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On The Welfare Theorem And Pareto Optimality In Small Economies

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***Abstract*—in small economies where exist high inequalities, not necessary relied to the country's wealth capacity, the equilibrium is dominated by agents' bad choices actions when young, thus is not Pareto optimal since prevail high poverty levels in the second period when adult as a consequence of the action chosen before highlighted by non formal jobs increase without resting income once old. However, the several states of nature options open to the household when young, including mostly bad results later-on compare to good results are states plurality that are non-observable action earlier and endowed of probabilities to bring high or low benefit able to yield an agent toward a poverty trap. Consequently, in contrast to the Walrasian economy, this paper uses Allais market economy concept to shows-off that, the re-allocation of the resources or of the surplus is able to make the equilibrium optimal in the Pareto sense for the welfare theorems of Arrow and Debreu to hold among population in small countries' economies and reduce inequalities as well as poverty.**

Keywords—equilibrium, welfare theorem, Pareto optimality, uncertainty on the future

I. INTRODUCTION

The Walrasian general equilibrium model optimal properties are summarized by the two following welfare theorems despite of the fact that, the Pareto optimality of a competitive equilibrium is strongly relied on the existence of a complete financial market, which is not the case in small countries where they are not developed enough, indeed, are incomplete markets so that, using Allais (1943) and Allais (1981), we show that, the surplus total distribution, can lead to Pareto optimality in small countries where prevail high inequalities mainly caused by actions chosen by households them self when young endowed of high probabilities to turn-out to become bad actions later-on when adult as well as when old, since they lead to non formal jobs without resting income. The parents' education and altruism should play strongly on the young choices for their future but are almost absent in the children future direction of the small countries since quantity prevail in the choice of the children rather than quality. Therefore, we agree that, each action chosen by a household, leads to an equilibrium, which is thus multiple for them all if taken at the same time in the whole small country. Moreover, those effective equilibria are not unique and far from the Pareto optimality leaving the welfare theorems play among the small country's inhabitants. Indeed, we prove in this article that, total allocation or surplus distribution, is the only way to reach the Pareto optimal equilibrium expected to establish.

Definition1: an equilibrium is Pareto optimal, if the set of feasible allocations is compact, so that, suitable continuity assumptions on preferences lies and when production is possible, compactness of the set of feasible allocations is needed

Definition2: the private ownership economy has an optimum if the following three assertions are satisfied i.e

(i) *$\forall\, n \in N$, $x_{n,h} = (x_{n,h}(w))_{w\in\Omega}) \subset R$ is closed and bounded from below*

(ii) *$\forall\, x_{n,h}' \in X$ yields, the set $\Gamma=\{\forall\, x_{n,h} \in X$ yields, $x_{n,h} \leq x'_{n,h}\}$ is closed and bounded*

(iii) *$\sum_{n\in N, h\in Z} (x_{n,h}) = X$ is closed and convex such that,*

$\sum\{x_{n,h}+(-x_{n,h})\}_{n\in N,\ h\in Z}=\{0\}$ *and* $w \in \{\sum_{n\in N} x_{n,h}(w) - \sum_{n\in N,\ h\in Z} x_{n,h}(w)\} \neq \emptyset \subset \Omega$ is the states' space i.e the space of options or actions that the household can choose to invest on when young

Where $x_{n,h}(w)=x_{n,h}^{w}$ is allocation of the agent $n\in N$ of type, $h\in Z$ such that, choices result are bad when $h<0$ and good when, $h\geq 0$

Lemma1: the first welfare theorem of Arrow and Debreu: *let ζ be a private ownerships economy endowed of N young agents of type, h indexed by n with the equilibrium, $(p_{n,h}^{w*}, x_{n,h}^{w*}, y_{n,h}^{w*}) = \lim_{w(j)\to w} \{(p_{n,h}^{w(j)}, x_{n,h}^{w(j}, y_{n,h}^{w(j)})\}_{n\in N, h\in Z}$, then, the equilibrium is Pareto optimal since, there doesn't exist another feasible allocation that yields a better situation i.e improving one agent situation without making worse the situation of another agents*

