Successive Approximation For Solution of Two-Parameter Stochastic Differential Equations

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***Abstract*—In this paper we discuss the successive approximations for solutions of two-parameter stochastic differential equations under non-Lipschitz conditions. Some results in [1] have been improved.**

Keywords—Two-parameter stochastial differential equation; Lipchitz condition; Successive approximation.

I. Introduction

Let $R_+^2=[0,+\infty)\times[0,+\infty)$, $\partial R_+^2=R_+\times\{0\}\bigcup\{0\}\times R_+$,for $Z=(s,t)\in R_+^2, R_Z=[0,s]\times[0,t]$,but for fixed $T\in R_+$,we promise $R_T=[0,T]\times[0,T]$. In R_+^2 we introduce a partial order $\le$ by writing $(s,t)\le(s',t')$ when $s\le s', t\le t'$.

Let (Q,F,P,F_z) be a probability space with a family of sub- σ -algebras $F_z, z\in R^2$,satisfying the usual conditions F_1-F_4 in [2] ,and $B_z, z\in R_+^2$ be a two-paraneter Brownian motion with $\partial B=0$ (that is $B(z)=0,\ z\in\partial B_+^2$) on (Q,F,P,F_z) consider the non-Markovian stochastic differential system in plane.

$$(\mathrm{I})\begin{cases} dX(z)=\alpha(z,X(z))dz+\beta(z,X(z))dB(z) & z\in R_+^2\setminus\partial R_+^2\\ X(z)=\varphi(z) & z\in\partial R_+^2\end{cases}$$

Under the assumption that the coefficients α and β satisfy a Lipschitz condition and a growth condition,[1] established that equation (I) has unique solution, moreover the solution can be approximated successively. It is well known that there are some examples where the coefficients do not satisfy the Lipschitz condition. Yet we can prove the existence of solutions [3],[4].But in there cases, it is not known whether the solutions can be approximated successively or not.

In the present paper we propose a condition weaker than Lipschitz's which quaranteed the successive approximation of solutions. The method in this paper is different from [1].

II. Main Results

Through out this paper, we denote $m(R_z)=s\cdot t$ for $z=(s,t)\in R_+^2$ which is Lebesgue measure of R_z.

Theorem let $\varphi(z), z\in R_+^2$ be a continuous F_z adapted two-parameter stochastic process on (Q,F,P,F_z) ,for every $T>0$, $E\sup_{z\in R_T}|\varphi(z)|^2<\infty$.

α,β are continuous on $R_+^2\times R$ and satisfy the following conditions.

$$(C_1)\quad \begin{matrix}|\alpha(z,x)-\alpha(z,y)|^2\le\psi_1\rho(|x-y|^2)\\ |\beta(z,x)-\beta(z,y)|^2\le\psi_2\kappa(|x-y|^2)\end{matrix}$$

Where ρ,κ are defined on $[0,+\infty)$, continuous non-decreasing and concave function with $\rho(0)=\kappa(0)=0$ .and

$$(C_2)\quad \int_{0^+}\frac{\mathrm{d}u}{\rho(u)+\kappa(u)}=+\infty$$

$\psi_1(z),\psi_2(z)$ are locally bounded functions on R_+^2.

Then there exists unique solution $X(z), z\in R_+^2$ of equation (I) and for every $T>0$

$$\lim_{k\to\infty}E\left[\sup_{z\in R_T}\left|X^{(k)}(z)-X(z)\right|^2\right]=0 \qquad (2.1)$$

$$X^{(0)}(z)=\varphi(z)$$

Where

$$X^{(k+1)}(z)=\varphi(z)+\int_{R_z}\alpha(\xi,\chi^{(k)}(\xi))d\xi+\int_{R_z}\beta(\xi,X^{(k)}(\xi))dB(\xi)$$

$$k=0,1,2,\cdots \qquad (2.2)$$

In order to prove this theorem we need the following lemma.

Lemma Let $X^{(m)}(z)$, $m=0,1,2\cdots$ be the successive approximations defined by (2.2) , under the conditions of the theorem, for every $T>0$

$$E\sup_{\xi\in R_T}\left|X^{(m)}(\xi)\right|^2\le C(T) \quad m=0,1,2,\cdots \qquad (2.3)$$

Where $C(T)$ is a positive constant depending on T only.

