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Two Extensions Of Discrete Carleman's Inequality

Yuanling Guo*
School of Mathematical Sciences,Jiangsu University,Zhenjiang,212013,PR China

***Abstract*-This article re-examines the generalized forms of the two Carleman inequalities, discusses their correctness and error and the best constants in detailand obtains a correct and relatively complete conclusion.**

Keywords—Carleman's Inequality;Best constant

I. Introduction

Carleman's inequality is actually the case of the power exponent $p \to +\infty$ of Hardy's inequality, and its discrete form is:

Lemma 1.1 [1] .Suppose $\{a_n\}$ is a sequence of non-negative numbers, $\sum_{n=1}^{\infty} a_n < +\infty$,then

(1) $\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{\frac{1}{n}} \le e \sum_{n=1}^{\infty} a_n$;

(2) Inequality takes the equal sign if and only if $a_n = 0, n = 1, 2, \cdots$;

(3) e is the best constant.

This inequality has the following two weighted extensions.

Lemma 1.2 [3] .Suppose $\{\lambda_n\}$ is a positive sequence, $\Lambda_n = \lambda_1 + \lambda_2 + \cdots + \lambda_n$ .Let $\{a_n\}$ be a non-negative sequence,

$\sum_{n=1}^{\infty} \lambda_n a_n < +\infty$,then

$$\sum_{n=1}^{\infty} \lambda_n (a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{\frac{1}{\Lambda_n}} \le e \sum_{n=1}^{\infty} \lambda_n a_n ,$$

Among them, e is the best constant.

Lemma 1.3 [3] .Suppose $\{\lambda_n\}$ is a positive sequence, $\Lambda_n = \lambda_1 + \lambda_2 + \cdots + \lambda_n$. Let $\{a_n\}$ be a non-negative sequence.

$\sum_{n=1}^{\infty} \lambda_n a_n < +\infty$,then

$$\sum_{n=1}^{\infty} \frac{\Lambda_n}{\lambda_n} (e^{\frac{\Lambda_n}{\lambda_n}} - 1)(a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{\frac{1}{\Lambda_n}} \le e \sum_{n=1}^{\infty} \lambda_n a_n ,$$

Among them, e is the best constant.

The strange thing is that although the literature [3] claims e to be the best constant for the inequalities in Lemma 1.2 and 1.3, it does not give a proof. Although the monographs [1] and [2] both include Lemma 1.2, they do not say that e is the best constant.

If Lemma 1.3 is correct, take $a_n = \begin{cases} 1, n = 1 \\ 0, n > 1 \end{cases}$, then $e - 1 \le e\lambda_1$,that is $\lambda_1 \ge \frac{e-1}{e}$ .But there is no such requirement in Lemma 1.3, so this lemma is wrong. In addition, when the monograph [2] records this inequality as

$$\sum_{n=1}^{\infty} \frac{n}{\lambda_n} (e^{\frac{\lambda_n}{\Lambda_n}} - 1)(a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{\frac{1}{\Lambda_n}} \le e \sum_{n=1}^{\infty} \lambda_n a_n ,$$

it shows that e is the best constant.Take $a_n = \begin{cases} 1, n=1 \\ 0, n>1 \end{cases}$.We can also find the mistake by substituting it in the verification.

After in-depth study of these two inequalities, the correct conclusion is found.

II.Main conclusion

Theorem2.1. Let$\{\lambda_n\}$ be a sequence of positive numbers, $\Lambda_n = \lambda_1 + \lambda_2 + \cdots + \lambda_n, n = 1, 2, \cdots$ .Let $\{a_n\}$ be a non-negative sequence, $\sum_{n=1}^{\infty} \lambda_n a_n < +\infty$,then

(1) $\sum_{n=1}^{\infty} \lambda_n (a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{\frac{1}{\Lambda_n}} \le e \sum_{n=1}^{\infty} \lambda_n a_n$;

(2) Inequality takes the equal sign if and only if $a_n = 0, n = 1, 2, \cdots$;

(3) If $\sum_{n=1}^{\infty} \frac{\lambda_n}{\Lambda_n}$ convergences, then e is not the best constant;

(4) If $\lim_{n\to\infty} \frac{\lambda_n}{\Lambda_n} = 0$,but $\sum_{n=1}^{\infty} \frac{\lambda_n}{\Lambda_n}$ divergences ,then e is the best constant;

(5) If e is the best constant, then $\inf_{n \in N^+} \frac{\lambda_n}{\Lambda_n} = 0$.

