Coupled Axial-Transverse Dynamic Analysis of Symmetric Sandwich Beams under Harmonic Bending Forces

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***Abstract*— The coupled axial-transverse steady state dynamic response of symmetric sandwich beam under harmonic bending forces is studied. Starting with Hamilton's variational principle, the dynamic governing coupled equations and related boundary conditions based on the Euler-Bernoulli beam theory are derived. The resulting governing equations for three-layered symmetric sandwich beams are solved exactly and the closed form solutions for steady state axial-transverse coupled dynamic response analysis are then investigated for cantilever, simply supported and fixed-fixed boundary conditions. The applicability of the present closed form solutions is established through numerical examples with different transverse harmonic loads. Numerical results are compared with Abaqus finite shell element and exact solutions available in the literature to assess the accuracy of the present solution. Additional results are provided to investigate the effects of the core thickness and length of the sandwich beam on the natural frequencies, quasi-static and steady state dynamic responses.**

Keywords*— *Closed form solution, Coupled axial-transverse response, Harmonic forces, quasi-static response, Steady state response.

I. INTRODUCTION AND OBJECTIVE

Sandwich structural beams and panels are one of the most important structural components frequently used in several industries such as aerospace, marine, sports equipment, automotive and civil applications due to their extremely high strength to low weight ratios that leads to reduction in the total weight, high flexural and transverse shear stiffness, good corrosion resistance, high damping capacities and excellent thermal insulation properties. In addition, their materials are capable of absorbing large amounts of energy due to impact loads which results in high structural crashworthiness. The most common sandwich structural beam is composed of two thin stiff faces with a thicker lightweight low stiffness core to separate the faces. Common materials used for the face layers are metals and composite while the core is often made of foam or honeycomb metal. A core material is required to achieve two essential tasks, (1) it must keep the face layers the correct distance apart, and (2) it must not allow one face to slide over the other. It is also very important that, although the core is weaker than the face layers, but it is strong enough to resist the bending, buckling and crushing failures.

Although the dynamic analysis of sandwich beams based on different beam theories was the subject of major research studies during the past few years, but most of these studies are limited to free vibrations of sandwich beams. Numerous scientists developed and presented the analytical and finite element solutions for free vibration response of sandwich beams. Among them, [1] and [2] employed the finite element method to analyze the flexural vibration characteristics of a curved sandwich beam with an elastic core. Reference [3] developed the dynamic stiffness method for the free vibration analysis of three-layered symmetric sandwich beams. In another work, [4] derived a tenth-order governing differential equations for three-layered asymmetric sandwich beam and studied the free vibration characteristics using the dynamic stiffness matrix. Reference [5] developed an exact dynamic stiffness matrix to study the free flexural vibration of three-layered sandwich beams of unequal faces. Reference [6] used a trigonometric shear deformation beam theory to investigate the flexural analysis of sandwich beams. The free vibration characteristic of various viscoelastic sandwich Euler-Bernoulli beams for different boundary conditions is studied by [7]. Reference [8] presented the trigonometric shear deformation theory considering the effect of the transverse shear deformation to study the static flexural analysis of the soft core sandwich beams. Their theory based on transverse shear deformation employed the principle of the virtual work to formulate the governing differential equations and the associated boundary conditions. They derived the closed-form solutions for the sandwich beams with simply supported boundary conditions using Navier solution technique. Reference [9] developed a dynamic stiffness matrix to analyze the coupled bending-longitudinal free vibration of three-layered sandwich beam.

It should be observed that the previous studies are mainly focused to free vibration analysis of sandwich beams, with no attention on studying the dynamic analysis of sandwich beams subjected to harmonic forces. Thus, the main purpose of this paper is to formulate the governing differential equations and boundary conditions of symmetric sandwich beams and provide the closed form exact solutions for sandwich beams of equal faces under various bending harmonic forces. The present general exact solution is appropriate and efficient in analyzing the steady state dynamic response of symmetric sandwich beams subjected to transverse harmonic excitations. It is also capable to capture the quasi-static response and predict the eigen-frequencies and eigen-modes of the sandwich beams.

II. Mathematical Formulation

The coordinate system and notations for the three-layered symmetric sandwich beam of length L and width b is illustrated in Fig. (1). The top and bottom layers have an equal thickness $h_1 = h_2 = h$, while the core layer thickness is h_2. The sandwich beam under bending harmonic force caused the sandwich beam to bend in the *x-z* plane only. The face layers are modelled as Euler-Bernoulli beam having only axial and bending rigidity and the core layer is assumed to have only shear rigidity in which the stresses in the core along the axial direction are neglected. All layers are assumed to have the same transverse displacement $w(x,t)$, whereas the centre line displacements of the top and bottom layers in the axial direction are $u_1(x,t)$ and $u_3(x,t)$, respectively.

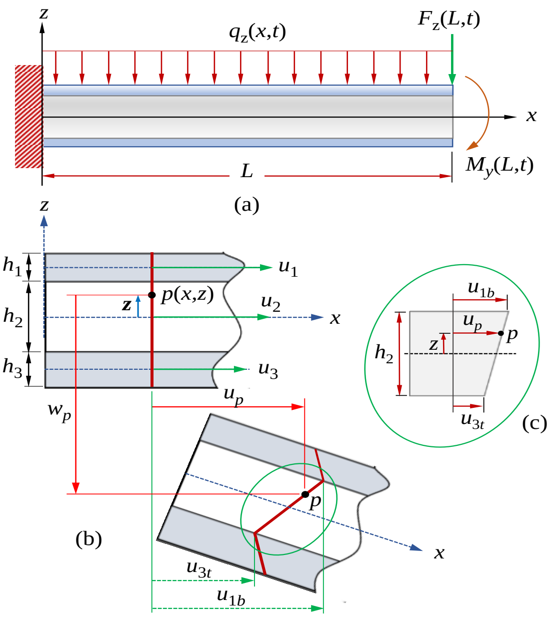

Fig. (1): *Sandwich beam under harmonic forces*

The sandwich beam model is based on the following assumptions [3]:

1. Displacements and strains are assumed small, then the theory of linear elasticity is applied.
2. The faces and core are made of homogeneous and isotropic materials.
3. Top and bottom face layers are considered as Euler-Bernoulli beams with axial and bending rigidities.
4. The three layers of sandwich beam are assumed to be perfectly bonded and there is no slippage between the layers during deformation.
5. The transverse normal strains in the three layers of the sandwich are negligible.
6. The transverse shear deformation within the face layers is negligible.