Proof: let $(p_{n,h}^{w*}, x_{n,h}^{w*}, y_{n,h}^{w*})$ be a Pareto superior equilibrium such that, $p_{n,h}^{w'} x_{n,h}^{w'} \geq p_{n,h}^{w*} x_{n,h}^{w*}$, then, there exists, $y_{n,h}^{w} \in R$ such that, $p_{n,h}^{w'} y_{n,h}^{w'} \geq p_{n,h}^{w*} y_{n,h}^{w*}$ which yields, $p_{n,h}^{w*} w_{n,h}^{*} = p_{n,h}^{w'}(\sum_{w'\in\Omega} x_{n,h}^{w'}) - p_{n,h}^{w*} \sum_{w\in\Omega} x_{n,h}^{w*} \geq 0$
Where, $p_{n,h}^{w} w_{n,h} < p_{n,h}^{w}(\sum_{w\in\Omega} x_{n,h}^{w'}) - p_{n,h}^{w'} \sum_{w\in\Omega} y_{n,h}^{w'} \geq 0$
meaning that,
$w(j) \neq \sum_{w\in\Omega} x_{n,h}^{w'} - \sum_{w\in\Omega} y_{n,h}^{w'}$
Indeed, $(p_{n,h}^{w'}, x_{n,h}^{w'}, y_{n,h}^{w'})$ is not feasible, thus the equilibrium is unique

Lemma2: the second welfare theorem of Arrow and Debreu: *let $(x_{n,h}^{w(j)}, y_{n,h}^{w(j)})_{n\in N, h\in Z}$ be multiple equilibria that converge to a pareto optimal equilibrium allocation, (x^{w*}, y^{w*}, p^{w*}) for a private ownership economy, ζ such that, $(x_{n,h})_{n\in N, h\in Z} = X$ is convex, then the set $\Gamma=\{x_{n,h} \in X$ such that, $x_{n,h}' \geq x_{n,h}\ \forall n\in N,\ h\in Z\}$ is also convex, there thus exist a price, p^{w*} such that, (x^{w*}, y^{w*}, p^{w*}) is pareto optimal for the economy, ζ*

Where, $x^*=(x_{n,h}^*)_{n\in N,h\in Z}$, $y^*=(y_{n,h}^*)_{n\in N,h\in Z}$ and $p^*=(p_{n,h}^*)_{n\in N,h\in Z}$ such that,
$(x_{n,h}^w)_{w\in\Omega} \to x^{w*}$ and $(y_{n,h}^w)_{w\in\Omega} \to y^{w*}$

Proof: let $(x_{n,h}^{w*}, y_{n,h}^{w*})$ be a Pareto optimal allocation for a private ownership, ζ such that, *for all* $n\in N$, $X_{n,h}^w$ is a convex set where $w\in\Omega$ and $\Gamma=\{x_{n,h}^{'w} \in X_{n,h}^w$ such that, $x_{n,h}^{w'} \geq x_{n,h}^w\}$ is convex. Therefore, a given $x_{n,h}^{w*} \in X_{n,h}^w$ yields to the existence of a price, $p_{n,h}^{w*}$ such that, $(p_{n,h}^{w*}, x_{n,h}^{w*}, y_{n,h}^{w*})$ is a quasi equilibrium of the economy, ζ.
Assuming, $G_{n,h}^w=\{\sum_{w(j)\in\Omega} x_{n,h}^{w(j)} + \sum_{w(j)\in\Omega} x_{m,h}^{w(j)} > x_{n,h}^{w*} + x_{m,h}^{w*} - y_{k,h}^{w*}\}_{m\neq n}$, then for a given $w(j)$ for which, $x_{n,h}^{w*}$ and $y_{n,h}^{w*}$ don't belong to $G_{n,h}^w$ then the equilibrium is nor Pareto optimal. But since, $G_{n,h}^w=\lim_{n\to\infty}(G_{n,h}^{w(j)})_{w(j)\in\Omega}$ then, there is a contradiction i.e a price exist such that, the equilibrium $(p_{n,h}^{w*}, x_{n,h}^{w*}, y_{n,h}^{w*})$ satisfies, $p_{n,h}^{w*}w^*=Inf\{p_{n,h}^{w(j)*}G_{n,h}^{w(j)}$ such that, $G_{n,h}^{w(j)} \in G_{n,h}^{w}\}$ exist

