4587E-03	-5.0900E-05
0.993	63	1.099557	0.621631809	2.012473	2.00E+00	2.001574	-1.0899E-02	-1.7188E-04
0.993	73	1.27409	0.652473375	2.132531	2.12E+00	2.120374	-1.2157E-02	-3.0336E-04
0.993	83	1.448623	0.680411091	2.24497	2.23E+00	2.231817	-1.3153E-02	-3.7712E-04
0.993	93	1.623156	0.705985973	2.351443	2.34E+00	2.33759	-1.3853E-02	-3.6680E-04

Table 6 The results of the computation for Log of Error versus M in degree for e = 0.993

	Log of Error	
M in degree	LOG($\|Eact - E'\|$)	LOG($\|$Eact − E $\|$)
3	-3.30	-4.47
13	-2.64	-3.65
23	-2.36	-4.37
33	-2.20	-7.61
43	-2.10	-6.33
53	-2.02	-4.29
63	-1.96	-3.76
73	-1.92	-3.52
83	-1.88	-3.42
93	-1.86	-3.44

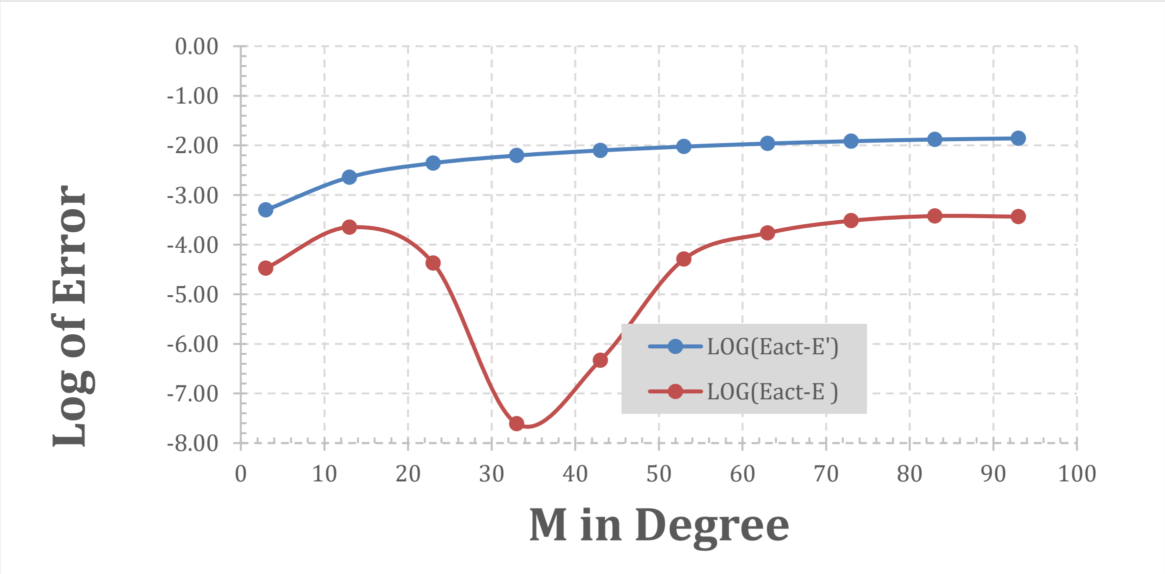

Figure 3 The graph of Log of Error versus M in degree for e = 0.993

4. Conclusion

A non-iterative approach to solve Kepler's equation for satellite with keplerian orbits is presented. The approach considered in the paper used onetime seeded secant method to enhance an already existing power series expansion solution result for estimating the eccentricity anomaly of keplerian orbits. The analytical expressions and algorithms for defining the power series expansion solution and the combined onetime seeded secant method and power series expansion solution are presented. The two methods were implemented in Matlab software and the results showed that in all the cases considered, the enhanced version had significant reduction in the estimation error when compared to the error obtained with the power series expansion solution alone.

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