Using the deformation continuity of sandwich beam, the displacements at the layer boundaries between top and bottom layers and core layer can be expressed as:

At $z = h_2/2$:

$$u_{1b} = u_1 + \frac{h_1}{2} w' \quad \text{and} \quad u_{2t} = u_2 + \frac{h_2}{2} w' \tag{1}$$

and at $z = -h_2/2$:

$$u_{3t} = u_3 - \frac{h_3}{2} w' \;\; \text{and} \;\; u_{2b} = u_2 - \frac{h_2}{2} w' \tag{2}$$

At continuity conditions, $u_{1b} = u_{2t}$ and $u_{3t} = u_{2b}$:

$$u_1 + \frac{h_1}{2} w' = u_2 + \frac{h_2}{2} w' \text{, and}$$

$$u_3 - \frac{h_3}{2} w' = u_2 - \frac{h_2}{2} w' \tag{3}$$

where the prime denotes differentiation with respect to *x*.

The axial displacement in the core layer at point *p* that located at height *z* from the core middle surface can be obtained (Figure 1) as:

$$\frac{u_p - u_{3t}}{\left(\frac{h_2}{2} + z\right)} = \frac{u_{1b} - u_{3t}}{h_2}$$

or,

$$u_p = \frac{(u_{1b} + u_{3t})}{2} + \frac{z}{h_2}(u_{1b} - u_{3t}) \tag{4}$$

Substituting equations (1) and (2) into equation (4), one obtains:

$$u_2 = \frac{(u_1 + u_3)}{2} + (h_1 - h_3)\frac{w'}{4} \tag{5}$$

The transverse displacement of arbitrary point $p(x,z)$ is given as:

$$w_p(x,z,t) = w(x,t) \tag{6}$$

The transverse shear strain of point *p* located in the core (layer 2) is:

$$\gamma_{xzp} = \frac{\partial w_p}{\partial x} + \frac{\partial u_p}{\partial z} \tag{7}$$

Substituting equations (4) and (6) into equation (7), gives:

$$\gamma_{xzp} = w' + \frac{(u_{1b} - u_{3t})}{h_2} \tag{8}$$

From equations (1) and (2), by substituting into equation (8), one obtains:

$$\gamma_{xzp} = w'\left[1+\frac{(h_1+h_3)}{2h_2}\right]+\frac{(u_1-u_3)}{h_2}$$
(9)

From the symmetry of the motion of the sandwich beam, in which the thicknesses of the faces are identical (i.e., $h_1=h_3=h$): $u_1(x,t)=-u_3(x,t)=u(x,t)$. On substitution, equation (9) becomes:

$$\gamma_{xzp} = w'\left(1+\frac{h}{h_2}\right)+\frac{2u}{h_2} \tag{10}$$

III. Total Potential Energy of Sandwich Beam

A. *Strain Energy of Sandwich Beam*

The strain energy due to axial stress in both faces is given by:

$$U_f = \int_0^\ell EA(u')^2 dx + \int_0^\ell EI(w'')^2 dx$$
(11)

where $A_i = t_i b$ and $I_i = bt^3/12$, (for $i=1,3$).

The strain energy of the core (layer 2) due to shear strain is given as:

$$U_2 = \frac{1}{2}\int_0^\ell \kappa G_2 A_2 (\gamma_{xzp})^2 dx$$
(12)

where κ is the shear correction factor, A_2 is the cross-section area of the core layer, and G_2 is the shear modulus of the core material.

Substituting equation (10) into equation (12), one obtains:

$$U_2 = \frac{1}{2}\int_0^\ell \kappa G_2 A_2 \left[w'\left(1+\frac{h}{h_2}\right)+\frac{2u}{h_2}\right]^2 dx$$
(13)

Then, the total strain energy *U* of the three-layered symmetric sandwich beam due to normal and shear strains are written as:

$$U = \int_0^\ell \left(EAu'^2 + EIw''^2 + \frac{\kappa G_2 A_2}{2}\left[w'\left(1+\frac{h}{h_2}\right)+\frac{2u}{h_2}\right]^2\right) dx \tag{14}$$

B. *Potential Energy of Applied Forces*

The symmetric sandwich beam subjected to dynamic bending forces, distributed transverse force $q_z(x,t)$, concentrated transverse forces $F_z(x_e,t)$ and bending moments $M_x(x_e,t)$ applied at the beam ends, i.e., $x_e = 0, L$ is considered as shown in Fig. (2).

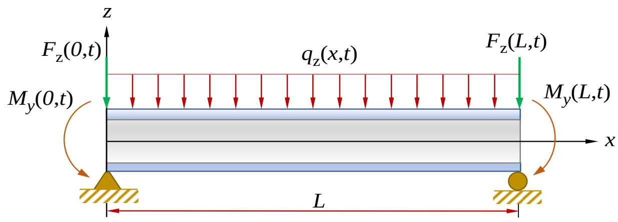

Fig. (2): *Three-layered sandwich beam under dynamic bending forces*

The potential energy of the external dynamic forces and moments is given by:

$$V = -\int_0^\ell q_z(x,t)\,w\,dx - \left[F_z(x,t)w\right]_0^\ell - \left[M_x(x,t)w'\right]_0^\ell$$
(15)

The total potential energy of the three-layered sandwich beam is:

$$\begin{aligned} \Pi = U+V = \int_0^\ell \Big(& EA(u')^2 + EI(w'')^2 \\ & + \frac{\kappa G_2 A_2}{2h_2^2}\left[2u + w'(h+h_2)\right]^2 - q_z w \Big) dx \\ & - \left[F_z(x,t)w(x,t)\right]_0^\ell - \left[M_x(x,t)w'(x,t)\right]_0^\ell \end{aligned}$$
(16)

C. *Kinetic Energy of Sandwich Beam*

It is assumed that the core layer is made of light material so that the core mass is negligible in comparison with the mass of the face layers [3], the kinetic energy of the sandwich beam is given by:

$$T_f = \int_0^\ell \rho A\left[(\dot{u})^2 + (\dot{w})^2\right] dx \tag{17}$$

where ρ is the density of the face material.

IV. Harmonic Vibration

The symmetric sandwich beam subjected to various harmonic bending forces as shown in Fig. (2) are given as:

$$q_z(x,t),\, F_z(x,t), M_x(x,t) = \left[\bar{q}_z(x),\, \bar{F}_z(x)\Big|_0^\ell, \bar{M}_x(x)\Big|_0^\ell\right] e^{i\Omega t} \tag{18}$$

where Ω is the exciting frequency of the applied harmonic forces, $i=\sqrt{-1}$ is the imaginary constant, $q_z(x,t)$ is the distributed transverse harmonic force along the sandwich beam axis, $F_z(x,t)$ and $M_x(x,t)$ are the end harmonic transverse forces and bending moments applied at the beam ends $(i.e., x=0, L)$.

Under the given applied harmonic forces and moments, the axial and transverse displacements are assumed to take the following form:

$$u(x,t),\ w(x,t) = \left[U(x),\ W(x)\right] e^{i\Omega t} \tag{19}$$

where $U(x)$ and $W(x)$ are the axial and transverse displacement amplitudes, respectively.