Allais (1943) defined a market economy as an economy in which agents make all possible advantageous transactions or choices. In contrast, in the Walrasian theory of market, agents don't trade through a single price system. A stable equilibrium then yields such as, no further surplus can be distributed. Then, Luenberger (1992b) proved that, Allais equilibrium concept can be proved to be optimal in the Pareto sense.
This paper aim is thus, to formalize Allais in a small economy and look for a Pareto optimal distribution of surplus or allocation. Pareto optimality is based on the welfare theorems announced above. Indeed, the economic environment can be presented now in order to show how great inequalities which prevail in small countries can be overcome for the unique equilibrium to exist and to be endowed of the welfare theorem properties i.e Pareto optimality and leave emerge an efficiency situation which is not necessarily an equity situation.
Section 2 presents the model which aim is to find the competitive equilibrium and section 3, aim is to look for the Pareto optimality character of the equilibrium based on the actions chosen by the agents when young which yield to an economic inefficiency making the surplus be re-allocated to the population as a necessity.

II. The Model

In an exchange economy with 3 periods of lives for the household, young, adult and old where Ω, the aggregate states of nature exist, C commodities and *N* inhabitants at each age level since the population is constant. Thus, the agents are indexed by $n\in N$. In the first period when young, each household indexed by, $n\in\{1, 2, ..., N\}$ of type, $h\in Z$ chooses an action which as an observable consequence later-on in her social life once adult since she can be in rich, medium or poor social class,…. Moreover, in the first period, each agent *n* of type, *h* chooses an non-observable action, $a_{n,h}^{w(j)} \in [0, 1]$. In the second period, each household faces states of nature such as pregnancy, early marriage, polygamy, human capital accumulation, job market entering, etc… and each state, she receives a bundle of endowments, $e_{n,h}^{w(j)} \in R_{++}^C$ since, $w=(w(1),w(2),..w(J))$, we have, $e_{n,h}^{w1} < e_{n,h}^{w2} < \ldots < e_{n,h}^{wJ}$ with the respective probabilities associated with the individual state of nature, $\pi_{n,h}^{w(j)}$ depends on the action chosen by the household in the first period. Individuals behave as if the individual shocks were independent across households of the same type.
Let, $\pi_{n,h}^{w(j)}: [0, 1) \to (0,1) \in C^\infty \subset R$ be the probability of the state of nature each such that, $\pi_{n,h}^{w(j)}=(\pi_{n,h}^1, \pi_{n,h}^2, \ldots \pi_{n,h}^J)$ and $\sum_{w\in\Omega}(\pi_{n,h}^{w(j)})_{1\leq j\leq J}=1$

Assumption1: $\partial\pi_{n,h}^{w(j)}/\partial a_{n,h}^{w(j)} > 0$, $\partial^2\pi_{n,h}^{w(j)}/\partial(a_{n,h}^{w(j)})^2 < 0$ for every action chosen
The first assumption means that, higher levels of action increase the likelihood of the good state of nature
The utility function is thus expressed such that, $U: R^{2\Omega C}$ $[0, 1)++ \to C^2$ i.e
$U_n(x_{n,h}, a_{n,h}^{w(j)})=\sum_{w(j)\in\Omega}[(\pi_n(a_{n,h}^{w(j)})u_n(x_{n,h})]-v_n(a_{n,h}^{w(j)})$ (1)
Where Ω is the state space such that, $w=(w(j)_{1\leq j\leq J})\in\Omega$, $\sum_{w(j)\in\Omega}\pi_n(a_{n,h}^{w(j)})=1$ and $\pi_n(a_{n,h}^{w(j)})$ is the probability for the nth agent of type h to choose the action, $a_{n,h}^{w(j)}$ among $w\in\Omega$
Assumption2: for all $(x_{n,h}, a_{n,h}) \in R_{++}^2 x[0, 1)$ it yields