Proof First, since ρ,κ non-negative concave function with $\rho(0)=\kappa(0)=0$. α,β are continuous in (z,x). We know that condition (C_1) implies the growth condition (ρ,κ are dominated by a linear function)

$$|\alpha(z,x)|^2+|\beta(z,x)|^2\le T_1(1+|x|^2) \quad z\in R_T \quad (2.4)$$

Where T_1 is a constant depending on T.

By the definition of $X^{(m)}(z)$,for any $z\in R_T$

$$E\sup_{\xi\in R_T}\left|X^{(m)}(\xi)\right|^2\le 3E\sup_{\xi\in R_T}|\varphi(\xi)|^2+3E\sup_{\xi\in R_T}\left[\int_{R_\xi}\alpha(\eta,X^{(m-1)}(\eta))d\eta\right]^2$$
$$+3E\sup_{\xi\in R_T}\left[\int_{R_\xi}\beta(\eta,X^{(m-1)}(\eta))dB(\eta)\right]^2 \qquad (2.5)$$

Utilizing Cauchy inequality and Doob-Carori inequality [2] the right side of (2.5)

$$\le 3E\sup_{\xi\in R_z}|\varphi(\xi)|^2+3m(R_z)E\int_{R_z}\left|\alpha(\eta,X^{(m-1)}(\eta))\right|^2d\eta$$

$$+48E\int_{R_\xi}\left|\beta(\eta,X^{(m-1)}(\eta))\right|^2dB(\eta)\le 3E\left[\sup_{\xi\in R_T}|\varphi(\xi)|^2\right]$$

$$+T_2\int_{R_z}E\left[\alpha^2(\eta,X^{(m-1)}(\eta))+\beta^2(\eta,X^{(m-1)}(\eta))\right]d\eta$$
$$\le 3E\sup_{\xi\in R_T}|\varphi(\xi)|^2+T_2\cdot T_1\int_{R_z}E\left[1+\left|X^{(m-1)}(\eta)\right|^2\right]d\eta$$

$$\le 3E\sup_{\xi\in R_T}|\varphi(\xi)|^2+T_3\int_{R_z}\left[1+E\sup_{\xi\in R_\eta}\left|X^{(m-1)}(\xi)\right|^2\right]d\eta$$

Where $T_2=\max\left\{48,3T^2\right\}$ $\quad T_3=T_1\cdot T_2$

Now we denote

$$h(T)=3E\sup_{\xi\in R_T}|\varphi(\xi)|^2 \quad A^{(m)}(z)=E\sup_{\xi\in R_T}\left|X^{(m)}(\xi)\right|^2$$

Then

$$A^{(m)}(z)\le h(T)+T_3\int_{R_z}[1+A^{(m-1)}(\eta)]d\eta$$

$$m=0,1,2.\cdots \; z\in R_T \qquad (2.6)$$

We shall show that

$$A^{(m)}(z)\le h(T)+(1+h(T))\sum_{k=1}^{m}\frac{T_3^k}{(k!)^2}[m(R_z)]^k$$

$$m=0,1,2.\cdots,\; z\in R_T \qquad (2.7)$$

holds by induction.

By the definition of $A^{(m)}(z)$,(2.7) holds for $m=0$ .Assume that (2.7) holds for m ,then we have for $m+1$ and $z=(s,t)\in R_T$

$$A^{(m+1)}(z)\le h(T)+T_3\int_{R_z}[1+A^{(m)}(\eta)]d\eta$$

$$\le h(T)+T_3\int_{R_z}\left\{1+h(T)+(1+h(T))\sum_{k=1}^{m}\frac{T_3^k}{(k!)^2}[m(R_z)]^k\right\}d\eta$$

$$=h(T)+(1+h(T))T_3m(R_z)+T_3(1+h(T))\sum_{k=1}^{m}\frac{T_2^k}{(k!)^2}\int_0^s\int_0^t s_1^k\cdot t_1^k ds_1dt_1$$

$$=h(T)+(1+h(T))T_3m(R_z)+(1+h(T))\sum_{k=1}^{m}\frac{T_3^k}{[(k+1)!]}[m(R_z)]^{k+1}$$

$$=h(T)+(1+h(T))\sum_{k=1}^{m+1}\frac{T_3^k}{(k!)^2}[m(R_z)]^k$$

Thus (2.7) holds for all m.

