G^2 Continuous Blending of Cubic PH Curve Under Arc Length Constraint

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***Abstract*—The Pythagorean Hodograph curve is a polynomial curve with parametric velocity that can be used to solve the calculation problems of curve arc length and isometric distance. This paper discusses the G^2 continuous blending of cubic PH curves under arc length constraint. With the specified free parameters, the first and last two points of the two curves (including the blending point) are known to construct a cubic PH curve such that the curve is G^2 continuous with shortest arc length. Therefore, the problem is transformed into an extreme value problem with the shortest arc length under G^2 continuous conditions. The objective function is established by Lagrange multiplier approach, and a set of nonlinear equations is obtained by using the extreme values of the G^2 conditions. Then, given the initial values of the 18 parameters, the interpolation curve with the shortest total arc length is selected by the iterative method. Finally, we get two public PH curves that satisfy G^2 continuous at their joint point with the shortest total arc length.**

Keywords—Cubic PH curve; G^2 continuous; Arc-length; Conditional extreme value; Lagrange multiplier method；Iteration method

I. INTRODUCTION

Computer Aided Geometric Design (CAGD）can be widely used in the modern industry, from the Design of aircraft, ships and automobiles, to the Design of engineering parts and molds, and even the Design of robots, to biomedical image processing [1]. The main research object of CAGD is geometric shape design in industrial product design. In 1990, Farouki and Sakkali first proposed the Pythagorean Hodographs (PH) curve with good properties, which overcame some shortcomings of general parametric curves, such as the PH curve with accurate isometric lines and the arc length can be accurately expressed in rational form [2]. These advantages provide a lot of convenience for product geometers in data processing, making PH curve application research a hot spot in CAGD. [3-4] introduce the geometric properties of the control polygon of odd-order PH curve, and [5] gives two methods to distinguish the cubic and quintic PH curves.

Subsequently, Farouki et al. studied the interpolation of PH curve and obtained many research results [6]. PH curve interpolation problems are generally divided into Hermite interpolation and spline interpolation, in Hermite interpolation can be divided into parametric continuity and geometric continuity [7]. Meek and Walton studied G^1 Hermite interpolation of cubic parametric curves [8]. Liu et al. made an in-depth study on the structure of PH curve [9-13]. These studies prove that the geometric continuous structure of the PH curve has become a hot spot, and in practical applications, the G^2 geometric continuity at the splicing point of cubic PH curves can meet the product requirements.

Roulier and Piper studied the construction of Bézier curves satisfying a given arc length by using the method of iterative numerical calculation [14]. For similar problems, Farouki et al. obtained a simpler and more effective construction by using the PH curve because the arc length of PH curve can be displayed [12]. Shetty et al. proposed different methods for the extension of B-Spline curve, so that the extended curve and the original curve can be C^2 continuous spliced at the splicing point [15]. Zhou studied the continuation of cubic Bézier curve under G^2 continuous constraints [16].

This paper also makes use of the property that the arc length of PH curve can be expressed in an accurate and rational formula, and focuses on the study of the curve with the shortest arc length for splicing with the constraint of arc length after the G^2 continuous PH curve of three times [11]. Lagrange multiplier method is used to set the initial value of all variables, and iteration method is used to select the optimal control points of the curve to satisfy the G^2 continuous at the splicing point. However, because the curve shape of continuation is not unique, in order to get a better shape of the curve, the arc length as the constraint makes the curve of continuation more ideal.

The rest of this paper is organized as follows. In the second part, some basic theorems and inferences about the cubic PH curve are proposed. The third part solves the problem. By constraining the shortest arc length, given the first and last end points of the two cubic PH curves (including the connection points), the two cubic PH curves can reach G^2 continuous at the splicing point. The forth part illustrates our method through a few examples. Finally, a brief conclusion is made in fifth part.

II. Cubic Pythagorean Hodograph curves E

In 1990, Farouki and Sakkalis first proposed the PH curve, which has an accurate isometric line and can be accurately calculated [2]. Based on the excellent properties of PH curve, the research on PH curve becomes the focus. Subsequently, Farouki added to the properties of PH curves and introduced the geometric properties of the control polygon of the cubic PH curve [3]. In 2015, Farouki presented two ways to identify the cubic PH curve [3, 5]. Liu et al. studied the formation of three PH curves based on three Bezier curves [9]. Buqing Su et al. studied the geometric relationship between control points when the PH curve was continuous [10, 11]. In 2014, Farouki et al. also studied the Hermite interpolation of the fifth, sixth and seventh order PH curves [12, 13].