(i) $\partial u(x_{n,h}^{w(j)})/\partial x_{n,h}^{w(j)} >> 0$

(ii) $\partial v_{n,h}^{w(j)}(0)/\partial a_{n,h}^{w(j)} = 0$; $\partial v_{n,h}^{w(j)}/\partial a_{n,h}^{w(j)} > 0$; $\lim_{a\to 1}\partial v_{n,h}^{w(j)}/\partial a_{n,h}^{w(j)} = +\infty$

(iii) $\partial^2 u_{n,h}/\partial(x_{n,h}^{w(j)})^2$ *is negative definite and* $\partial^2 v_n/\partial(a_{n,h}^{w(j)})^2$ *is strictly positive*

(iv) $cl=\{y_{n,h}^{w(j)} \in R_{++}$ *such that,* $u_n(y_{n,h}^{w(j)}) > u_{n,h}(x)\}$

(i)show-off, the increasing character of the utility function in regard to the consumption demanded, (ii) guarantees that higher levels of action reduce the household utility, the level, v is risk taken by the household for having chosen a given strategy. (iii) guarantees the existence of the equilibrium and (iv) guarantees its unicity.
Where, $(x_{n,h}^{w(j)}, a_{n,h}^{w(j)})$ is the couple of commodity acquired by the nth agent and the corresponding action she chooses when young
There exist I>1 firms which strategies' space is given by,
$\{y_{n,h}^w: [0, 1]x(1,2,\ldots N) \to R^{2\Omega}\}$
Where, $y_{n,h}(w(j))$ specifies the firm offer to household, *n* of type *h* in each possible state of nature, w (j)∈Ω with payoff contingent upon the individual state of nature. For simplicity, we refer to the value of the firm as, $y_{n,h}^{w(j)}$ is the payoff contingent upon the individual state of nature and action chosen.
Suppose a household accept the payoff proposed and given the commodity price, p^w in the second period, then the household *n* of type *h* solves the problem

$$Max_{n\in N}\{u_{n,h}(x_{n,h}, a_{n,h}^{w(j)})\}$$
$$\text{subject to}$$
$$p^w(x_{n,h} - e_{n,h}^{w(j)}) + p_1^w y_{n,h}^{w(j)} = 0$$
$$\textit{for all } w\in\Omega$$

Where, p_1^w is the price of the 1st commodity in the second period
Since, a household accept or reject a given firm offer, $y_{n,h}^{w(j)}$, we can announce the first proposition

Proposition1: *for each price,* p^w *and contract,* $y_{n,h}^{w(j)}$ *satisfying,* $p^w e_{n,h}^{w(j)} - y_{n,h}^{w(j)} > 0$, *there is an unique equilibrium solution to the household problem,* $(x_{n,h}^{w*}, a_{n,h}^{w*})$ *where* $h\in Z$ *and* $w\in\Omega$
(See the appendix for proof)

III Looking for a Pareto Optimal Equilibrium

The purpose now, is to find an income allocation which shows that, the equilibrium converges to the Pareto optimal solution i.e , $((x_{n,h}^{w*}, a_{n,h}^{w*}), y_{n,h}^{w*})$