Since the present formulation is proposed to capture only the coupled axial-transverse bending steady state dynamic response of the three-layered symmetric sandwich beam, thus, the axial and transverse displacement functions suggested in equation (19) neglect the transient response of the sandwich beam.

V. Dynamic Differential Equations

In order to derive the dynamic differential equations and the associated boundary conditions, the Hamilton's variational principle can be applied. The principle required that:

$$\delta \int_{t_1}^{t_2}\left(\delta T_f - \delta \Pi\right)dt = 0, \quad \text{for} \quad \delta u(x,t)\Big|_{t_1}^{t_2} = \delta w(x,t)\Big|_{t_1}^{t_2} = 0 \tag{20}$$

where δ is the variational operator, t_1 and t_2 define the time interval.

From Equations (18-19) and by substituting into energy expressions (16-17), the resulting equations into Hamilton's equation (20), performing integration by parts and collecting the terms in δu and δw yields the governing differential equations for coupled axial-transverse response of symmetric sandwich beams under harmonic bending forces as:

$$EAU'' + \left(\rho A\Omega^2 - \frac{2\kappa G_2 A_2}{h_2^2}\right)U - \frac{\kappa G_2 A_2}{h_2^2}(h+h_2)W' = 0 \tag{21}$$

$$-2\rho A\Omega^2 W + 2EIW^{iv} - \frac{\kappa G_2 A_2 (h+h_2)^2}{h_2^2}W'' - \frac{2\kappa G_2 A_2 (h+h_2)}{h_2^2}U' = \vec{q}_z(x)$$

(22)

It is noted that, the governing differential equations (21-22) are similar to those derived by [3] for free vibration of symmetric sandwich beams with two differences: (a) the presence of non-zero forcing functions on the right hand sides of equation (22), and (b) the time dependence of the coupled equations for motion has been eliminated as direct outcome of the substitution made in equations (21-22), (c) the inertia terms are now specialized for the case of harmonic bending forces. The associated natural and essential boundary conditions are:

$$\left[2EAU'\delta U\right]_0^L = 0 \tag{23}$$

$$\left[\left(2EIW''' - \frac{\kappa G_2 A_2 (h+h_2)}{h_2^2}\left[2U + (h+h_2)W'\right] + \bar{F}_z\right)\delta W\right]_0^L = 0$$

$$\left(\left[2EIW'' - \bar{M}_x\right]\delta W'\right)_0^L = 0 \tag{24-25}$$

A. Exact Homogeneous Solution of Coupling Equations

The exact homogeneous solutions of the coupled axial-transverse bending equations (21-22) are obtained by setting the right-hand side of the equations equal to zero, i.e., $\bar{q}_z(x) = 0$. The homogeneous solutions of the axial and transverse displacement functions to take the following forms:

$$\{\chi(x)\}_{2\times 1} = \begin{Bmatrix} U_h(x) \\ W_h(x) \end{Bmatrix}_{2\times 1} = \begin{Bmatrix} \bar{A}_i \\ \bar{B}_i \end{Bmatrix}_{2\times 1} e^{\beta_i x}$$

(26)

Substituting equation (26) into coupled equation (21-22), yields in matrix form:

$$\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}_{2\times 2} \begin{Bmatrix} \bar{A}_i \\ \bar{B}_i \end{Bmatrix}_{2\times 1} = \{0\}_{2\times 1}$$

(27)

in which

$$T_{11} = \left(EA\beta_i^2 - \frac{2\kappa G_2 A_2}{h_2^2} + \rho A\Omega^2\right),$$

$$T_{12} = T_{21} = -\frac{2\kappa G_2 A_2\left(h+h_2\right)}{h_2^2}\beta_i,$$

$$T_{22} = \left(2EI\beta_i^4 - \frac{\kappa G_2 A_2}{h_2^2}\left(h+h_2\right)^2\beta_i^2 - 2\rho A\Omega^2\right),$$

$\left\langle \bar{A}_i \quad \bar{B}_i \right\rangle_{1\times 2}$ is a vector of constants corresponding to root β_i. For a non-trivial solution, the determinant of the bracketed matrix in equation (27) is set to zero, leading to the sixth order polynomial equation of the form $q_3\beta_i^6 + q_2\beta_i^4 + q_1\beta_i^2 + q_o = 0$, in which q_o through q_3 are constants arising from the expansion of the determinant of 2×2 matrix and depend upon cross-section properties, sandwich material constants and exciting frequency Ω and are given as: $q_3 = E^2 AI$,

$$q_2 = \left[\rho A\Omega^2 EI - \frac{\kappa G_2 A_2 E}{2h_2^2}\left(I - A\left(h+h_2\right)^2\right)\right],$$

$$q_1 = -\rho A\Omega^2\left[\frac{\kappa G_2 A_2}{2h_2^2}\left(h+h_2\right)^2 + EA\right], \text{ and}$$

$$q_o = -\rho A\Omega^2\left(\rho A\Omega^2 - \frac{2\kappa G_2 A_2}{h_2^2}\right).$$

The exact homogeneous solutions for the axial and transverse displacements $U(x)$ and $W(x)$ are written as:

$$U_h(x) = \sum_{i=1}^{6}\bar{A}_i e^{\beta_i x}, \quad W_h(x) = \sum_{i=1}^{6}\bar{B}_i e^{\beta_i x}$$

(28)

Equation (28) has twelve unknown integration constants $\bar{A}_i, \bar{B}_i$ for $i = 1,2,\ldots,6$, but only six boundary conditions are provided in equation (23-25). It is necessary to reduce the two sets $\bar{A}_i$ and $\bar{B}_i$ of unknown integration constants to six independent boundary conditions by writing the set of constants $\bar{B}_i$ in terms of the constants of the other set $\bar{A}_i$. By back-substitution into equation (27), one can relate constants $\bar{B}_i$ through $\bar{B}_i = \bar{G}_i \bar{A}_i$, where:

$$\bar{G}_i = \frac{\left(\rho A \Omega^2 h_2^2 + EA\beta_i^2 h_2^2 - 2\kappa G_2 A_2\right)}{\kappa G_2 A_2 (h + h_2)\beta_i}$$

The exact homogeneous solutions for axial $U(x)$ and transverse displacements $W(x)$ presented in equation (28) govern the axial-transverse coupled steady state response are obtained as:

$$\{\chi_h(x)\}_{2\times 1} = \left[\bar{G}\right]_{2\times 6} [E(x)]_{6\times 6} \{\bar{A}\}_{6\times 1}$$