Now we have

$$A^{(m)}(z)\le h(T)+(1+h(T))\sum_{k=1}^{m}\frac{T_3^k}{(k!)^2}[m(R_z)]^k$$

$$\le(1+h(T))\sum_{k=0}^{m}\frac{T_3^k}{(k!)^2}T^{2k}\le(1+h(T))\sum_{k=0}^{m}\frac{(T_3\cdot T^2)^k}{k!}\le C(T)$$

where $C(T)=(1+h(T))e^{T_3\cdot T^2}$

Take $z=(T,T)$ then the lemma is obtained.

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The proof of the theorem.

Let $L_1(T), L_2(T)$ be the bounds of ψ_1, ψ_2 on R_T .For

$z \in R_T$,by the definition of $X^{(n)}(z)$ and lemma, we

have

$$E[\sup_{\xi\in R_z}\left|X^{(n)}(\xi)-X^{(m)}(\xi)\right|^2] \le 2I_1+2I_2$$

Where

$$I_1 = E\sup_{\xi\in R_z}\left\{\int_{R_\xi}[\alpha(\eta,X^{(n-1)}(\eta))-\alpha(\eta,X^{(m-1)}(\eta))]d\eta\right\}^2$$
$$\le Em(R_z)\int_{R_z}\left|\alpha(\eta,X^{(n-1)}(\eta))-\alpha(\eta,X^{(m-1)}(\eta))\right|^2 d\eta$$
$$\le T^2E\int_{R_z} L_1(T)\rho(\left|X^{(n-1)}(\eta)-X^{(m-1)}(\eta)\right|^2)d\eta$$
$$\le T^2L_1(T)\int_{R_z} E\rho(\sup_{\xi\in R_\eta}\left|X^{(n-1)}(\xi)-X^{(m-1)}(\xi)\right|^2)d\eta$$

$$I_2 = E\sup_{\xi\in R_z}\left\{\int_{R_\xi}[\beta(\eta,X^{(n-1)}(\eta))-\beta(\eta,X^{(m-1)}(\eta))]dB(\eta)\right\}^2$$
$$\le 16\int_{R_z} E\left|\beta(\eta,X^{(n-1)}(\eta))-\beta(\eta,X^{(m-1)}(\eta))\right|^2 d\eta$$
$$\le 16L_2(t)\int_{R_z}\kappa(E\sup_{\xi\in R_\eta}\left|X^{(n-1)}(\xi)-X^{(m-1)}(\xi)\right|^2)d\eta$$

Let $H(u)=\rho(u)+\kappa(u)$

Then H is non-decreasing and continuous. Utilizing Fatou lemma and Lemmma in this paper, for every fixed natural number k

$$\sup_{m,n\ge k} E\sup_{\xi\in R_z}\left|X^{(n)}(\xi)-X^{(m)}(\xi)\right|^2$$

$$\le T_4\int_{R_z} H(\sup_{m,n\ge k} E\sup_{\xi\in R_\eta}\left|X^{(n-1)}(\xi)-X^{(m-1)}(\xi)\right|^2)d\eta \quad (2.8)$$

Where $T_4=\max\left\{T^2L_1(T), 16L_2(T)\right\}$

Denote $V_k(z)=\sup_{m,n\ge k} E\sup_{\xi\in R_z}\left|X^{(n)}(\xi)-X^{(m)}(\xi)\right|^2$

then $V_k(z)$ is non-increasing about k .

Thus there exists function $V(z)$ definited on R_T such that

$$V_k(z)\to V(z) \quad (as\, k\to\infty \quad z\in R_T) \quad (2.9)$$

By (2.8) ,it is clear that

$$V_k(z)\le T_4\int_{R_z} H(V_{k-1}(\eta))d\eta \quad k\ge 1 \quad (2.10)$$

Let $k\to\infty$, using the dominated convergence theorem we have

$$V(z)\le T_4\int_{R_z} H(V(\eta))d\eta \quad k\ge 1 \quad (2.11)$$

By the condition (C_2) and Lemma 2.3 in [5], $V(z)=0,\ z\in R_T$, then

$$V_k(z)\to 0,\ (as\, k\to\infty\ z\in R_T) \quad (2.12)$$

Which is

$$\lim_{k\to\infty}\sup_{m,n\ge k}\sup_{\xi\in R_z}\left|X^{(n)}(\xi)-X^{(m)}(\xi)\right|^2=0 \quad z\in R_T \ (2.13)$$