Given N agents indexed by $n\in \{1, 2, ..., N\}$ of type $h\in Z$ has an endowment, $y_{n,h}^{w}$ at the beginning of the second period, where $y_{n,h}^{w} \in Z$ i.e can be $y_{n,h}^{w} <0$ or $y_{n,h}^{w} \geq$ since, $y_{n,h}^{w}=(y_{n,h}^{w(1)}, y_{n,h}^{w(2)}, ..., y_{n,h}^{w(J)})_{h\in Z}$ there thus, exist, a threshold, where income becomes positive and remains negative before i.e $y_{n,h}^{w} < y_{n,h}^{w*} <0$ then $h<0$ and $y_{n,h}^{w*} \geq y_{n,h}^{w} \geq 0$ then, $h\geq 0$

Each household has a utility function, u_{nh} defined from R_{+}^{C} to R assumed to be continuous, strictly increasing and strictly quasi-concave.

Let the feasible allocations to be expressed such that, $FF(w)$ i.e

$$FF(w)=\{x_{n,h} \in R_{+}^{CI} \text{ such that, } \sum_{n\in N} p^{w}(x_{n,h}^{w} - e_{n,h}^{w}) + p_1^{w} y_{n,h}^{w} = \sum_{n\in N} y_{n,h}^{w} \text{ for all } n, h\}$$

Where, $x_{n,h} = p^{w}(x_{n,h}^{w} - e_{n,h}^{w}) + p_1^{w} y_{n,h}^{w}$

We use now, Allais (1943, 1981) to introduce the concept of distributable surplus or the benefit function according to Luenberger, (1992)

Definition3: let, $b_{n,h}^{w(j)}$ be the benefit function corresponding to the utility function, $u_{n,h}^{w(j)}$ according to the state w chosen of the nth agent of type h defined such as,

$$b_{n,h}^{w(j)}(x_{n,h}^{w}, u_{n,h}, g_{n,h}) = \max\{\beta^{w} \text{ such that, } u_{n,h}(x_{n,h}^{w} - \beta^{w} g_{n,h}) \geq u_{n,h}, x_{n,h} - \beta^{w} g_{n,h}\}$$

Where, $g_{n,h} \in R_{++}$ and $g_{n,h} \neq 0$

If the constraint is not feasible, the set Max $\{b_{n,h}^{w(j)}(x_{n,h}, u_{n,h}, g_{n,h})\}_{n\in N, h\in Z} = b_{n,h}^{w(j)}(x_{n,h}^{w}, u_{n,h}, g_{n,h}) \rightarrow -\infty$

The benefit function measures, the maximum an individual, n of type h is willing to give-up in order to move at a higher utility level The trading process is based on the maximization of the total distributable surplus or total benefit of the country in order to achieve pareto optimality.

The set of individually rational is defined as follows,

$$IR(y_{n,h}^{w(j)}) = \{x_{n,h}^{w(j)} \in FA(w) \text{ such that, } u_{n,h}(x_{n,h}^{w(j)}) \geq u_{n,h}(y_{n,h}^{w(j)})\ \forall n\in N, \forall h\in Z\}$$
$$\text{for all } y_{n,h}^{w(j)} \in Z$$

The allocations, $x_{n,h}^{w(j)} = (x_{1n,h}^{w(1)}, x_{2n,h}^{w(2)}, ..., x_{Mn,h}^{w(J)})$ is the solution of the following problem

$$Max_{n\in N, h\in Z} \{\sum_{w(j)\in\Omega} b_{n,h}^{w(j)}(x_{n,h}^{w(j)}, u_n(x_{n,h}^{w(j)-1}))\}$$
$$\text{st}$$
$$x_{n,h}^{w(j)} \in IR(x_{n,h}^{w(j)-1}) \qquad (P)$$

Hence, at each stage of the trading process, the allocation maximizes the total benefit function, thus we show now, that, this trading process, converge to a Pareto optimal allocation

Proposition2: *let $(x_{n,h}^{w(j)})_{w(j)\in\Omega}$ be a sequence of allocations such that, $x_{n,h}^{w*}$ is a solution of the problem (P), then $x_{n,h}^{w*}$ is Pareto optimal*