Theorem 2.2. Let $\{\lambda_n\}$be a sequence of positive numbers, $\Lambda_n = \lambda_1 + \lambda_2 + \cdots + \lambda_n, n = 1, 2, \cdots$ .Let $\{a_n\}$ be a non-negative sequence, $\sum_{n=1}^{\infty} \lambda_n a_n < +\infty$, then

(1) $\sum_{n=1}^{\infty} \Lambda_n (e^{\frac{\lambda_n}{\Lambda_n}} - 1)(a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{\frac{1}{\Lambda_n}} \le e \sum_{n=1}^{\infty} \lambda_n a_n$;

(2) Inequality takes the equal sign if and only if $a_n = 0, n = 1, 2, \cdots$;

(3) If $\sum_{n=1}^{\infty} \frac{\lambda_n}{\Lambda_n}$ convergences, then e is not the best constant;

(4) If $\lim_{n\to\infty} \frac{\lambda_n}{\Lambda_n} = 0$,but $\sum_{n=1}^{\infty} \frac{\lambda_n}{\Lambda_n}$ divergences ,then e is the best constant;

Note 1. The inequality in Theorem 2.2 is actually an enhanced version of the inequality in Theorem 2.1. Therefore, if e is the best constant in Theorem 2.1, it must also be the best constant in Theorem 2.2.

III．Proof of main conclusion

1.Proof of Theorem 2.1

(1) Take any positive sequence $\{c_n\}$,for any positive integer, according to the weighted mean inequality, we have

$$(a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{\frac{1}{\Lambda_n}} = \left[\frac{(c_1 a_1)^{\lambda_1} (c_2 a_2)^{\lambda_2} \cdots (c_n a_n)^{\lambda_n}}{c_1^{\lambda_1} c_2^{\lambda_2} \cdots c_n^{\lambda_n}} \right]^{\frac{1}{\Lambda_n}}$$

$$\le \frac{\frac{\lambda_1}{\Lambda_n} c_1 a_1 + \frac{\lambda_2}{\Lambda_n} c_2 a_2 + \cdots + \frac{\lambda_n}{\Lambda_n} c_n a_n}{(c_1^{\lambda_1} c_2^{\lambda_2} \cdots c_n^{\lambda_n})^{\frac{1}{\Lambda_n}}} = \frac{\sum_{j=1}^{n} \lambda_j c_j a_j}{\Lambda_n (c_1^{\lambda_1} c_2^{\lambda_2} \cdots c_n^{\lambda_n})^{\frac{1}{\Lambda_n}}}$$

And the inequality takes the equal sign if and only if $c_1 a_1 = c_2 a_2 = \cdots = c_n a_n$. Therefore,

$$\sum_{n=1}^{\infty}\lambda_n(a_1^{\lambda_1}a_2^{\lambda_2}\cdots a_n^{\lambda_n})^{\frac{1}{\Lambda_n}} \le \sum_{n=1}^{\infty}\frac{\lambda_n}{\Lambda_n(c_1^{\lambda_1}c_2^{\lambda_2}\cdots c_n^{\lambda_n})^{\frac{1}{\Lambda_n}}}\sum_{j=1}^{n}\lambda_j c_j a_j$$

$$=\sum_{j=1}^{\infty}\lambda_j\left[c_j\sum_{n=j}^{\infty}\frac{\lambda_n}{\Lambda_n(c_1^{\lambda_1}c_2^{\lambda_2}\cdots c_n^{\lambda_n})^{\frac{1}{\Lambda_n}}}\right]a_j$$

Select appropriately $\{c_n\}$ so that $c_j\sum_{n=j}^{\infty}\frac{\lambda_n}{\Lambda_n(c_1^{\lambda_1}c_2^{\lambda_2}\cdots c_n^{\lambda_n})^{\frac{1}{\Lambda_n}}} < e,\ j=1,2,3,\cdots$,then

$$\sum_{n=j}^{\infty}\frac{\lambda_n}{\Lambda_n(c_1^{\lambda_1}c_2^{\lambda_2}\cdots c_n^{\lambda_n})^{\frac{1}{\Lambda_n}}} < \frac{e}{c_j},\ j=1,2,3,\cdots$$

Further demand that $\{c_n\}$ is monotonous and undiminished.Let $\lim_{n\to\infty}c_n = c$,then $c\in(0,+\infty]$.