Next, we will propose some basic theorems and inferences about the cubic PH curve. First, we give the structural relationship of the control vertices of the PH curve.

A planar Bézier curve $r(t)$ with control points P_i, $i=0,1,\cdots,n$, can be expressed as

$$r(t)=\sum_{i=0}^{n}P_iB_i^n(t)$$

where $B_i^n(t)$ is Bernstein basis function.

Theorem 1 ([2]) The cubic Bézier curve $r(t)$ is a PH curve if and only if its control points satisfy the following relations:

$$\begin{cases} P_1=P_0+\frac{1}{3}(u_0{}^2-v_0{}^2,2u_0v_0) \\ P_2=P_1+\frac{1}{3}(u_0u_1-v_0v_1,u_0v_1+u_1v_0) \\ P_3=P_2+\frac{1}{3}(u_1{}^2-v_1{}^2,2u_1v_1) \end{cases} \quad (1)$$

where (u_0,v_0) and (u_1,v_1) are real numbers.

Corollary 2 ([11]) The arc length of cubic PH curve can be expressed as a rational formula:

$$S=S(1)=\frac{1}{3}(u_0{}^2+v_0{}^2+u_0u_1+v_0v_1+u_1{}^2+v_1{}^2) \quad (2)$$

where u_0, v_0, u_1 and v_1 are real numbers.

Corollary 3 ([10]) Suppose that the control points of two cubic PH curves $L(t)$ and $R(t)$ are $\{P_i\}_{i=0}^3$ and $\{Q_i\}_{i=0}^3$ respectively, $L(t)$ and $R(t)$ are spliced, with P_3 and Q_3 as splicing points. When the two curves reach G^2 continuous at the splicing point, the control points satisfy the following relations:

$$\begin{cases} Q_0=P_3 \\ Q_1=P_3+\alpha(P_3-P_2) \\ Q_2=P_3+\alpha(P_3-P_2)-\alpha^2(P_2-P_1)+r(P_3-P_2) \end{cases} \quad (3)$$

where $\alpha>0$, r are real numbers.

III. G^2 Continuous Shortest Splicing SF Cubic PH Curves

A. Description of the problem

This paper presents a method for constructing G^2 continuous cubic PH curves based on arbitrary control polygons. Given the beginning and end points of the curve (including join points) of two cubic PH curves. We can define different interior points for each side of the control polygon, so that the curve is G^2 continuous at the splice points. By using the shortest arc length as the constraint, the whole curve is more optimized. In the following work, the control point expression of PH curve was first given, and the curve algorithm was derived when the curve reached G^2 continuous. Finally, the solution was carried out under the constraint of arc length.

Let the control vertices of the two cubic Bezier curves $L(t)$ and $R(t)$ are $\{P_i\}_{i=0}^3$ and $\{Q_i\}_{i=0}^3$ respectively, namely

$$L(t)=\sum_{i=0}^{3}P_iB_i^3(t),\ R(t)=\sum_{i=0}^{3}Q_iB_i^3(t),\ t\in[0,1]$$

When the two curves are cubic PH curves, their control vertices all satisfy equation (1), and when the curve is G^2 continuous at the splicing point, the control vertices should satisfy equation (3).

For the two cubic PH curves of given starting and ending points, suppose that

$$P_0=(x_0,y_0),\ P_3=Q_0=(x_3,y_3),\ Q_3=(x_4,y_4) \quad (4)$$

and the control vertex of curve $L(t)$ satisfies the following relationship:

$$\begin{cases} P_1=P_0+\frac{1}{3}(u_0{}^2-v_0{}^2,2u_0v_0) \\ P_2=P_1+\frac{1}{3}(u_0u_1-v_0v_1,u_0v_1+u_1v_0) \\ P_3=P_2+\frac{1}{3}(u_1{}^2-v_1{}^2,2u_1v_1) \end{cases} \quad (5)$$

the control vertex of curve $R(t)$ satisfies the following relationship:

$$\begin{cases} Q_1=Q_0+\frac{1}{3}(s_0{}^2-t_0{}^2,2s_0t_0) \\ Q_2=Q_1+\frac{1}{3}(s_0s_1-t_0t_1,s_0t_1+s_1t_0) \\ Q_3=Q_2+\frac{1}{3}(s_1{}^2-t_1{}^2,2s_1t_1) \end{cases} \quad (6)$$

where u_i, v_i, s_i and t_i, $i=0,1$ are real numbers. From equation (2), we have the total arc length of the two curves is

$$S=S_L+S_R=\frac{1}{3}(u_0{}^2+v_0{}^2+u_0u_1+v_0v_1+u_1{}^2+v_1{}^2+s_0{}^2+t_0{}^2$$
$$+s_0s_1+t_0t_1+s_1{}^2+t_1{}^2) \qquad (7)$$

Meanwhile, the control points of the two curves satisfy equation (3). Let g_0, g_1, g_2, g_3 are

$$\begin{cases} g_0=P_3-(x_3,y_3) \\ g_1=Q_3-(x_4,y_4) \\ g_2=P_3+\alpha(P_3-P_2)-Q_1 \\ g_3=P_3+\alpha(P_3-P_2)-\alpha^2(P_2-P_1)+r(P_3-P_2)-Q_2 \end{cases} \qquad (8)$$

Since this paper is the splicing of two cubic PH curves discussed on the plane, a total of 8 equations are satisfied in equation (8), which are denoted as:

$$g_{01}=0,\ g_{02}=0,\ g_{11}=0,\ g_{12}=0,\ g_{21}=0,\ g_{22}=0,$$
$$g_{31}=0,\ g_{32}=0.$$

So the problem is to make the total arc length of curves $L(t)$ and $R(t)$ shortest under the constraints of equations (4), (5), (6) and (8), that is

$$\min\ (S_L+S_R)$$

s.t. Equations (4), (5), (6) and (8) are valid.

B. Establish and solve the objective function

In this paper, two cubic Bézier curves are G^2 continuous at the splicing point, so the control points need to satisfy equation (3). Meanwhile, the control vertex of two Bézier curves satisfies equation (5) and equation (6), which are cubic PH curves. After the combination of equations (3), (5) and (6), the cubic PH curves can be made G^2 continuous. Next, our main work is how to choose the free quantity u_0, v_0, u_1, v_1, s_0, t_0, s_1, t_1, α, γ to make the two obtained PH curves have the shortest arc length.

From the arc length formula of the cubic PH curve, the constraint condition is that the G^2 continuous is reached at the spline point, and the minimum arc length is the objective function. According to equations (7) and (8), the Lagrangian function is established:

$$L=S_L+S_R+\lambda_1g_{01}+\lambda_2g_{02}+\lambda_3g_{11}+\lambda_4g_{12}+\lambda_5g_{21}+\lambda_6g_{22}$$
$$+\lambda_7g_{31}+\lambda_8g_{32}$$
$$=\frac{1}{3}(u_0{}^2+v_0{}^2+u_0u_1+v_0v_1+u_1{}^2+v_1{}^2+s_0{}^2+t_0{}^2+s_0s_1+t_0t_1$$
$$+s_1{}^2+t_1{}^2)+\lambda_1g_{01}+\lambda_2g_{02}+\lambda_3g_{11}+\lambda_4g_{12}+\lambda_5g_{21}+\lambda_6g_{22}$$
$$+\lambda_7g_{31}+\lambda_8g_{32} \qquad (9)$$

Then, equations (4), (5) and (6) are substituted into equation (9), and the Lagrange multiplier method is applied to solve the equation, suppose that

$$q_1=\frac{\partial}{\partial u_0}L,\ q_2=\frac{\partial}{\partial v_0}L,\ q_3=\frac{\partial}{\partial u_1}L,\ q_4=\frac{\partial}{\partial v_1}L,\ q_5=\frac{\partial}{\partial s_0}L,$$
$$q_6=\frac{\partial}{\partial t_0}L,\ q_7=\frac{\partial}{\partial s_1}L,\ q_8=\frac{\partial}{\partial t_1}L,\ q_9=\frac{\partial}{\partial \alpha}L,\ q_{10}=\frac{\partial}{\partial r}L。$$