Take $z=(T,T)$,we obtain

$$E\sup_{\xi\in R_T}\left|X^{(n)}(\xi)-X^{(m)}(\xi)\right|^2\to 0 \quad (as\ m,n\to\infty) \ (2.14)$$

Then there exists a square intergrable process $X(z)$ such that [1]

$$E\sup_{z\in R_T}\left|X^{(n)}(z)-X(z)\right|^2\to 0 \quad (as\ n\to\infty) \ (2.15)$$

If we put

$$Y_n=\sup_{z\in R_T}\left|X^{(n)}(z)-X(z)\right|^2$$

Then we have

$$EY_n^2\to 0 \quad (as\ n\to\infty).$$

It follows that there exists $\{n_k\}\subset\{n\}$ such that

$$Y_{n_k}\to 0 \quad a.s\ (as\ k\to\infty)$$

Therefore $X^{(n_k)}(z)\to X(z)$ uniformly in $z\in R_T$ as $k\to\infty$ .Since $X^{(n)}(z)$ is continuous and F_z adapted, then so is in $x(z)$.

Now we rove that $X(z)$ is the solution of equation (I) .It is sufficient to show that for $z\in R_T$

$$E\left|\left[\psi(z)+\int_{R_z}\alpha(\xi,X(\xi))d\xi+\int_{R_z}\beta(\xi,X(\xi))dB(\xi)\right]-X(z)\right|^2=0 \quad (2.16)$$

The left side of (2.16)

$$\le 2E\left|X^{(n+1)}(z)-X(z)\right|^2$$
$$+2E\left|X^{(n+1)}(z)-\psi(z)-\int_{R_z}\alpha(\xi,X(\xi))d\xi-\int_{R_z}\beta(\xi,X(\xi))dB(\xi)\right|^2$$
$$=2J_1^{(n)}+2J_2^{(n)}$$

From (2.15) , $J_1^{(n)} \to 0 \, (as \; n \to \infty)$

$$J_2^{(n)} = E\left|\int_{R_z}\left[\alpha(\xi, X(\xi)) - \alpha(\xi, X^{(n)}(\xi))\right]d\xi + \int_{R_z}\left[\beta(\xi, X(\xi)) - \beta(\xi, X^{(n)}(\xi))\right]dB(\xi)\right|^2$$

$$\leq 2T^2 L_1(T)\int_{R_z} \rho(E\left|X(\xi) - X^{(n)}(\xi)\right|^2 d\xi)$$

$$+2L_2(T)\int_{R_z} \kappa(E\left|X(\xi) - X^{(n)}(\xi)\right|^2 d\xi)$$

By Fatou lemma $J_{1=2}^{(n)} \to 0 \, (as \; n \to \infty)$. Therefore $X(z)$ is a solution of equation (I).

Finally we prove the uniqueness for (I)

Let $X'(z), X''(z)$ be two solutions of (I) ,and T be an arbitrary positive number, $z \in R_T$ then

$$E\left|X'(z) - X''(z)\right|^2 \leq 2E\left|\int_{R_z}[\alpha(\xi, X'(\xi)) - \alpha(\xi, X''(\xi))]d\xi\right|^2$$

$$+2E\left|\int_{R_z}[\beta(\xi, X'(\xi)) - \beta(\xi, X''(\xi))]d\xi\right|^2$$

$$\leq 2T^2\int_{R_z} E\left|\alpha(\xi, X'(\xi)) - \alpha(\xi, X''(\xi))\right|^2 d\xi$$

$$+2\int_{R_z} E\left|\beta(\xi, X'(\xi)) - \alpha(\xi, X''(\xi))\right|^2 d\xi$$

$$\leq T_5\int_{R_z}\left[\rho(E\left|X'(\xi) - X''(\xi)\right|^2) + \kappa(E\left|X'(\xi) - X''(\xi)\right|^2)\right]d\xi$$

where

$$T_5 = \max\left\{2T^2 L_1(T), 2L_2(T)\right\}$$

By Lemma 2.3 in [5]

$$E\left|X'(z) - X''(z)\right|^2 = 0, \qquad z \in R_T$$

Since T is arbitrary we obtain $X'(z) = X''(z) \quad a.s$

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