(29)

in which $\langle \chi_h(x) \rangle_{1\times 2} = \langle U_h(x) \; W_h(x) \rangle_{1\times 2}$, $[E(x)]_{6\times 6}$ is a diagonal matrix consisting of the exponential functions $e^{\beta_i x}$ (for $i = 1,2,3,\ldots,6$), $\langle \bar{A} \rangle_{1\times 6} = \langle \bar{A}_1 \; \bar{A}_2 \; \bar{A}_3 \; \ldots \; \bar{A}_6 \rangle_{1\times 6}$ is the vector of unknown constants to be determined from the boundary conditions of the sandwich beam, and

$$\left[\bar{G}\right]_{2\times 6} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ \bar{G}_1 & \bar{G}_2 & \bar{G}_3 & \bar{G}_4 & \bar{G}_5 & \bar{G}_6 \end{bmatrix}_{2\times 6}.$$

B. *Particular Solution for Sandwich Beam*

For a symmetric sandwich beam under distributed transverse force, $q_z(x) = \bar{q}_z \, e^{i\Omega t}$, the corresponding particular solution $\langle \chi_P \rangle_{1\times 2}$ is obtained as:

$$\langle \chi_P \rangle_{1\times 2} = \langle U_p \;\; W_p \rangle_{1\times 2} = \left\langle 0 \;\; -\bar{q}_z / \rho A \Omega^2 \right\rangle_{1\times 2}$$

(30)

The total coupled axial-transverse bending steady state response is obtained by adding the homogeneous solution in equation (29) to the particular solution in equation (30), one obtains:

$$\{\chi(x)\}_{2\times 1} = \left[\bar{G}\right]_{2\times 6} [E(x)]_{6\times 6} \{\bar{A}\}_{6\times 1} + \{\chi_p\}_{2\times 1}$$

(31)

C. *Exact Solution of Cantilever Sandwich Beam*

A sandwich cantilever beam subjected to distributed harmonic force $q_z(x,t) = \bar{q}_z e^{i\Omega t}$, concentrated force $F_z(L,t) = \bar{F}_z e^{i\Omega t}$ and concentrated bending moment $M_x(\ell,t) = \bar{M}_x e^{i\Omega t}$ applied at the cantilever free end is considered. The unknown constants are obtained from the following boundary conditions at both ends:

$U(0) = W(0) = W'(0) = U'(L) = 0$,

$$2EIW'''(L) - \frac{\kappa G_2 A_2 (h + h_2)}{h_2^2}\left[2U(L) + (h + h_2)W'(L)\right] = -\bar{F}_z(L)$$

and $2EIW''(L) = M_x(L)$.

Imposing the displacement functions in (28) into the boundary conditions, the total steady state solution for a symmetric sandwich cantilever beam becomes:

$$\{\chi_c(x)\}_{2\times 1} = \left[\bar{G}\right]_{2\times 6} [E(x)]_{6\times 6} [\Psi_c]^{-1}_{6\times 6} \{\bar{Q}_c\}_{6\times 1} + \{\chi_p\}_{2\times 1} \quad (32)$$

where

$$[\Psi_c]_{6\times 6} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ \bar{G}_1 & \bar{G}_2 & \bar{G}_3 & \bar{G}_4 & \bar{G}_5 & \bar{G}_6 \\ \beta_1\bar{G}_1 & \beta_2\bar{G}_2 & \beta_3\bar{G}_3 & \beta_4\bar{G}_4 & \beta_5\bar{G}_5 & \beta_6\bar{G}_6 \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 \\ \eta_1 & \eta_2 & \eta_3 & \eta_4 & \eta_5 & \eta_6 \\ \mu_1 & \mu_2 & \mu_3 & \mu_4 & \mu_5 & \mu_6 \end{bmatrix}_{6\times 6}$$

$\langle \bar{Q}_c \rangle_{1\times 8} = \langle -U_p \;\; -W_p \;\; 0 \;\; 0 \;\; -\bar{F}_z(L) \;\; \bar{M}_x(L) \rangle_{1\times 8}$.

$\alpha_i = \beta_i e^{\beta_i L}$, $\mu_i = 2EI\beta_i^2 \bar{G}_i e^{\beta_i L}$ and

$$\eta_i = \left(2EI\bar{G}_i\beta_i^3 - \frac{\kappa G_2 A_2 (h + h_2)}{h_2^2}\left[2 + (h + h_2)\bar{G}_i\beta_i\right]\right) e^{\beta_i L}.$$

D. *Exact Solution for Simply-supported Sandwich Beam*

A simply-supported sandwich beam subjected to distributed harmonic transverse force $q_z(x,t) = \bar{q}_z e^{i\Omega t}$, and bending moments $M_x(x_e,t) = \bar{M}_x e^{i\Omega t}$ at sandwich beam ends $(x_e = 0, L)$ is considered. The boundary conditions are: $U(0) = W(0) = 0$, $2EIW''(0) = M_x(0)$, $U'(L) = W(0) = 0$, and $2EIW''(L) = -M_x(L)$.

From equation (31), by substituting into the above boundary conditions, the total closed form steady state solution for simply-supported symmetric sandwich beam is determined as:

$$\{\chi_s(x)\}_{2\times 1} = \left[\bar{G}\right]_{2\times 6} [E(x)]_{6\times 6} [\Psi_s]^{-1}_{6\times 6} \{\bar{Q}_s\}_{6\times 1} + \{\chi_p\}_{2\times 1} \quad (33)$$

in which,

$$[\Psi_s]_{6\times 6} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ \bar{G}_1 & \bar{G}_2 & \bar{G}_3 & \bar{G}_4 & \bar{G}_5 & \bar{G}_6 \\ \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_6 \\ \bar{\alpha}_1 & \bar{\alpha}_2 & \bar{\alpha}_3 & \bar{\alpha}_4 & \bar{\alpha}_5 & \bar{\alpha}_6 \\ \bar{\eta}_1 & \bar{\eta}_2 & \bar{\eta}_3 & \bar{\eta}_4 & \bar{\eta}_5 & \bar{\eta}_6 \\ \bar{\mu}_1 & \bar{\mu}_2 & \bar{\mu}_3 & \bar{\mu}_4 & \bar{\mu}_5 & \bar{\mu}_6 \end{bmatrix}_{6\times 6},$$

$\langle \bar{Q}_s \rangle_{1\times 6} = \langle -U_p \;\; -W_p \;\; \bar{M}_x(0) \;\; 0 \;\; -W_p \;\; -\bar{M}_x(L) \rangle_{1\times 6}$,

where $\lambda_i = 2EI\beta_i^2 \bar{G}_i$, $\bar{\alpha}_i = 2EA\beta_i e^{\beta_i L}$, $\bar{\eta}_i = \bar{G}_i e^{\beta_i L}$ and $\bar{\mu}_i = 2EI\beta_i^2 \bar{G}_i e^{\beta_i L}$.