Proof, let $(u_{n,h}^{w(j)})$ be the utility along the sequence $(x_{n,h}^{w(j)}, a_{n,h}^{w(j)}))_{w(j)\in\Omega}$ be the nth agent possible allocations in regard to the strategy chosen before, then there exist is a $u_{n,h}^{w*}$ and $x_{n,h}^{*}$ such that, $lim_{j\rightarrow 1}(u_{n,h}^{w(j)})_{w\in\Omega} \rightarrow u_{n,h}^{w*}$ and $Lim_{j\rightarrow 1}(x_{n,h}^{j})_{j\in\Omega} \rightarrow x_{n,h}^{w*}$ because $(u_{n,h}^{w(j)})_{1\leq j\leq M}$ belongs to a compact set and by continuity, thus converge. Indeed, there exists, a unique $x_{n,h}^{*} \in FF(w)$ *such that,* $(x_{n,h}^{w(j)})_{n\in N, h\in Z, w\in\Omega} \rightarrow x_{n,h}^{w*}$ is the existence of the equilibrium allocations.

Assuming that the limit is not unique, then there exists another, $x_{n,h}^{w*'}$ such that, $u_{n,h}^{w*} = u(x_{n,h}^{w*}) = u(x_{n,h}^{*'})$ *where* $x_{n,h}^{w*} \neq x_{n,h}^{w*'}$ and then, by strict concavity of the utility function, $u_{n,h}^{w}(\lambda x_{n,h}^{*} + (1-\lambda)x_{n,h}^{*'}) \geq u_{n,h}^{w}(x_{n,h}^{*})$ *for all* $\lambda\in(0, 1)$ $\forall n\in N, \forall w\in\Omega$ *and* $\forall h\in Z$*, yields,*

$\sum_{n=1}^{N} b_{n,h}^{w}(\lambda x_{n,h}^{*} + (1-\lambda)x_{n,h}^{*'}, u_{n,h}) > 0$ since $b_{n,h}^{w}(\lambda x_{n,h}^{*} + (1-\lambda)x_{n,h}^{*'}, u_{n,h}) \geq 0$

But, $u_{n,h}^{w*}$ cannot be a limit utility allocation of the trading sequence, since $\lambda x_{n,h}^{w*} + (1-\lambda)x_{n,h}^{w*'}$ yields a higher benefit and is includes in the same compact set, therefore, $x_{n,h}^{w*}$ is unique such that, $(u_{n,h}^{w(j)})_{w(j)\in\Omega} \rightarrow u_{n,h}^{w*}$ and $(x_{n,h}^{w(j)}) \rightarrow x_{n,h}^{w*} \in FF$, thus $u_{n,h}^{w(j)}(x_{n,h}^{w*}) = u_{n,h}^{w*}$ and $(x_{n,h}^{w(j)})_{w(j)\in\Omega} \rightarrow x_{n,h}^{w*}$ then yields to the fact that, IR is a correspondence on $FF(w)$. Since the function IR is defined from $FF(w)$ *to* $FF(w)$, where $FF(w)$ is a compact set, then IR is closed.

Therefore, given an allocation $(y_{n,h}^{w(j)})_{w(j)\in\Omega}$ such that, $(y_{n,h}^{w(j)})_{w(j)\in\Omega} \rightarrow y_{n,h}^{w*} \in IR(y_{n,h}^{w})$ and $x_{n,h}^{w(j)} \in IR$ such that, *if* $x_{n,h}^{w(j)} = y_{n,h}^{w(j)}$ then they converge to the same limit. Otherwise, *if* $x_{n,h}^{w(j)} \neq y_{n,h}^{w(j)}$ then, there exists $\varepsilon>0$ such that, $n\in N, h\in Z$ yields, $IR(y_{n,h}^{w(j)}) \cap B\varepsilon(x_{n,h}^{w(j)}) = \emptyset$ where $B_{\varepsilon}(x_{n,h}^{w(j)})$ is a closed ball of center $x_{n,h}^{w(j)}$ and of radius, ε

