Upper Bound for the Least Poisson Ratio of Orthotropic Materials

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***Abstract*—A recent excellent article, 'Poisson's ratio bounds in orthotropic materials…' by Mentrasti, et al has drawn our attention to Chapter II of Vijayakumar's five-decade-old Doctoral Thesis. In this Chapter II, bounds for Poisson's ratios of orthotropic material are considered by applying the well-known energy and volume criteria with emphasis on upper bounds for the least Poisson's ratio. Assuming that all six Poisson's ratios are positive, true for structural materials, none of them is greater than unity and three or more of them are less than 1/2. The upper bound for the least Poisson's ratio from the volume criteria gives a better upper bound than that of energy criteria.**

Keywords—Elasticity; Orthotropy; Poisson Ratios; Upper Bounds

1. INTRODUCTION

Ever since Hooke enunciated the law of proportionality of stress and strain, elastic properties of materials received considerable attention from experimental and theoretical physicists and engineers [1, 2, 3]. Recent trends in modern technology and rapid progress in the material sciences demand a thorough investigation of the elastic behaviour of anisotropic materials.

It is well known that the positive-definiteness of quadratic strain-energy function of a given material within the elastic range imposes certain restrictions on the possible values of elastic constants. Knowledge of these restrictions and the consequent relationships between elastic constants is of significance for experimental and theoretical investigations. They are useful in separating out extraneous values which sometimes appear when elastic constants are deduced from sound wave measurements. Further, these restrictions are helpful in restricting the compilation of analytical and design information on structural components to physically sensible ranges of material constants.

In a recent article [4], non-trivial constraints in a particularly expressive form of the domain of existence, named Tetrahedron-Ellipsoid locus, are presented in the space of three independent Poisson's ratios. Several notable restrictions for the elastic behaviour of orthotropic composite materials have been established. An extensive, experimental investigation is carried out to detect several Poisson's ratios directly measured by means of physical strain gauges and Digital Image Correlation.

In view of the above mentioned article [4], a relook at Chapter II entitled, 'Bounds for Poisson's ratios of orthotropic materials', in Vijayakumar's doctoral thesis [5] has become relevant now. In this Chapter II, bounds for Poisson's ratios in relation to Young's moduli, $E_1 \leq E_2 \leq E_3$ without any loss of generality, are presented with emphasis on upper bounds, ν_e and ν_v, for the least Poisson's ratio by the well-known energy and volume criteria, namely,

(1) The strain energy density function in terms of stresses of a given material is positive for an arbitrary stress system
(2) The first order change in volume of an element of a given medium is positive for an arbitrary tensile stress system.

An earlier attempt of this nature by Bert [6] was highly restrictive and contained a basic error in the analysis. In chapter II of doctoral thesis [5], positive least Poisson's ratio is mathematically identified and upper bounds (v_e, v_v,) are derived. Validity of these upper bounds is verified with some experimental data of elastic constants [7] reported earlier.

II SOME BASIC RELATIONS

Let 0-1, 0-2, 0-3 be the three mutually orthogonal axes of elastic orthotropic body of a given material and let E_1, E_2, E_3 be the corresponding Young's moduli. From the well-known quadratic form of strain-energy density functions, the stress-strain relationships can be put in the form, (using the summation convention over the repeated indices, i, j, k = 1, 2, 3)

$$\varepsilon_i = (\sigma_i / E_i) - \tfrac{1}{2} [v_{ji} (\sigma_j / E_j) + v_{ki} (\sigma_k / E_k)],$$

$$i \neq j \neq k \qquad (1)$$

$$\gamma_{ij} = \gamma_{ji} = \tau_{ij} / G_{ij}, \qquad i \neq j \qquad (2)$$

$$v_{ij} / E_i = v_{ji} / E_j \qquad i \neq j \qquad (3)$$

For given Young's moduli, we have from relations (3) that three of the six Poisson's ratios are independent. To specify the least Poisson's ratio, one should, therefore, bring in two more relations among them. Bert [6] specified these two relationships by assuming $v_{ij} = v_{ik}$ and on this restrictive physical basis, determined the maximum variation of the least Poisson's ratio by both energy and volume criteria. (It is to be noted, however, that some of the numerical results [6] are not correct on account of a basic error in his analytical procedure). In our analysis, we shall use only the mathematical condition that it is the least Poisson's ratio to provide the necessary relationships.