E. Exact Solution for fixed-fixed Sandwich Beam

A fixed-fixed symmetric sandwich beam subjected to distributed harmonic transverse force $q_z(x,t)=\bar{q}_z e^{i\Omega t}$ is considered. The boundary conditions for given beam are $U(0)=W(0)=W'(0)=U(L)=W(L)=W'(L)=0$.

The total closed-form steady state solution for fixed-fixed symmetric sandwich beam under harmonic transverse force is then obtained as:

$$\{\chi_F(x)\}_{2\times1}=\left[\bar{G}\right]_{2\times6}[E(x)]_{6\times6}[\Psi_F]^{-1}_{6\times6}\{\bar{Q}_F\}_{6\times1}+\{\chi_p\}_{2\times1} \tag{34}$$

where

$$[\Psi_F]_{6\times6}=\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ \bar{G}_1 & \bar{G}_2 & \bar{G}_3 & \bar{G}_4 & \bar{G}_5 & \bar{G}_6 \\ \zeta_1 & \zeta_2 & \zeta_3 & \zeta_4 & \zeta_5 & \zeta_6 \\ \xi_1 & \xi_2 & \xi_3 & \xi_4 & \xi_5 & \xi_6 \\ \varphi_1 & \varphi_2 & \varphi_3 & \varphi_4 & \varphi_5 & \varphi_6 \\ \tau_1 & \tau_2 & \tau_3 & \tau_4 & \tau_5 & \tau_6 \end{bmatrix}_{6\times6},$$

$\left\langle \bar{Q}_s \right\rangle_{1\times6}=\left\langle -U_p \quad -W_p \quad 0 \quad -U_p \quad -W_p \quad 0 \right\rangle_{1\times6}$,

$\zeta_i=\beta_i\bar{G}_i$, $\xi_i=e^{\beta_i L}$, $\varphi_i=\bar{G}_i e^{\beta_i L}$, and $\tau_i=\beta_i\bar{G}_i e^{\beta_i L}$.

VI. NUMERICAL RESULTS AND DISCUSSION

Although the exact closed-form solution developed in the present formulation provides the steady state dynamic axial-transverse coupled response of three-layered symmetric sandwich beams under various harmonic bending forces, it can also capture the quasi-static axial-transverse coupled response under the given harmonic forces when adopting a very low exciting frequency $\Omega\approx0.01\omega_1$ compared to the first natural frequency ω_1 of the given sandwich beam. To validate the accuracy of the present solution, two examples are presented. The results based on the present formulation are compared with available exact solutions in the literature and established Abaqus finite shell element.

A. Example (1): Symmetric sandwich beam - Verification

In order to show the validity and accuracy of the present closed-form solution, a cantilever symmetric sandwich beam of length $L=0.9114mm$ is subjected to concentrated force $F_z(L,t)=200e^{i\Omega t}N$ applied at the free end. The following sandwich beam date taken from the literature [3] are used in the present analysis: the top and bottom layers have the same thickness $h=0.4572mm$, core thickness $h_2=12.7mm$, the width of the sandwich $b=25.4mm$, $EA=31500N$, $EI=1.362Nm^2$, $\kappa G_2A_2=1050N$, $m=1.225\times10^{-3}kg/m$.

Extracting Natural Frequencies

Multiple steady-state dynamic response analyses under distributed transverse force $F_z(L,t)=200e^{i\Omega t}N$ are performed for an exciting frequency f varying from nearly zero to 1000Hz. The coupled axial-transverse natural frequencies are then extracted from the peaks of the displacements-frequency relationships. The transverse $W_{\max}$ and axial $U_{\max}$ displacements at the tip of cantilever sandwich beam against the exciting frequency are shown in Fig. (3). Peaks on the diagrams indicate resonance and then indicators of the natural frequencies of the sandwich beam. It is observed that the transverse displacement peaks are matched with the axial displacement peaks indicating that the mode shapes are indeed coupled. The first four natural frequencies extracted at the peaks in Fig. (3) are provided in Table (1). The natural frequencies of the symmetric sandwich beam predicted using the present solution are compared to those published by Banerjee [3] and other results reported in Ahmed [1]. As a general observation, the present results for natural frequencies are exactly matched with the corresponding results of Banerjee [3]. It is also noted that, the frequencies predicted by both present and Banerjee solutions are differed from 7.39% to -9.59% from those based on Ahmed results. In addition, the first four steady state modes for normalized transverse and axial displacements are illustrated in Figs. (3b and 3d), respectively.

Table (1): Natural frequencies of the three-layered symmetric sandwich beam

Freq. No.	**Natural frequencies in (*Hz*)**			Error %	
	Banerjee (2003) [1]	Ahmed (1971) [2]	Present Solution [3]	=[1-3]/1	=[2-3]/2
1	31.46	33.97	31.46	0.00%	7.39%
2	193.7	200.5	193.7	0.00%	3.39%
3	529.2	517.0	529.2	0.00%	-2.36%
4	1006	918.0	1006	0.00%	-9.59%

B. Example (2): Quasi-state and dynamic analyses

In order to study the effects of core thicknesses and sandwich beam lengths on the natural frequencies, quasi-static and steady state dynamic responses, a symmetric sandwich cantilever beam subjected to distributed harmonic force $q_z(x,t)=100e^{i\Omega t}N/m$ is considered as shown in Fig. (4). The sandwich beam has a width $b=40mm$ and identical face thicknesses $h=2mm$, while the core thickness is varied from 20mm to 70mm and sandwich length is varied from 0.8m to 2.4m. The material properties for faces are: $E=70GPa$, $\rho=2700kg/m^3$, and the core properties are:

$G_2=80GPa$, and $\rho_2=100g/m^3$.

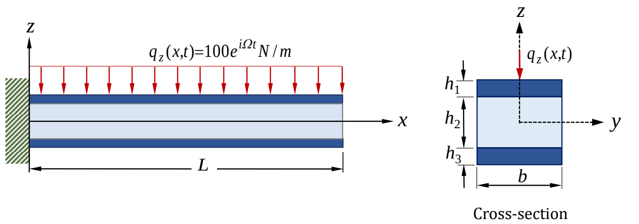

Fig. (4): *Sandwich cantilever beam under distributed harmonic transverse force*

Natural frequencies

The steady state dynamic analyses of symmetric sandwich cantilever beam under the given distributed harmonic force is investigated in order to extract the coupled axial-transverse natural frequencies. The first five natural frequencies (in *Hz*) extracted from the steady state responses presented in Table (2) are conducted based on the present closed-form solution for core thickness $h_2 = 50mm$ and for various sandwich lengths $(L = 0.8, 1.2, 1.6, 2.0, 2.4m)$.