Let $(y_{n,h}^{w(j,q)})_{w(j)\in\Omega}$ be a sub sequence of $(y_{n,h}^{w(j)})$ such that, $IR(y_{n,h}^{w(j),q}) \cap B\varepsilon(x_{n,h}^{w(j)}) = \emptyset$ for all $q\in\{1,2,...,M\}$ and considering, $x_{n,h}^{w(j)'} = \lambda y_{n,h}^{w(j)} + (1-\lambda)x_{n,h}^{w(j)}$ with $\lambda\in(0,1)$, then by strict concavity of the utility function, $U_{n,h}^{w}(x_{n,h}') \geq U_{n,h}^{w}(y_{n,h}^{w(j)})$ *for* all $n\in N$ *and* $h\in Z$ thus, it is possible to take λ and $x_{n,h}^{w(j)''}$ such that, $d(x_{n,h}^{w(j)}, x_{n,h}^{w(j)''}) < \varepsilon/2$ *so* that, *all* $z_{n,h}$ which belongs to the open ball of center $x_{n,h}^{w(j)''}$ and of radius, ε' for all $y_{n,h}^{w(j)}$ i.e $B_{\varepsilon'}(x_{n,h}^{w(j)''})$ is such that, $B_{\varepsilon'}(x_{n,h}^{w(j)''}) \subset B_{\varepsilon}(x_{n,h}^{w*})$. Indeed, since $(y_{n,h}^{w(j)}) \rightarrow y_{n,h}^{w*}$ then, there exists M such that, $\forall m\geq M$ yields, $(y_{n,h}^{w,m}) \in B_{\varepsilon'}(y_{n,h}^{w})$, thus $U_{n,h}^{w}(y_{n,h}^{w,m}) < U_{n,h}^{w}(z_{n,h}^{m})$ $\forall n \in N$, $\forall m \in M$, $\forall z_{n,h} \in B_{\varepsilon'}(x_{n,h}^{w(j)''})$ indeed, $IR(y_{n,h}^{w(j)}) \cap B_{\varepsilon}(x_{n,h}^{w(j)}) \neq \emptyset$ $\forall q\geq N$, $\forall w(j)\in\Omega$, thus is a contradiction, since $IR(y_{n,h}^{w(j),q}) \cap B_{\varepsilon}(x_{n,h}^{w*}) = \emptyset$, $\forall q\geq N$, indeed, $IR(y_{n,h}^{w(j)}) \cap B_{\varepsilon}(x_{n,h}^{w(j)}) = \emptyset$ i.e unicity is ensured

Finally, to show that, $x_{n,h}^{w*}$ is a Pareto optimal allocation, we define, $V(x_{n,h}^{w(j)-1})$ such that

$$V(x_{n,h}^{w(j)-1}) = Max_{x_{n,h}}\{\sum b_{n,h}^{w(j)}(x_{n,h}^{w(j)}, u_n(x_{n,h}^{w(j)-1}))_{w(j)\in\Omega}$$
$$\text{st}$$
$$x_{n,h}^{w(j)} \in IR(x_{n,h}^{w(j)-1})\}$$

Since $IR(,)$ is a compact-valued and continuous correspondence, $V(,)$ is continuous. Then, $V(x_{n,h}^{w(j)})_{w(j)\in\Omega, n\in N, h\in Z} \rightarrow V(x_{n,h}^{w*})$