Before proceeding with the analysis, it is convenient to introduce some additional notation:

$$v_{ij} / E_i = \alpha_{ij} \qquad i \neq j = 1, 2, 3 \qquad (4)$$

$$K_{ij} = \sqrt{(E_i / E_j)} \qquad i \neq j = 1, 2, 3 \qquad (5)$$

$$A_{ij} = \alpha_{ij} \sqrt{(E_i E_j)} = \sqrt{(v_{ij} v_{ji})} = v_{ij} / K_{ij}$$

$$i \neq j = 1, 2, 3 \qquad (6)$$

From equations (3), we have $\alpha_{ij} = \alpha_{ji}$ and from equations (6), $A_{ij} = A_{ji}$ from which it is clear that if A_{ij} are bounded, all Poisson's ratios are also bounded.

III ENERGY CRITERION

By the energy criterion, the strain-energy density

$$U_0 = \tfrac{1}{2} \Sigma\, \sigma_i \varepsilon_i + \tfrac{1}{4} \Sigma\, \tau_{ij} \gamma_{ij} \qquad (7)$$

is positive for any arbitrary stress system. Using relations (1), (2) and (3), U_0 can be written as

$$U_0 = \tfrac{1}{2} U_1 + \tfrac{1}{4} U_2$$

in which

$$U_1 = \Sigma (\sigma_i^2 / E_i) + \Sigma\, \alpha_{ij} \sigma_i \sigma_j, \qquad (8a)$$

$$U_2 = \Sigma \tau_{ij}^2 / G_{ij} \qquad (8b)$$

Since $U_2 \geq 0$ due to G_{ij} being positive, $U_0 \geq 0$ if and only if $U_1 \geq 0$ for arbitrary stresses [σ_1, σ_2, σ_3].

Using relations (6) and the transformations

$$\sigma_i' = \sigma_i / \sqrt{E_i} \qquad (9)$$

$$\sigma_1'' = \sigma_i' - A_{12} \sigma_2' - A_{31} \sigma_3', \qquad (10)$$

we have, henceforth superscript ^{2} of a quantity denoting square of the quantity,

$$U_1 = \sigma_1''^2 + (1 - A_{12}^2) \sigma_2'^2 + (1 - A_{23}^2) \sigma_3'^2 - 2(A_{23} + A_{31} A_{12}) \sigma_2' \sigma_3' \qquad (11)$$

which has to be positive for arbitrary stresses. Then, it follows that

$$A_{12}^2 \leq 1 \quad \text{and} \quad A_{23}^2 \leq 1 \qquad (12)$$

If $A_{12}^2 = 1$, say, it would also follow that

$$A_{23} + A_{31} A_{12} = 0, \qquad (13)$$

so that one of the A_{ij} is negative.

If one assumes that all v_{ij} are positive (a valid assumption for orthotropic structural materials in their aggregate forms), no $A_{ij} < 0$ and $A_{12}^2 \neq 1$. Hence, one concludes that $A_{12} < 1$. By the symmetry in U_1, it follows that all A_{ij} are less than unity. Hence, referring to equations (5) and (6), one concludes that

$$v_{ji}^2 < E_i / E_j \qquad (14)$$

To seek further information, affect another transformation

$$\sigma_2'' = \sigma_2' - (A_{23} + A_{31} A_{12})\, \sigma_3' / (1 - A_{12}^2) \qquad (15)$$

so that one may rewrite

$$U_1 = \sigma_1''^2 + (1 - A_{12}^2)\, \sigma_2''^2 +$$

$$+ [1 - (A_{12}^2 + A_{23}^2 + A_{31}^2 + 2 A_{12} A_{23} A_{31})]\, \sigma_3'^2 / (1 - A_{12}^2) \qquad (16)$$

The condition $U_1 \geq 0$ now yields the constraint

$$A_{12}^2 + A_{23}^2 + A_{31}^2 + 2 A_{12} A_{23} A_{31} \leq 1 \qquad (17)$$

Considering the case $A_{12} = A_{23} = A_{31} = A_0$, it is readily seen that at least one of them cannot exceed A_0 and that $A_0 = \frac{1}{2}$.