If $c=+\infty$,then $\sum_{n=j}^{\infty}(\frac{1}{c_n}-\frac{1}{c_{n+1}})=\frac{1}{c_j},\ j=1,2,\cdots$.

If $c<+\infty$,then $\sum_{n=j}^{\infty}(\frac{1}{c_n}-\frac{1}{c_{n+1}})=\frac{1}{c_j}-\frac{1}{c}<\frac{1}{c_j},\ j=1,2,\cdots$.

Therefore, $\sum_{n=j}^{\infty}(\frac{1}{c_n}-\frac{1}{c_{n+1}})\le\frac{1}{c_j},\ j=1,2,\cdots$.Choose the monotonous positive sequence to satisfy

$$\sum_{n=j}^{\infty}\frac{\lambda_n}{\Lambda_n(c_1^{\lambda_1}c_2^{\lambda_2}\cdots c_n^{\lambda_n})^{\frac{1}{\Lambda_n}}} < e\sum_{n=j}^{\infty}(\frac{1}{c_n}-\frac{1}{c_{n+1}}),\ j=1,2,\cdots$$

.

Then

$$\frac{\lambda_n}{\Lambda_n(c_1^{\lambda_1}c_2^{\lambda_2}\cdots c_n^{\lambda_n})^{\frac{1}{\Lambda_n}}} < e(\frac{1}{c_n}-\frac{1}{c_{n+1}}),\ n=1,2,\cdots.$$

Find such a sequence $\{c_n\}$ of positive numbers.

Take $d_1=0$, $d_n\ge 0, n=2,3,\cdots$, $c_n=e^{\sum_{j=1}^{n}d_j}$,then $c_1=1$,

$$c_1^{\lambda_1}c_2^{\lambda_2}\cdots c_n^{\lambda_n}=\prod_{i=1}^{n}c_i^{\lambda_i}=\prod_{i=1}^{n}(e^{\sum_{j=1}^{i}d_j})^{\lambda_i}=\prod_{i=1}^{n}e^{\lambda_i\sum_{j=1}^{i}d_j}=e^{\sum_{i=1}^{n}\lambda_i\sum_{j=1}^{i}d_j}$$

,

$$\sum_{i=1}^{n}\lambda_i\sum_{j=1}^{i}d_j=\sum_{j=1}^{n}d_j\sum_{i=j}^{n}\lambda_i=\sum_{j=1}^{n}(\Lambda_n-\Lambda_{j-1})d_j=\Lambda_n\sum_{j=1}^{n}d_j-\sum_{j=1}^{n}\Lambda_{j-1}d_j$$
$$=\Lambda_n\sum_{j=2}^{n}d_j-\sum_{j=2}^{n}\Lambda_{j-1}d_j$$

Here, supplementary agreethat $\Lambda_0=0$ .Take $d_j=\frac{\lambda_{j-1}}{\Lambda_{j-1}},\ j=2,3,\cdots$,then

$$c_n=e^{\sum_{j=1}^{n}d_j}=e^{\sum_{j=2}^{n}d_j}=e^{\sum_{j=2}^{n}\frac{\lambda_{j-1}}{\Lambda_{j-1}}}=e^{\sum_{j=1}^{n-1}\frac{\lambda_j}{\Lambda_j}},$$

$$\sum_{i=1}^{n}\lambda_i\sum_{j=1}^{i}d_j=\Lambda_n\sum_{j=2}^{n}\frac{\lambda_{j-1}}{\Lambda_{j-1}}-\sum_{j=2}^{n}\lambda_{j-1}=\Lambda_n\sum_{j=1}^{n-1}\frac{\lambda_j}{\Lambda_j}-\sum_{j=1}^{n-1}\lambda_j=\Lambda_n\sum_{j=1}^{n-1}\frac{\lambda_j}{\Lambda_j}-\Lambda_{n-1}$$

,

then

$$c_1^{\lambda_1}c_2^{\lambda_2}\cdots c_n^{\lambda_n}=e^{\Lambda_n\sum_{j=1}^{n-1}\frac{\lambda_j}{\Lambda_j}-\Lambda_{n-1}},$$

then

$$(c_1^{\lambda_1}c_2^{\lambda_2}\cdots c_n^{\lambda_n})^{\frac{1}{\Lambda_n}}=e^{\sum_{j=1}^{n-1}\frac{\lambda_j}{\Lambda_j}-\frac{\Lambda_{n-1}}{\Lambda_n}}=e^{\sum_{j=1}^{n-1}\frac{\lambda_j}{\Lambda_j}-\frac{\Lambda_n-\lambda_n}{\Lambda_n}}=e^{\sum_{j=1}^{n}\frac{\lambda_j}{\Lambda_j}-1}=e^{-1}e^{\sum_{j=1}^{n}\frac{\lambda_j}{\Lambda_j}}$$