there are a total of 18 free quantities, which are

$$u_0,\ u_1,\ v_0,\ v_1,\ s_0,\ s_1,\ t_0,\ t_1,\ \alpha,\ r,$$
$$\lambda_1,\ \lambda_2,\ \lambda_3,\ \lambda_4,\ \lambda_5,\ \lambda_6,\ \lambda_7,\ \lambda_8$$

and 18 equations need to be satisfied, which are

$$g_{01}=0,\ g_{02}=0,\ g_{11}=0,\ g_{12}=0,\ g_{21}=0,\ g_{22}=0,$$
$$g_{31}=0,\ g_{32}=0, q_1=0,\ q_2=0,\ q_3=0,\ q_4=0, q_5=0,$$
$$q_6=0,\ q_7=0,\ q_8=0,\ q_9=0,\ q_{10}=0。$$

Because the calculation of nonlinear system is too complex, it is very difficult to calculate 18 free variables directly. By giving the initial values of the 18 degrees of freedom, we selected the optimal parameters satisfying the system through iteration. Finally, the arc lengths and control points of the two PH curves were obtained by substituting equations (5) and (6).

C. Specific algorithm

The algorithm for constructing a cubic PH curve with the shortest arc length and G2 continuous at the splicing point is as follows:

The algorithm constructs the PH curve of the shortest G^2 continuous three arc lengths

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Input: control vertex P_0, $P_3(Q_0)$, Q_3

Output: all free quantities of two cubic PH curves

a) Given the control point parameters of two cubic PH curves: u_i, v_i, s_i, t_i, i=1,2;

b) Lagrange function is established with equations (4), (5), (6) and (8) as constraints and equation (7) as objective function;

c) A nonlinear system with 18 free variables and 18 equations is established, and the initial values of these 18 free variables are given
$\Lambda=\{u_i,\ v_i,\ s_i,\ t_i,\ \alpha,\ \gamma,\ \lambda_j\},\ i=0,1;\ j=1,\cdots,8$
Through the iterative method, the optimal solution is obtained:
$\bar{\Lambda}=\{\bar{u}_i,\ \bar{v}_i,\ \bar{s}_i,\ \bar{t}_i,\ \bar{\alpha},\ \bar{\gamma},\ \bar{\lambda}_j\},\ i=0,1;\ j=1,\cdots 8;$

d) Solve the control points P_i, Q_i, $i=0,\cdots,3$ of the cubic PH curve by substituting $\bar{\Lambda}$ into equations (5) and (6).

e) Calculate the total arc length by substituting

$\bar{\Lambda}$ into equation (7) and draw the curve.

IV. EXAMPLE COMPARISON

In this section, we list some examples to construct the shortest concatenation of two cubic PH curves when they reach G^2 continuous. And compare different coordinates and different initial values. We set the output to 5 bits precision.

A. Example one

Suppose that the first and last endpoints of the two cubic PH curves (including the splicing point) are $P_0=(-2,-3)$, $P_3=Q_0=(0,0)$, $Q_3=(3,4)$, They reach G^2 continuous at the splicing point and the total arc length is the shortest. Input initial values

$$\Lambda_1=\{1;\ 0.6;\ 1.2;\ 2.1;\ -2.7;\ 0.5;\ 0.7;\ 0.2;\ 0.5;\ 2;\ 0.9;\ 1.1;\ 2.4;\ 2;\ 0.7;\ 1.3;\ -2;\ 1.1\},$$

After 32 iterations, the optimal solution is obtained

$$\bar{\Lambda}_1=\{\bar{u}_0,\ \bar{u}_1,\ \bar{v}_0,\ \bar{v}_1,\ \bar{s}_0,\ \bar{s}_1,\ \bar{t}_0,\ \bar{t}_1\}=\{0.0760,\ 2.8949,\ -0.5413,\ 1.8155,\ -3.2332,\ 3.7388,\ -2.0276,\ 1.5242\}$$

By substituting $\bar{\Lambda}_1$ into equations (5) and (6), the control points are obtained:

$P_0=(-2,-3)$, $P_1=(-2.0957,\ -3.0274)$,
$P_2=(-1.6948,\ -3.5038)$, $P_3=(-2.7127\text{e-}05,\ 2.4310\text{e-}05)$,
$Q_1=(2.1141,\ 4.3704)$, $Q_2=(-0.8852,\ 0.2008)$,
$Q_3=(3.0000,\ 4.0000)$.