The inequality (17) can be restated as constraints

$$(v_{ij}^2 / K_{ij}^2 + v_{jk}^2 / K_{jk}^2 + v_{ki}^2 / K_{ki}^2 + 2 v_{ij} v_{jk} v_{ki}) \leq 1, \quad i \neq j \neq k = 1, 2, 3 \qquad (18)$$

with any three independent Poisson's ratios. Since we have assumed that all $\nu_{ij} \geq 0$, these relations imply that

$$v_{ij}^2 / K_{ij}^2 + v_{jk}^2 / K_{jk}^2 \leq 1, \; i \neq j \neq k = 1, 2, 3 \qquad (19)$$

Therefore, for a given set of ratios of Young's moduli, variation of any two Poisson's ratios is subjected to the corresponding inequality in equation (19). For specified values of these two ratios, the variations of the remaining Poisson's ratio from the relations (18) are given by

$$v_{ki} / K_{ki} = v_{ik} / K_{ik} \leq [(1 - v_{ij}^2 / K_{ij}^2)(1 - v_{jk}^2 / K_{jk}^2)]^{1/2} - (v_{ij} / K_{ij})(v_{jk} / K_{jk}) \qquad (20)$$

Further, the inequality sign in the equation (18) is satisfied for

$$v_{ij} / K_{ij} = (v_{jk} / K_{jk}) = (v_{ki} / K_{ki}) = \tfrac{1}{2} \qquad (21)$$

Hence,

$$\min (v_{ij}, v_{jk}, v_{ki}) \leq \tfrac{1}{2} \max (K_{ij}, K_{jk}, K_{ki}) \quad i \neq j \neq k = 1, 2, 3 \qquad (22)$$

To obtain an upper bound for the least Poisson's ratio, we stipulate, without any loss of generality, that $E_1 \leq E_2 \leq E_3$. Then from equation (2), the least Poisson's ratio is one among v_{12}, v_{23} and v_{13}. The corresponding constraint in equation (18) is

$$[(v_{12} / K_{12})^2 + (v_{23} / K_{23})^2 + (v_{13} / K_{13})^2 + 2 v_{12} v_{23} v_{13} / K_{13}^2] \leq 1 \qquad (18a)$$

Considering the equality relation in the above, with $v_{12} = v_{23} = v_{13} = v$ and $K_{13} = (K_{12} K_{23})$, we obtain

$$\Phi (v) = [2 v^3 + (1 + K_{12}^2 + K_{23}^2) v^2 - K_{12}^2 K_{23}^2] = 0 \qquad (23)$$

For given values of K_{12} and K_{13} which are not greater than unity, it is shown below that the equation (23) possesses only one positive root 'v_e', the upper bound to the least Poisson's ratio by the energy criteria. For this purpose, above cubic equation, with $K_{13}^2 = E_1 / E_3$, $K_{12}^2 = E_1 / E_2$, $K_{23}^2 = E_2 / E_3$, and using $K_{13} = K_{12} K_{23}$, is expressed as

$$[2 v^3 / (K_{13} K_{12} K_{23}) + (1/K_{13}^2 + 1/K_{12}^2 + 1/K_{23}^2) v^2 - 1 = 0 \qquad (24)$$

Note that isotropic material corresponds to $K_{13} = 1$. In this case, equation (24) becomes $(2 v^3 + 3 v^2 - 1) = 0$ which is

$$(1 + v)^2 (2 v - 1) = 0 \qquad (25)$$

It shows that $v = \frac{1}{2}$ is the only one positive root. Hence, energy criteria also gives the same well known upper bound of the Poisson's ratio by the volume criteria. It is likely that mod $|v| < 1$ if Young's moduli of orthotropic material are close to the isotropic modulus E.

For $K_{13} \neq 1$, the equation (24) can be put in the form, with

$a = 2 / (K_{13}K_{12}K_{23})$, $b = \frac{1}{3}\,[1/K_{13}^2 + 1/K_{12}^2 + 1/K_{23}^2]$, and $x = (a\,v + b)$,

$$x^3 + 3Hx + G = 0 \qquad (26)$$

in which $H = -b^2$ and $G = -a^2 + 2b^3$.

The nature of roots of equation (26) is dependent [6] on the expression q^2 which is $[-(G^2 + 4H^3)]$.

We show, now, that $q^2 = 0$ for isotropic materials (i.e., for $E_1 = E_2 = E_3$), and is positive for other materials.