,

Therefore,

$$\frac{\lambda_n}{\Lambda_n(c_1^{\lambda_1}c_2^{\lambda_2}\cdots c_n^{\lambda_n})^{\frac{1}{\Lambda_n}}}=\frac{\frac{\lambda_n}{\Lambda_n}}{e^{-1}e^{\sum_{j=1}^{n}\frac{\lambda_j}{\Lambda_j}}}<e\frac{e^{\frac{\lambda_n}{\Lambda_n}}-1}{e^{\sum_{j=1}^{n}\frac{\lambda_j}{\Lambda_j}}}=e\left(\frac{1}{e^{\sum_{j=1}^{n-1}\frac{\lambda_j}{\Lambda_j}}}-\frac{1}{e^{\sum_{j=1}^{n}\frac{\lambda_j}{\Lambda_j}}}\right)=e\left(\frac{1}{c_n}-\frac{1}{c_{n+1}}\right)$$

.

Take $c_n = e^{\sum_{j=1}^{n-1} \frac{\lambda_j}{\Lambda_j}}$,then

$$c = \lim_{n\to\infty} c_n = \lim_{n\to\infty} e^{\sum_{j=1}^{n-1} \frac{\lambda_j}{\Lambda_j}} = e^{\lim_{n\to\infty}\sum_{j=1}^{n-1} \frac{\lambda_j}{\Lambda_j}} = e^{\sum_{n=1}^{\infty} \frac{\lambda_n}{\Lambda_n}},$$

If $\sum_{n=1}^{\infty} \frac{\lambda_n}{\Lambda_n}$ convergences,then $c < +\infty$. If $\sum_{n=1}^{\infty} \frac{\lambda_n}{\Lambda_n}$ divergences,then $c = +\infty$.

$$\sum_{n=1}^{\infty} \lambda_n (a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{\frac{1}{\Lambda_n}} \le \sum_{j=1}^{\infty} \lambda_j \left[c_j \sum_{n=j}^{\infty} \frac{\lambda_n}{\Lambda_n (c_1^{\lambda_1} c_2^{\lambda_2} \cdots c_n^{\lambda_n})^{\frac{1}{\Lambda_n}}} \right] a_j$$

$$\le e\sum_{j=1}^{\infty} \lambda_j \left[c_j \sum_{n=j}^{\infty} (\frac{1}{c_n} - \frac{1}{c_{n+1}}) \right] a_j$$

If $c < +\infty$,then

$$\sum_{n=1}^{\infty} \lambda_n (a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{\frac{1}{\Lambda_n}} \le e\sum_{j=1}^{\infty} \lambda_j \left[c_j \sum_{n=j}^{\infty} (\frac{1}{c_n} - \frac{1}{c_{n+1}}) \right] a_j = e\sum_{j=1}^{\infty} \lambda_j c_j (\frac{1}{c_j} - \frac{1}{c}) a_j$$

$$= e\sum_{j=1}^{\infty} \lambda_j (1 - \frac{c_j}{c}) a_j \le e\sum_{j=1}^{\infty} \lambda_j (1 - \frac{c_1}{c}) a_j = e(1 - \frac{1}{c}) \sum_{j=1}^{\infty} \lambda_j a_j \le e\sum_{j=1}^{\infty} \lambda_j a_j$$

If $c = +\infty$,then

$$\sum_{n=1}^{\infty} \lambda_n (a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{\frac{1}{\Lambda_n}} \le e\sum_{j=1}^{\infty} \lambda_j \left[c_j \sum_{n=j}^{\infty} (\frac{1}{c_n} - \frac{1}{c_{n+1}}) \right] a_j = e\sum_{j=1}^{\infty} \lambda_j a_j$$

In summary, $\sum_{n=1}^{\infty} \lambda_n (a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{\frac{1}{\Lambda_n}} \le e\sum_{n=1}^{\infty} \lambda_n a_n$.

(2) The sufficiency is obvious and the proof necessity is as follows.