The total arc length is obtained by substituting $\bar{\Lambda}_1$ into equation (7). In this case, the corresponding cubic PH curve is shown in fig.1. Obviously, the error generated P_3 and Q_3 by interpolation is almost negligible.

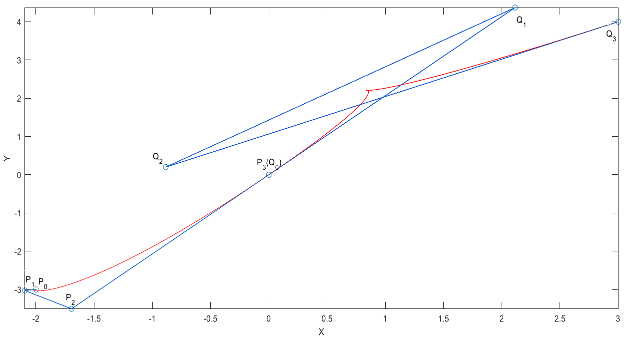


Fig. 1. *Example of initial value is* Λ_1

B. Example two

Suppose that the three points of two cubic PH curves are $P_0=(0.1,0.2)$, $P_3=Q_0=(0.3,0.6)$ and $Q_3=(0.9,1.4)$ respectively. Find the curve path with the shortest total arc length and G^2 continuous at the splicing point. Given initial value

$$\Lambda_2=\{1;\ 0.6;\ 1.2;\ 2.1;\ -2.7;\ 0.5;\ 0.7;\ 0.2;\ 0.5;\ 2;\ 0.9;\ 1.1;\ 2.4;\ 2;\ 0.7;\ 1.3;\ -2;\ 1.1\},$$

After 108 iterations, the optimal solution is

$$\bar{\Lambda}_2=\{-1.1119,\ 0.7733,\ -0.7029,\ 0.4174,\ -0.5960,\ 1.7588,\ -0.3270,\ 0.8851\}$$。

Substitute $\bar{\Lambda}_2$ into (5) and (6) respectively to get the control points $\{P_i\}_{i=0}^3$ and $\{Q_i\}_{i=0}^3$. Substitute into equation (7) to obtain the total arc length $S_{L+R}=1.4502$. The corresponding cubic PH curve is shown in fig.2.

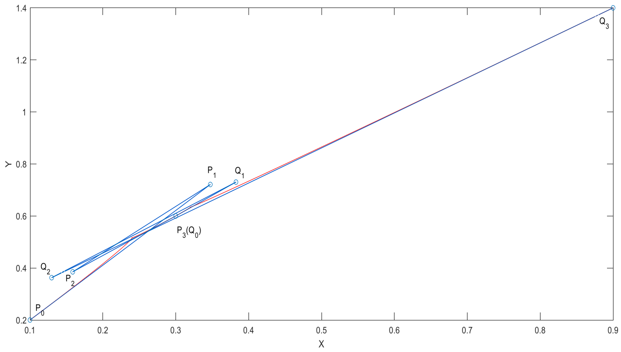


Fig. 2. *Example of initial value is* Λ_2

C. Example three

It is known that the first and last two endpoints of two cubic PH curves $L(t)$ and $R(t)$ are $P_0=(0,0)$, $Q_3=(2,0)$ and $P_3=Q_0=(1,1)$, making the curves G^2 continuous and the total arc length shortest when splicing. Given two initial values, respectively

a) $\Lambda_3=\{0.1;\ 1;\ 0.1;\ 1;\ 0;\ 0.1;\ 0.1;\ 0;\ 1;\ 1;\ 1;\ 1;\ 0;\ 0;\ 0;\ 1;\ 0.1;\ 0.1\}$

After 109 iterations, the optimal solution could be obtained

$$\bar{\Lambda}_3=\{\bar{u}_0,\ \bar{u}_1,\ \bar{v}_0,\ \bar{v}_1,\ \bar{s}_0,\ \bar{s}_1,\ \bar{t}_0,\ \bar{t}_1\}=\{1.7051,\ 0.3130,\ 0.4778,\ 0.5338,\ 0.2134,\ 1.7746,\ -0.1246,\ -0.7081\}$$

Substitute them into (5) and (6) respectively to get the control points

$P_1=(0.8930,\ 0.5431)$, $P_2=(0.9859,\ 0.8964)$,
$Q_1=(0.9336,\ 0.9900)$, $Q_2=(1.0304,\ 0.8660)$.