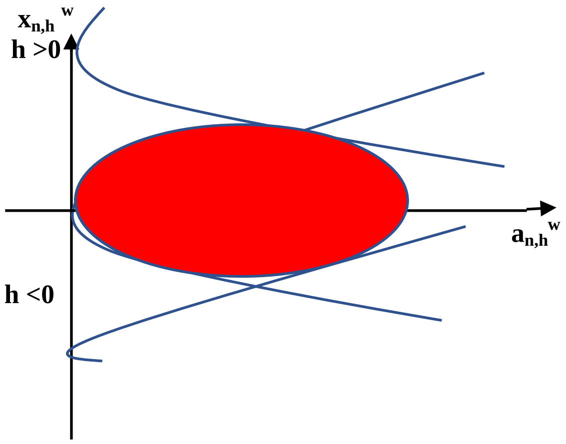
By definition, $b_{n,h}^{w(j)}(x_{n,h}^{w(j)'}, u_n(x_{n,h}^{w(j)'}) = 0\ \forall n\in N, \forall h\in Z$*, then* $V(x_{n,h}^{w(j)'}) = 0$*, thus* $x_{n,h}^{w(j)'}$ solves

$$Max_{xn}\{\sum_{w(j)\in\Omega} b_{n,h}^{w(j)}(x_{n,h}^{w(j)}, u_{n,h}(x_{n,h}^{w(j)-1}))$$
$$\text{st}$$
$$\sum_{n=1}^{N} b_{n,h}^{w(j)}(x_{n,h}^{w(j)'}, u_n(x_{n,h}^{w(j)'}) = 0$$
$$x_{n,h}^{w(j)} \in IR(x_{n,h}^{w(j)-1})\}$$

Assuming that, $x_{n,h}^{w*}$ is not Pareto optimal, then there exists, $y_{n,h}^{w(j)} \in FF(w)$ such that, $u_{n,h}^{w(j)}(y_{n,h}^{w(j)}) > u_{n,h}^{w(j)}(x_{n,h}^{w*})$ $\forall n\in N, \forall h\in Z, w(j)\in\Omega$

Since utility functions are strictly increasing, it is possible to choose, $y_{n,h}^{w(j)} >> 0$ for all $n\in N$, $\forall h\in Z$, there exists $y_{n,h}^{w(j)} \in IR(x_{n,h}^{w(j)'})$ such that, $\sum_{n=1}^{N} b_{n,h}^{w(j)}(y_{n,h}^{w(j)}, u_{n,h}(x_{n,h}^{w(j)'}))_{w(j)\in\Omega} = 0$ is a contradiction

Figure1: the Pareto optimality of the equilibrium



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Appendix

Proof of proposition1

Given a contract $y_{n,h}^{w}$, consider the problem, (P) such that,

$Max\{u_{n,h}(x_{n,h})\ st\ p^{w} x_{n,h} = p^{w} e_{n,h}^{w} + y_{n,h}^{w}\}$

Let $x_{n,h}^{w(j)}$ be the unique solution of the problem, where uniqueness is provided by the strict concavity of the utility function (assumption2) and the convexity of the budget constraint

Consider the system, $\alpha+\sum_{w\in\Omega}(D\pi_t(a_{n,h}^{w(j)}))(u_{n,h}(x_{n,h}(p, y_{n,h}^{w(j)}))-u_{n,h}(x_{n,h}^{w(j)2}(p, y_{n,h}^{w})-Dv_{n,h}(a_{n,h}^{w(j)}) =+$ and $min\{\alpha, a_{n,h}^{w(j)}\}=0$ which describes the household optimal choice of action.

If $\sum_{w\in\Omega} u_{n,h}(x_{n,h}(p, y_{n,h}^{w(j)}))-u_{n,h}(x_{n,h}(p, y_{n,h}^{w(j)}))\leq 0$

Then the system has a unique solution, $a_{n,h}=0$

If $\sum_{w\in\Omega} u_{n,h}(x_{n,h}(p, y_{n,h}^{w(j)}))-u_{n,h}(x_{n,h}(p, y_{n,h}^{w(j)}))>0$

Then the system has a solution with, $a_{n,h}\neq 0$ *and* $\alpha=0$

By the implicit function theorem, the solution is unique referred such as,

$(x_{n,h}(p, y_{n,h}^{w}), a_{n,h}(p, y_{n,h}^{w}))$

The solution can't be multiple because of the strict concavity of the utility function

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