When the inequality takes the equal sign,we have

$$\sum_{j=1}^{\infty} \lambda_j \left[c_j \sum_{n=j}^{\infty} \frac{\lambda_n}{\Lambda_n (c_1^{\lambda_1} c_2^{\lambda_2} \cdots c_n^{\lambda_n})^{\frac{1}{\Lambda_n}}} \right] a_j = e\sum_{j=1}^{\infty} \lambda_j \left[c_j \sum_{n=j}^{\infty} (\frac{1}{c_n} - \frac{1}{c_{n+1}}) \right] a_j$$

.

But for any positive integer n, $\lambda_n > 0$, $c_n > 0$, $a_n \ge 0$,

$$\frac{\lambda_n}{\Lambda_n (c_1^{\lambda_1} c_2^{\lambda_2} \cdots c_n^{\lambda_n})^{\frac{1}{\Lambda_n}}} < e(\frac{1}{c_n} - \frac{1}{c_{n+1}}),$$

Therefore, $a_n = 0, n = 1, 2, \cdots$.

(3) If $\sum_{n=1}^{\infty} \frac{\lambda_j}{\Lambda_j}$ convergences,then $c < +\infty$.According to (1),we have

$$\sum_{n=1}^{\infty} \lambda_n (a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{\frac{1}{\Lambda_n}} \le e(1 - \frac{1}{c}) \sum_{j=1}^{\infty} \lambda_j a_j,$$

So e is not the best constant.

(4) $c_n = e^{\sum_{j=1}^{n-1} \frac{\lambda_j}{\Lambda_j}}, n = 1, 2, \cdots$ .Due to $\frac{\lambda_n}{\Lambda_n} \in (0,1]$,the convergence of $\sum_{n=1}^{\infty} \frac{\lambda_n}{c_n}$ is the same as that of $\sum_{n=1}^{\infty} \lambda_n e^{-\sum_{j=1}^{n} \frac{\lambda_j}{\Lambda_j}}$.

When $n = 1$, $\sum_{j=1}^{n} \frac{\lambda_j}{\Lambda_j} = 1$.

When $n > 1$,

$$\sum_{j=1}^{n} \frac{\lambda_j}{\Lambda_j} = 1 + \sum_{j=2}^{n} \frac{\lambda_j}{\Lambda_j} = 1 + \sum_{j=2}^{n} \frac{\Lambda_j - \Lambda_{j-1}}{\Lambda_j} = 1 + \sum_{j=2}^{n} \int_{\Lambda_{j-1}}^{\Lambda_j} \frac{dx}{\Lambda_j}$$

$$< 1 + \sum_{j=2}^{n} \int_{\Lambda_{j-1}}^{\Lambda_j} \frac{dx}{x} = 1 + \int_{\Lambda_1}^{\Lambda_n} \frac{dx}{x} = 1 + \ln \frac{\Lambda_n}{\Lambda_1}$$

So $\sum_{j=1}^{n} \frac{\lambda_j}{\Lambda_j} \le 1 + \ln \frac{\Lambda_n}{\Lambda_1}$ .Then,

$$\lambda_n e^{-\sum_{j=1}^{n} \frac{\lambda_j}{\Lambda_j}} \ge \lambda_n e^{-(1+\ln\frac{\Lambda_n}{\Lambda_1})} = e^{-1} \Lambda_1 \frac{\lambda_n}{\Lambda_n}.$$

$\sum_{n=1}^{\infty}\frac{\lambda_n}{\Lambda_n}$ divergences,so $\sum_{n=1}^{\infty}\lambda_n e^{-\sum_{j=1}^{n}\frac{\lambda_j}{\Lambda_j}}$ divergences,then $\sum_{n=1}^{\infty}\frac{\lambda_n}{c_n}$ divergences.