Substitute into equation (7) to obtain the total arc length $S_{L_1+R_1}=2.8286$. The corresponding PH curve is shown in fig.3(a).

b) $\Lambda_4=\{0.5;\ 0.8;\ 1.2;\ 1.5;\ 2;\ -0.5;\ 2;\ 1;\ 3;\ 0.5;\ 1;\ 1;\ 1.2;\ 2.1;\ 3;\ 0.4;\ 1;-2\}$

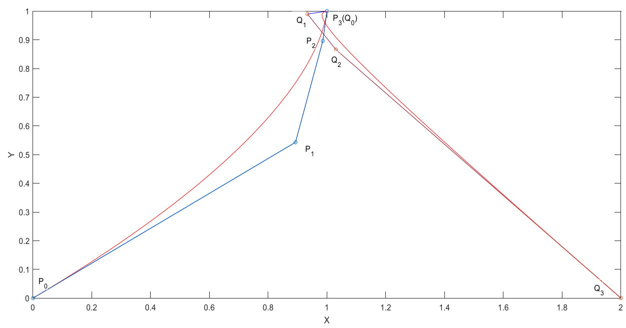
After 23 iterations,

$$\overline{\Lambda}_4 = \{\overline{u}_0, \overline{u}_1, \overline{v}_0, \overline{v}_1, \overline{s}_0, \overline{s}_1, \overline{t}_0, \overline{t}_1\} = \{-0.5454, 2.4472, -1.5459, 1.1720, 3.5448, -3.8335, 1.6976, 2.0686\}$$

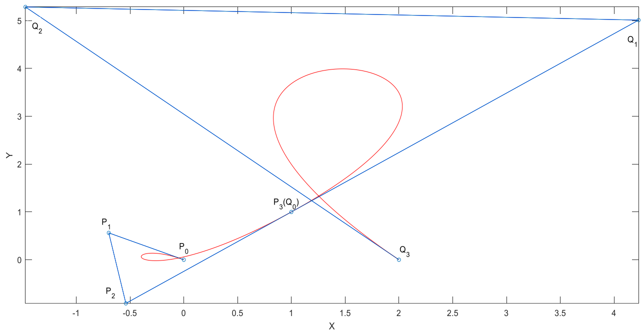
Substitute them into (5) and (6) respectively to get the control points

$P_1 = (-0.6974, 0.5621)$, $P_2 = (-0.5384, -0.9120)$,
$Q_1 = (4.2279, 5.0118)$, $Q_2 = (-1.4723, 5.2868)$.

Substitute it into equation (7) to obtain the total arc length $S_{L_2+R_2} = 10.4160$. The corresponding PH curve is shown in fig.3(b).



(a). *Example of initial value is* Λ_3



(b). *Example of initial value is* Λ_4

Fig. 3. PH curves of different initial values were selected

Obviously, although figure (a) and figure (b) are the same endpoints P_0, $,P_3,Q_3$ in fig.3, their initial values are different, and the resulting PH curve has different paths and arc lengths.

D. Remark

- The quintic PH curve or even the higher PH curve can also be expressed in the form of Bézier curve, giving its control points relationship. Moreover, G^2 continuous splicing of curves under the constraint of the shortest arc length can still be obtained by using the method in this paper, but will establish a nonlinear system with more variables.
- We discussed the cubic PH curve of G^2 splicing. If the shortest cubic PH curve of G^3 splicing is obtained, G^3 continuous conditions need to be added. Solving the equation will become more complicated and the amount of calculation is too large, so we need to further study how to optimize it.
- How to simplify the calculation of nonlinear equations is one of our future research directions.

V. Conclution

In this paper, the splicing of cubic PH curves is studied. When the G^2 continuous is satisfied, the control points and corresponding curve of the curve are obtained by taking the shortest arc length as the objective function. In the process of research, it is found that the free parameters increase the flexibility of the curve. When different initial values are defined, the generated curve shape and arc length are also different. However, we need to further study how to select the appropriate initial values to make the arc length shorter and whether the curve can achieve higher order geometric continuity.

The advantage of this paper lies in the simple form of the objective function, which can not only get the shortest curve through iteration, but also be compared easily. When the splicing points are G^2 continuous, the selected curve has a more ideal shape through the restriction of arc length, so as to meet the needs of users.

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