Let $B_{ij} = (1/K_{ij})^{2/3}$, (i, j = 1, 2, 3). Then,

$$q^2 = -(G^2 + 4H^3) = (4a^2b^3 - a^4) = [16 / (K_{13}K_{12}K_{23})^4]\,[(b / B_{13}B_{12}B_{23})^3 - 1] \qquad (27)$$

Therefore, $q^2 \geq 0$ according as $b \geq B_{13}B_{12}B_{23}$. However,

$$2B_{13}B_{12}B_{23} = [B_{13}^3 + B_{12}^3 + B_{23}^3 - 3B_{13}B_{12}A_{23}]$$

$$= [(B_{13} + B_{12} + B_{23})\{(B_{13} - B_{12})^2 + (B_{12} - B_{23})^2 + (B_{23} - B_{13})^2\}] \qquad (28)$$

From relation (18a) we have that at least one of the Poisson's ratios v_{12}, v_{23}, v_{13} is not greater than 'v_e', that is, v_e is an upper bound for the least Poisson's ratio. Figure 1 shows the dependence of v_e on E_1/E_3 (= K_{13}^2), the ratios of Young's moduli.

1V VOLUME CRITERION

By the volume criterion, the first order change in volume $\Delta V_0 = (\varepsilon_1 + \varepsilon_2 + \varepsilon_3)$ is to be positive for arbitrary tensile stress system. Using relations (1) and (2), we rewrite ΔV_0 with (i ≠ j ≠ k =1, 2, 3), as

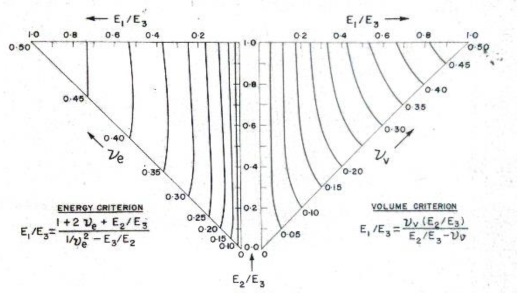

FIG. 1-UPPER BOUNDS FOR LEAST POISSON'S RATIO. ν_e BY ENERGY CRITERION & ν_v BY VOLUME CRITERION. ($E_1 \leq E_2 \leq E_3$)

$$\Delta V_0 = \sum \left(1 - \tfrac{1}{2}v_{ij} - \tfrac{1}{2}v_{ik}\right)(\sigma_i / E_i) \qquad (29)$$

This is positive for arbitrary tensile stress system, if and only if

$$N_i = \tfrac{1}{2}(v_{ij} + v_{ik}) \leq 1, \quad i \neq j \neq k = 1, 2, 3$$

That is,

$$N_1 = \tfrac{1}{2}(v_{12} + v_{13}) \leq 1, \qquad (30a)$$

$$N_2 = \tfrac{1}{2}(v_{21} + v_{23}) \leq 1, \qquad (30b)$$

$$N_3 = \tfrac{1}{2}(v_{31} + v_{32}) \leq 1, \qquad (30c)$$

It can be seen from the above relations that the volume criterion by itself yields no bounds for individual Poisson's ratios but only for the two dimensional Poisson parameters $N_i \leq 1$.

However, by assuming $\nu_{ij} \geq 0$ as before, one can obtain upper bounds for individual Poisson's ratios. Then, from inequalities (30), we have that none of the six Poisson's ratios is greater than unity and at least three of them are not greater than (1/2).

To get an upper bound for the least Poisson's ratio we stipulate again that $E_1 \leq E_2 \leq E_3$ and rewrite the relations (30) as

$$v_{12} + v_{13} \leq 1, \; [v_{12} / K_{12}^2] + v_{23} \leq 1,$$

$$[v_{13} / K_{13}^2] + [\nu_{23} / K_{23}^2] \leq 1, \qquad (31)$$

The above relations imply that

$$v_{12} \leq (K_{12})^2 = E_1/E_2, \quad v_{23} \leq (K_{23})^2 = E_2/E_3,$$

$$v_{13} \leq (K_{13})^2 = E_1/E_3 \qquad (32)$$

In view of these relations, the first relation in (30) can be expressed as

$$\nu_{12} + \nu_{13} \leq \min [1, (K_{12}^2 + K_{13}^2)] \quad (33)$$

Hence, for specified values of two of these Poisson's ratios ν_{12}, ν_{23}, ν_{13} subjected to the corresponding inequality in (22), one can easily obtain the maximum variation for the remaining Poisson's ratios. Further, we also have that

$$v_{12} \text{ (or } v_{13}) \leq \tfrac{1}{2} \min [1, (K_{12}^2 + K_{13}^2)], \text{ or}$$

$$v_{12} \text{ (or } v_{23}) \leq K_{12}^2 / (1 + K_{12}^2), \text{ or}$$

$$v_{13} \text{ (or } v_{23}) \leq (K_{13}^2 K_{23}^2) / (K_{13}^2 + K_{23}^2]$$

$$\leq K_{13}^2 / (1 + K_{12}^2) \quad (34)$$

Since the least Poisson's ratio, 'v', is one among v_{12}, v_{23}, v_{13}, we have from above,

$$v \leq \min [\tfrac{1}{2}, \tfrac{1}{2} (K_{12}^2 + K_{13}^2), K_{12}^2 / (1 + K_{12}^2), K_{13}^2 / (1 + K_{12}^2)] \quad (35)$$