Take any positive integer N ,define the sequence $a_n=\begin{cases}\frac{1}{c_n}, n=1,2,\cdots,N\\ 0,\ n>N\end{cases}$,then

$$\sum_{n=1}^{\infty}\lambda_n(a_1^{\lambda_1}a_2^{\lambda_2}\cdots a_n^{\lambda_n})^{\frac{1}{\Lambda_n}}=\sum_{n=1}^{N}\frac{\lambda_n}{(c_1^{\lambda_1}c_2^{\lambda_2}\cdots c_n^{\lambda_n})^{\frac{1}{\Lambda_n}}},$$

$$\sum_{n=1}^{\infty}\lambda_n a_n=\sum_{n=1}^{N}\frac{\lambda_n}{c_n},$$

$\sum_{n=1}^{\infty}\frac{\lambda_n}{c_n}$ divergences,so $\left\{\sum_{n=1}^{N}\frac{\lambda_n}{c_n}\right\}$ is a sequence of positive numbers that is strictly monotonically increasing towards infinity.According to Stokes' formula,

$$\lim_{N\to\infty}\frac{\sum_{n=1}^{\infty}\lambda_n(a_1^{\lambda_1}a_2^{\lambda_2}\cdots a_n^{\lambda_n})^{\frac{1}{\Lambda_n}}}{\sum_{n=1}^{\infty}\lambda_n a_n}=\lim_{N\to\infty}\frac{\sum_{n=1}^{N}\frac{\lambda_n}{(c_1^{\lambda_1}c_2^{\lambda_2}\cdots c_n^{\lambda_n})^{\frac{1}{\Lambda_n}}}}{\sum_{n=1}^{N}\frac{\lambda_n}{c_n}}$$

$$=\lim_{N\to\infty}\frac{\frac{\lambda_N}{(c_1^{\lambda_1}c_2^{\lambda_2}\cdots c_N^{\lambda_N})^{\frac{1}{\Lambda_N}}}}{\frac{\lambda_N}{c_N}}=\lim_{N\to\infty}\frac{e^{\sum_{j=1}^{N-1}\frac{\lambda_j}{\Lambda_j}}}{e^{\sum_{j=1}^{N}\frac{\lambda_j}{\Lambda_j}-1}}=e\lim_{N\to\infty}e^{-\frac{\lambda_N}{\Lambda_N}}=e$$

So e is the best constant.

(5) Let $r=\inf_{n\in N^+}\frac{\lambda_n}{\Lambda_n}$,then $r\in[0,1)$ .Take any non-negative sequence $\{a_n\}$,so that $\sum_{n=1}^{\infty}\lambda_n a_n<+\infty$,then

$$\sum_{n=1}^{\infty}\lambda_n(a_1^{\lambda_1}a_2^{\lambda_2}\cdots a_n^{\lambda_n})^{\frac{1}{\Lambda_n}}\le\sum_{j=1}^{\infty}\lambda_j\left[c_j\sum_{n=j}^{\infty}\frac{\lambda_n}{\Lambda_n(c_1^{\lambda_1}c_2^{\lambda_2}\cdots c_n^{\lambda_n})^{\frac{1}{\Lambda_n}}}\right]a_j$$

.

For any positive integer j ,due to $\frac{x}{e^x-1}$ strictly monotonically decreasing on $(0,+\infty)$,then

$$c_j\sum_{n=j}^{\infty}\frac{\lambda_n}{\Lambda_n(c_1^{\lambda_1}c_2^{\lambda_2}\cdots c_n^{\lambda_n})^{\frac{1}{\Lambda_n}}}=c_j\sum_{n=j}^{\infty}\frac{\frac{\lambda_n}{\Lambda_n}}{e^{\sum_{j=1}^{n}\frac{\lambda_j}{\Lambda_j}-1}}=ec_j\sum_{n=j}^{\infty}\frac{\frac{\lambda_n}{\Lambda_n}}{e^{\frac{\lambda_n}{\Lambda_n}}-1}\frac{e^{\frac{\lambda_n}{\Lambda_n}}-1}{e^{\sum_{j=1}^{n}\frac{\lambda_j}{\Lambda_j}}}$$

$$\le ec_j\sum_{n=j}^{\infty}\frac{r}{e^r-1}\left(\frac{1}{e^{\sum_{j=1}^{n-1}\frac{\lambda_j}{\Lambda_j}}}-\frac{1}{e^{\sum_{j=1}^{n}\frac{\lambda_j}{\Lambda_j}}}\right)=\frac{r}{e^r-1}ec_j\sum_{n=j}^{\infty}\left(\frac{1}{c_n}-\frac{1}{c_{n+1}}\right)\le\frac{r}{e^r-1}e$$

.