Table (2): Natural frequencies (in *Hz*) of symmetric sandwich beam for different beam lengths

Freq. No. (Hz)	Natural frequencies in Hz				
	L=0.8m	L=1.2m	L=1.6m	L=2.0m	L=2.4m
1	95.02	46.69	27.35	17.85	12.53
2	358.6	206.1	134.1	93.73	68.83
3	725.4	440.1	299.7	217.9	165.4
4	1068	672.7	473.7	354.4	275.8
5	1413	905.4	649.6	495.1	391.9

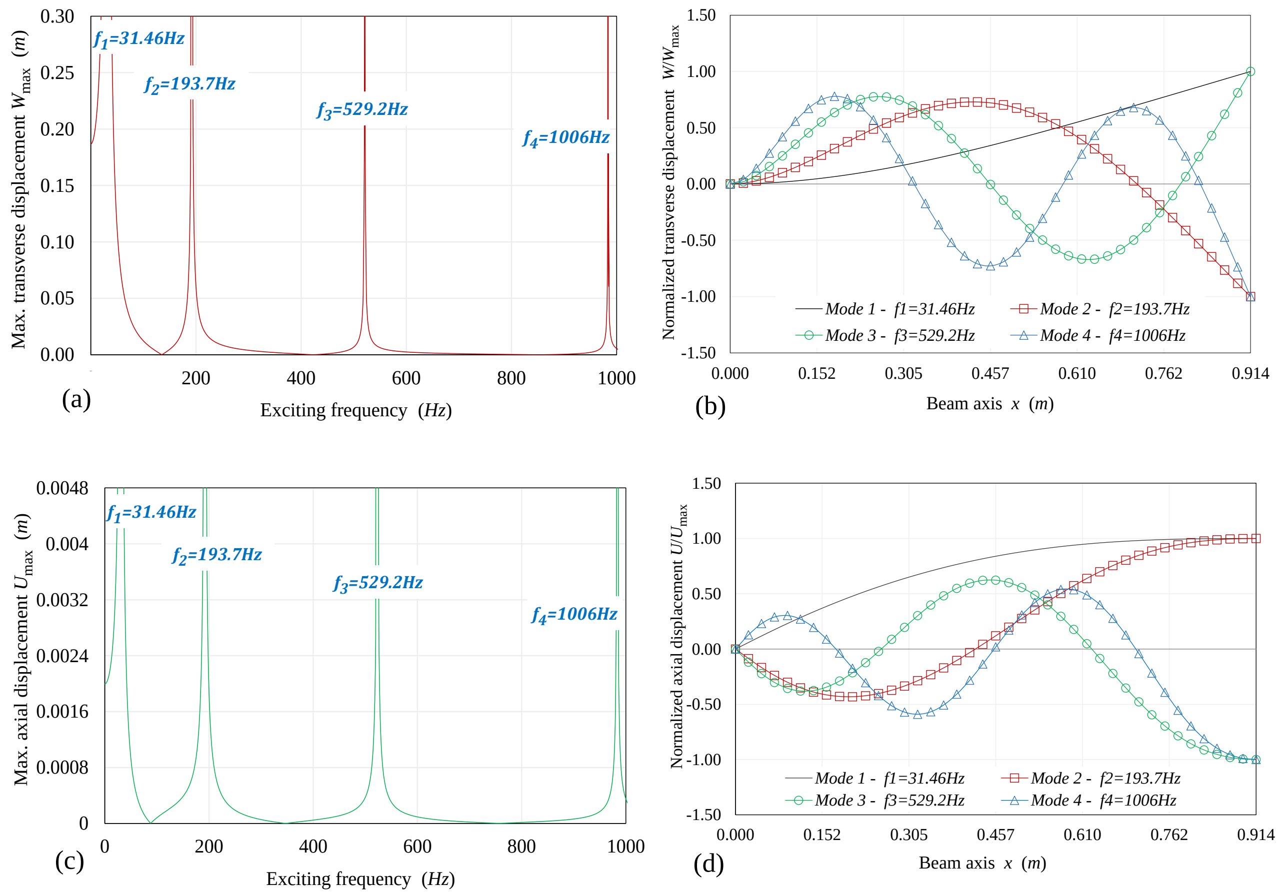

Fig. (3): *Natural frequencies and mode vibration of symmetric sandwich cantilever beam*

It is noted from Table (2) that, the natural frequencies decrease with increase of the sandwich beam length. Furthermore, the first five natural frequencies of the symmetric sandwich beam obtained using the present formulation against the sandwich beam length are shown in Fig. (5). It is observed that, as sandwich length is increased the natural frequency is decreased for various steady state mode numbers. This may be because of natural frequency is inversely proportional to the sandwich beam length.

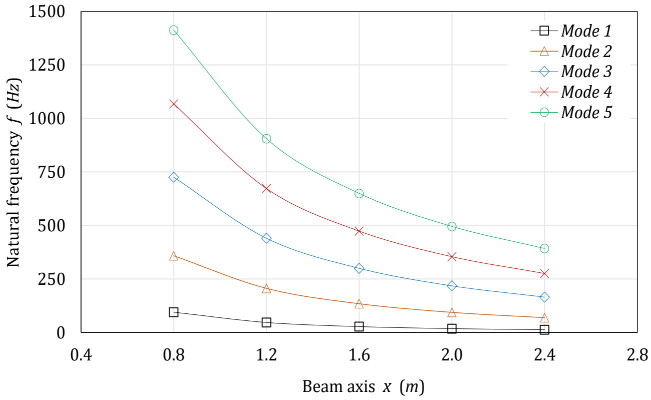

Fig. (5): *Natural frequencies of symmetric sandwich beam for different sandwich lengths (h_2=50mm)*

Quasi-static Analysis

Based on the present formulation, the quasi-static response of the three-layered symmetric sandwich cantilever beam under given harmonic distributed force $q_z(x,t)=100e^{i\Omega t}\,N/m$ with very low exciting frequency $\Omega=0.0628\,rad/\sec$ is approached. The static response results for the transverse and axial displacement functions for sandwich beam length $L=1.20m$ and core thickness $h_2=50mm$ are shown in Fig. (6). Results are based on (a) the closed-form solution developed in the present study, and (b) Abaqus finite shell solution. The static transverse and axial displacements results predicted by using the present formulation are observed to nearly coincide with those based on the Abaqus finite shell element solution.