From above inequality, with $E_1 \leq E_2 \leq E_3$ without any loss of generality, one concludes that

$$v \leq K_{13}^2 / (1 + K_{12}^2), \text{ that is,}$$

$$v \leq (1/E_3) / [(1/E_1) + (1/E_2)] \quad (36)$$

Hence an upper bound, 'v_v', for the least Poisson's ratio from volume criterion [8] is

$$v_v = (1/E_3) / [(1/E_1) + (1/E_2] \quad (37)$$

The dependence of v_v on E_1 / E_3 and E_2 / E_3 is shown in Figure 1.

Referring to equations (20, 23), it can be easily shown that $\Phi (v_v) \leq 0$ and $\Phi (v_e) = 0$, while $\Phi (v)$ increases monotonically from,

$(- E_1 / E_3)$ to

$$\tfrac{1}{4} (2 + E_1 / E_2 + E_2 / E_3 - E_1 / E_3) \quad (38)$$

as v increases from 0 to $\frac{1}{2}$.

Hence, $v_v \leq v_e \leq \frac{1}{2}$ so that the volume criterion yields a better upper bound for the least Poisson's ratio.

Illustrative Example: Consider experimental values of elastic constants of Beech wood [9]:

Young's Moduli: $[E_1, E_2, E_3] =$
$[1/878, 1/447, 1/72.6]\ 10^{13}$ dynes /cm^2

$[\alpha_{12}, \alpha_{23}, \alpha_{31}] = [\alpha_{21}, \alpha_{32}, \alpha_{13}]$
$= [325, 33, 38]\ 10^{-13}$ cm^2 / dyne

From the above data, one obtains all six Poisson's ratios:

$v_{13} = 38/878$, $v_{31} = 38/72.6$, $v_{12} = 325/878$, $v_{21} = 325/447$, $v_{23} = 33/447$, $v_{32} = 33/72.6$.

In this case, we have that four (v_{13}, v_{12}, v_{23}, v_{32}) of the six Poisson's ratio are less than ½ and v_{13} is the least. From the volume criterion, we have $(v_{13})_{max}$ is from

$\min [v_{23}, v_{12}, (K_{13}^2 - K_{12}^2 v_{23}), (1 - v_{12}), K_{13}^2] =$
$\min [(33 / 447), (325, 39.6, 553, 72.6) / 878] =$
$39.6 / 878$

Upper bound $v_v = (1/E_3) / [(1/ E_1) + (1/E_2)]$

$$= \frac{72.6}{1325} > (v_{13})_{max} \left(= \frac{39.6}{878}\right)$$

Since v_{23} and v_{12} are greater than v_v for this material, one can easily conclude, without measuring either v_{13} or v_{31} that v_{13} is the least Poisson's ratio and $[(K_{13})^2 - (K_{12})^2 v_{23}] = \frac{39.6}{878}$ is an upper bound for v_{13} which is 38/878.

CONCLUDING REMARKS

We have obtained the bounds of Poisson's ratios of orthotropic materials, using energy and volume criterion. By energy criterion, we have shown that all Poisson's ratios are bounded and $v_{ij} \leq K_{ij}$. Volume criterion yields that only the two-dimensional Poisson parameters N_i are bounded by unity, but does not yield bounds for individual Poisson's ratios.

Assuming $v_{ij} \geq 0$, it has been shown that volume criterion yields more restrictive variations of v_{ij} and upper bounds for v_{ij} are less than the corresponding bounds by the energy criterion. With the same assumption, volume criterion is shown to yield that three of the six Poisson's ratios are not greater than 1/2, and none of the ratios can exceed unity or K_{ij} [$=\sqrt{(E_i / E_j)}$]. Also, an upper bound v_v by volume criterion for the maximum variation of the least Poisson's ratio is shown to be, with $E_1 \leq E_2 \leq E_3$, $v_v = (1/E_3) / [(1/E_1) + (1/E_2)]$

Finally, one notes that the validity of any set of Young's moduli and Poisson's ratios deduced by experiment or otherwise are subject to the bounds in Figure 1 and more generally to the inequalities (18) and (20).

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