Therefore,

$$\sum_{n=1}^{\infty}\lambda_n(a_1^{\lambda_1}a_2^{\lambda_2}\cdots a_n^{\lambda_n})^{\frac{1}{\Lambda_n}}\le\frac{r}{e^r-1}e\sum_{j=1}^{\infty}\lambda_j a_j.$$

Due to $\frac{r}{e^r-1}e<e$, e is not the best constant.This is contradictory.So $\inf_{n\in N^+}\frac{\lambda_n}{\Lambda_n}=0$.#

2.Proof of Theorem 1.2

(1) Still take the positive sequence $\{c_n\}$ as $c_n=e^{\sum_{j=1}^{n-1}\frac{\lambda_j}{\Lambda_j}}, n=1,2,\cdots$.,let $c=\lim_{n\to\infty}c_n$,then $c\in(0,+\infty]$ .According to the weighted mean inequality,

$$\sum_{n=1}^{\infty}\Lambda_n(e^{\frac{\lambda_n}{\Lambda_n}}-1)(a_1^{\lambda_1}a_2^{\lambda_2}\cdots a_n^{\lambda_n})^{\frac{1}{\Lambda_n}}=\sum_{n=1}^{\infty}\Lambda_n(e^{\frac{\lambda_n}{\Lambda_n}}-1)\left[\frac{(c_1a_1)^{\lambda_1}(c_2a_2)^{\lambda_2}\cdots(c_na_n)^{\lambda_n}}{c_1^{\lambda_1}c_2^{\lambda_2}\cdots c_n^{\lambda_n}}\right]^{\frac{1}{\Lambda_n}}$$

$$\le\sum_{n=1}^{\infty}\Lambda_n(e^{\frac{\lambda_n}{\Lambda_n}}-1)\frac{\frac{\lambda_1}{\Lambda_n}c_1a_1+\frac{\lambda_2}{\Lambda_n}c_2a_2+\cdots+\frac{\lambda_n}{\Lambda_n}c_na_n}{(c_1^{\lambda_1}c_2^{\lambda_2}\cdots c_n^{\lambda_n})^{\frac{1}{\Lambda_n}}}=\sum_{n=1}^{\infty}(e^{\frac{\lambda_n}{\Lambda_n}}-1)\frac{\sum_{j=1}^{n}\lambda_jc_ja_j}{(c_1^{\lambda_1}c_2^{\lambda_2}\cdots c_n^{\lambda_n})^{\frac{1}{\Lambda_n}}}$$

$$=\sum_{j=1}^{\infty}\lambda_j\left[c_j\sum_{n=j}^{\infty}\frac{e^{\frac{\lambda_n}{\Lambda_n}}-1}{(c_1^{\lambda_1}c_2^{\lambda_2}\cdots c_n^{\lambda_n})^{\frac{1}{\Lambda_n}}}\right]a_j$$

For any positive integer j,

$$c_j\sum_{n=j}^{\infty}\frac{e^{\frac{\lambda_n}{\Lambda_n}}-1}{(c_1^{\lambda_1}c_2^{\lambda_2}\cdots c_n^{\lambda_n})^{\frac{1}{\Lambda_n}}}=c_j\sum_{n=j}^{\infty}\frac{e^{\frac{\lambda_n}{\Lambda_n}}-1}{e^{\sum_{k=1}^{n}\frac{\lambda_k}{\Lambda_k}-1}}=ec_j\sum_{n=j}^{\infty}\left(\frac{1}{e^{\sum_{k=1}^{n-1}\frac{\lambda_k}{\Lambda_k}}}-\frac{1}{e^{\sum_{k=1}^{n}\frac{\lambda_k}{\Lambda_k}}}\right)$$

$$=ec_j\sum_{n=j}^{\infty}\left(\frac{1}{c_n}-\frac{1}{c_{n+1}}\right)=\begin{cases}ec_j(\frac{1}{c_j}-\frac{1}{c}), & c<+\infty\\ e, & c=+\infty\end{cases}\le e$$

So $\sum_{n=1}^{\infty}\Lambda_n(e^{\frac{\lambda_n}{\Lambda_n}}-1)(a_1^{\lambda_1}a_2^{\lambda_2}\cdots a_n^{\lambda_n})^{\frac{1}{\Lambda_n}}\le e\sum_{n=1}^{\infty}\lambda_na_n$.

(2) The sufficiency is obvious and the proof necessity is as follows. We divide the discussion into two situations, $c<+\infty$ and $c=+\infty$.

First, $c<+\infty$.