Core thickness Effect on Quasi-static response

To study the influence of core thickness on the quasi-static response, the transverse and axial displacements of symmetric sandwich beam having length 1.20m are plotted for different core thicknesses. Fig. (7) show the effect of increase of the core thickness on the static response. It is observed that, as the core thickness increase, both the transverse and axial displacements are decreased. It is also observed that the maximum value of the static transverse displacement is decreased about 86% (from 20.44mm to 2.929mm) and the axial

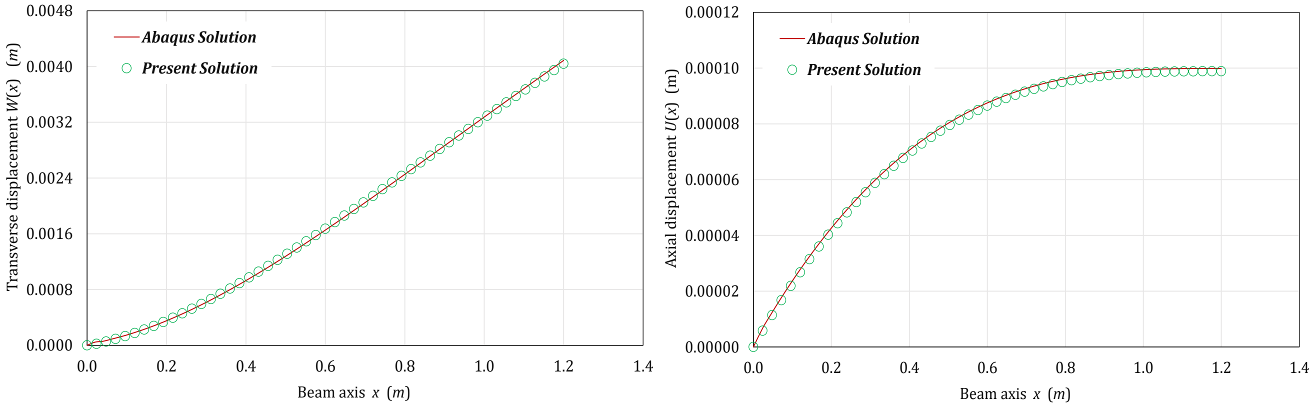

Fig. (6): *Static response of symmetric sandwich cantilever beam (L=1.2m and h_2=50mm)*

displacement is decreased about 64% (from 0.2331mm to 0.0829mm) when the core thickness changed (from 20mm to 60mm). For quasi-static response, it can be conclude that the increase of the core thickness leads to an increase in the sandwich beam stiffness.

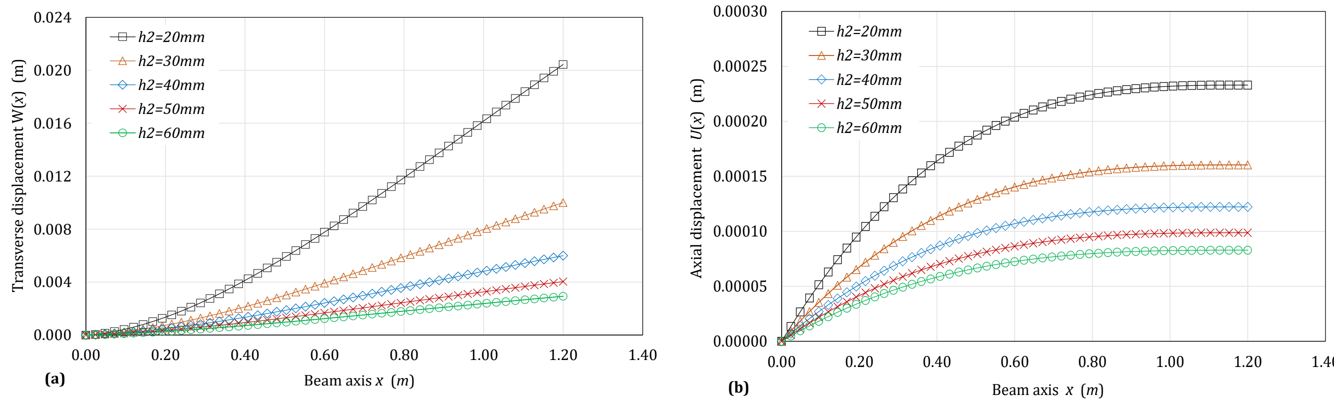

Fig. (7): *Quasi-static response of symmetric sandwich beam of length L=1.2m under harmonic distributed transverse force*

Effect of Sandwich length on Quasi-static response

Another parametric study related to the study of variation of the sandwich beam length versus the quasi-static response are provided for various core thicknesses in Table (3). Fig. (8) illustrates the influence of sandwich beam length on the maximum transverse and axial displacement functions. The mentioned results reveal that, the quasi-static responses increase by increasing the sandwich beam length and decrease by increasing the core thickness.

Table (3): Quasi-static response of three-layered symmetric sandwich cantilever beam under distributed harmonic transverse force

L (*m*)	h_2=20*mm*		h_2=30*mm*		h_2=40*mm*		h_2=50*mm*		h_2=60*mm*	
	W_{max} (*mm*)	U_{max} (*mm*)	W_{max} (*mm*)	U_{max} (*mm*)	W_{max} (*mm*)	U_{max} (*mm*)	W_{max} (*mm*)	U_{max} (*mm*)	W_{max} (*mm*)	U_{max} (*mm*)
0.8	4.371	6.907	2.214	4.756	1.370	3.625	0.949	2.929	0.706	2.457
1.2	11.89	22.45	5.918	15.45	3.606	11.78	2.463	9.519	1.811	7.985
1.6	42.55	54.39	20.69	37.45	12.33	28.55	8.255	23.07	5.953	19.35
2.0	111.7	107.1	53.73	73.71	31.72	56.19	21.03	45.40	15.03	38.08
2.4	242.8	185.6	116.2	127.8	68.23	97.43	45.01	78.72	32.01	66.03

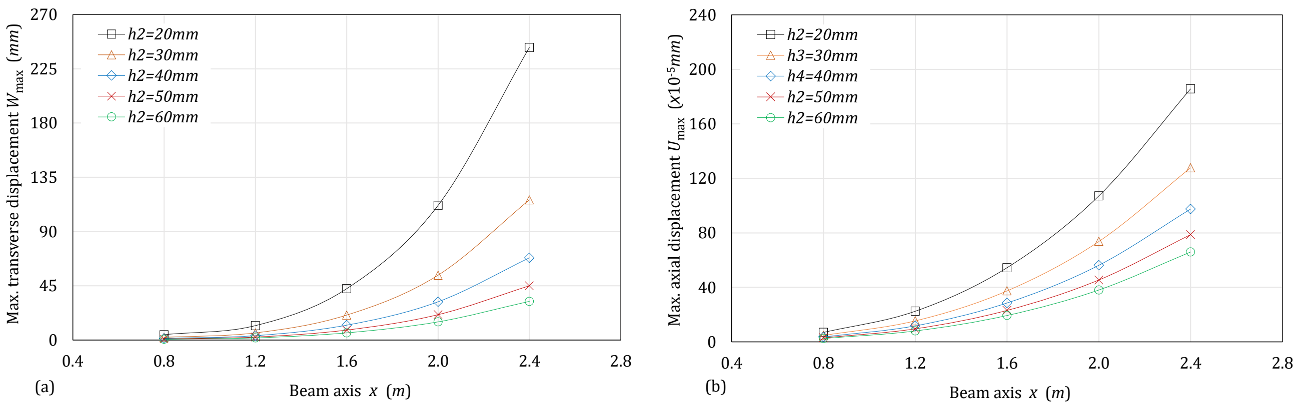

Fig. (8): *The influence of sandwich beam length and core thickness on the Quasi-static response*

Dynamic response analysis

The steady state dynamic response results of transverse and axial displacement functions versus the sandwich beam axis are plotted in Fig. (9) for exciting frequency $\Omega = 377\, rad/\sec$. Figure (9) is provided for sandwich beam length $L = 1.20m$ and core thickness $h_2 = 50mm$. Results based on the present closed-form solution are slightly differ from those based on Abaqus S4R shell model. The differences are attributed to shear deformation effects which are captured in Abaqus shell solution but not in the present formulation.