For any positive integer j,

$$c_j\sum_{n=j}^{\infty}\frac{e^{\frac{\lambda_n}{\Lambda_n}}-1}{(c_1^{\lambda_1}c_2^{\lambda_2}\cdots c_n^{\lambda_n})^{\frac{1}{\Lambda_n}}}\le ec_j(\frac{1}{c_j}-\frac{1}{c})=e(1-\frac{c_j}{c})\le e(1-\frac{1}{c})$$

,

Then,

$$\sum_{n=1}^{\infty}\Lambda_n(e^{\frac{\lambda_n}{\Lambda_n}}-1)(a_1^{\lambda_1}a_2^{\lambda_2}\cdots a_n^{\lambda_n})^{\frac{1}{\Lambda_n}}\le e(1-\frac{1}{c})\sum_{j=1}^{\infty}\lambda_ja_j.$$

Inequality takes the equal sign,so $e(1-\frac{1}{c})\sum_{n=1}^{\infty}\lambda_na_n=e\sum_{n=1}^{\infty}\lambda_na_n$,namely $\sum_{n=1}^{\infty}\lambda_na_n=0$.

$\lambda_n>0, a_n\ge 0, n=1,2,\cdots$,so $a_n=0, n=1,2,\cdots$.

Second , $c=+\infty$.

When $c=+\infty$, $\sum_{n=1}^{\infty}\frac{\lambda_n}{\Lambda_n}$ divergences.

According to the necessary and sufficient conditions oftaking the equal sign ofthe weighted mean inequality,

$$c_1a_1=c_2a_2=c_3a_3=\cdots.$$

If a_n are not all 0 ,then a_n are not 0 at all.We assume $1=c_1a_1=c_2a_2=c_3a_3=\cdots$,then

$a_n=\frac{1}{c_n}, n=1,2,\cdots$,so $\sum_{n=1}^{\infty}\frac{\lambda_n}{c_n}$ convergences.But in the proof of Theorem 1.1, we have already shown $\sum_{n=1}^{\infty}\frac{\lambda_n}{c_n}$ divergences.This is contradictory .So

$a_n=0, n=1,2,\cdots$.

In summary, the inequality takes the equal sign if and only if $a_n=0, n=1,2,\cdots$.

(3) $\sum_{n=1}^{\infty}\frac{\lambda_n}{\Lambda_n}$ convergences,so $c<+\infty$.In the proof of (2), we get $\sum_{n=1}^{\infty}\Lambda_n(e^{\frac{\lambda_n}{\Lambda_n}}-1)(a_1^{\lambda_1}a_2^{\lambda_2}\cdots a_n^{\lambda_n})^{\frac{1}{\Lambda_n}}\le e(1-\frac{1}{c})\sum_{j=1}^{\infty}\lambda_ja_j$ .Then $e(1-\frac{1}{c})<e$,Therefore, e is not the best constant.

(4) $\lim_{n\to\infty}\frac{\lambda_n}{\Lambda_n}=0$,but $\sum_{n=1}^{\infty}\frac{\lambda_n}{\Lambda_n}$ divergences. According to Theorem 1.1, e is the best constant.

Let

$$A=\left\{\{a_n\}\left|a_n\geq 0,n=1,2,\cdots,0<\sum_{n=1}^{\infty}\lambda_n a_n<+\infty\right.\right\}$$,th

en

$$e=\sup_{\{a_n\}\in A}\frac{\sum_{n=1}^{\infty}\lambda_n(a_1^{\lambda_1}a_2^{\lambda_2}\cdots a_n^{\lambda_n})^{\frac{1}{\Lambda_n}}}{\sum_{n=1}^{\infty}\lambda_n a_n}\leq\sup_{\{a_n\}\in A}\frac{\sum_{n=1}^{\infty}\Lambda_n(e^{\frac{\lambda_n}{\Lambda_n}}-1)(a_1^{\lambda_1}a_2^{\lambda_2}\cdots a_n^{\lambda_n})^{\frac{1}{\Lambda_n}}}{\sum_{n=1}^{\infty}\lambda_n a_n}\leq e$$

,

Therefore,

$$\sup_{\{a_n\}\in A}\frac{\sum_{n=1}^{\infty}\Lambda_n(e^{\frac{\lambda_n}{\Lambda_n}}-1)(a_1^{\lambda_1}a_2^{\lambda_2}\cdots a_n^{\lambda_n})^{\frac{1}{\Lambda_n}}}{\sum_{n=1}^{\infty}\lambda_n a_n}=e,$$

Therefore, e is the best constant.#

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