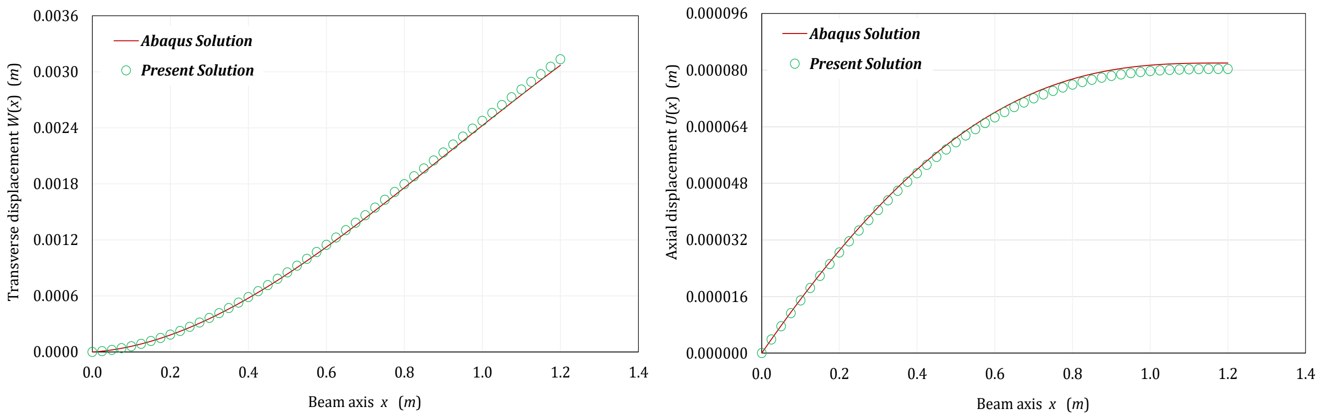

Fig. (9): *Steady state dynamic response of symmetric sandwich cantilever beam (L=1.2m and h*$_2$*=50mm)*

Effect of Core thickness on Dynamic response

The steady state dynamic response of three-layered symmetric sandwich cantilever beam of 1.20m length subjected to harmonic force $q_z(x,t) = 100e^{i\Omega t}\, N/m$ with exciting frequency $\Omega \approx 440\, rad/\sec$ is investigated. The dynamic analysis focuses on the effect of varying core thickness from 30mm to 70mm on the steady state response. The transverse and axial displacements are presented in Fig. (10) for various values of core thicknesses $(h_2 = 30, 40, 50, 60 \text{ and } 70mm)$. It is noted that, as the core thickness increased, the steady state transverse and axial displacements are increased. It is also observed that, the effect of the core thickness is more significant when the core thickness is greater than 50mm $(i.e., h_2 > 50mm)$. This leads to conclude that, for steady state response, the increase of the core thickness provides to decrease the stiffness of the sandwich beam.

Effect of Sandwich length on Dynamic response

The influence of length of the sandwich beam on the steady state response is investigated as shown in Fig. (11). As can be seen from Fig. (11), the steady state results indicate that as the sandwich length increases, both the transverse and axial displacements increase.

VII. Summary and Conclusion

Based on the Hamilton variational principle, the dynamic differential equations and associated boundary conditions governing the coupled axial-transverse response for symmetric sandwich beams subjected to general harmonic bending forces are derived. Exact expressions for the analytical closed-form solutions for axial and transverse responses are formulated for cantilever, simply-supported and fixed-fixed symmetric sandwich beams. The present exact closed form solutions are efficient in capturing the quasi-static and steady state dynamic responses of symmetric sandwich beams under harmonic bending forces. It is also capable of extracting the axial-transverse coupled natural frequencies and steady state axial-transverse bending modes. Comparison with established Abaqus finite shell element and exact solutions available in the literature exhibits the validity and accuracy of the present closed form solutions. Additionally, numerical results provided in the present study investigated the effects of the core thickness and sandwich beam length on the natural frequencies, quasi-static and steady state responses.

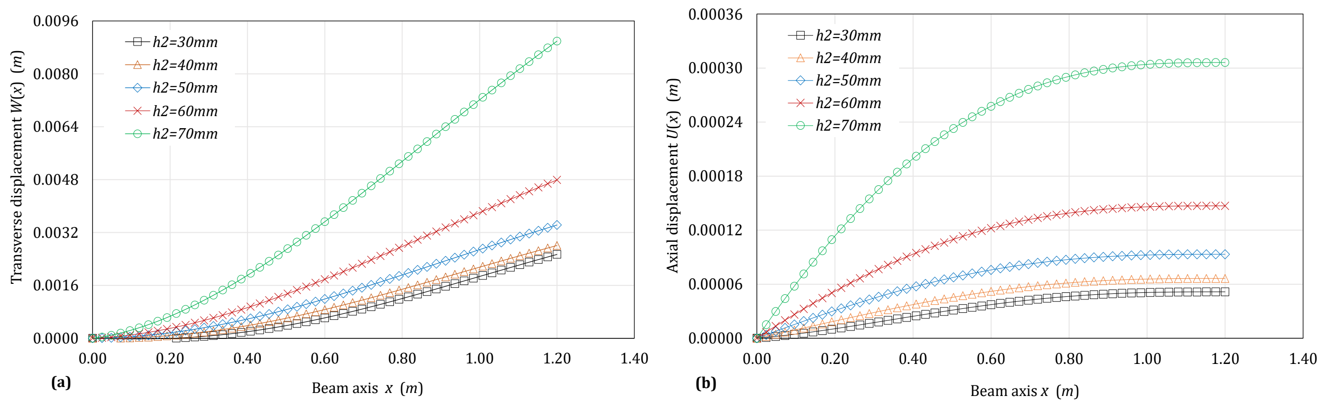

Fig. (10): *Steady state dynamic response of symmetric sandwich beam of length L=1.2m under harmonic distributed transverse force with* $\Omega \approx 440\, rad/\sec$

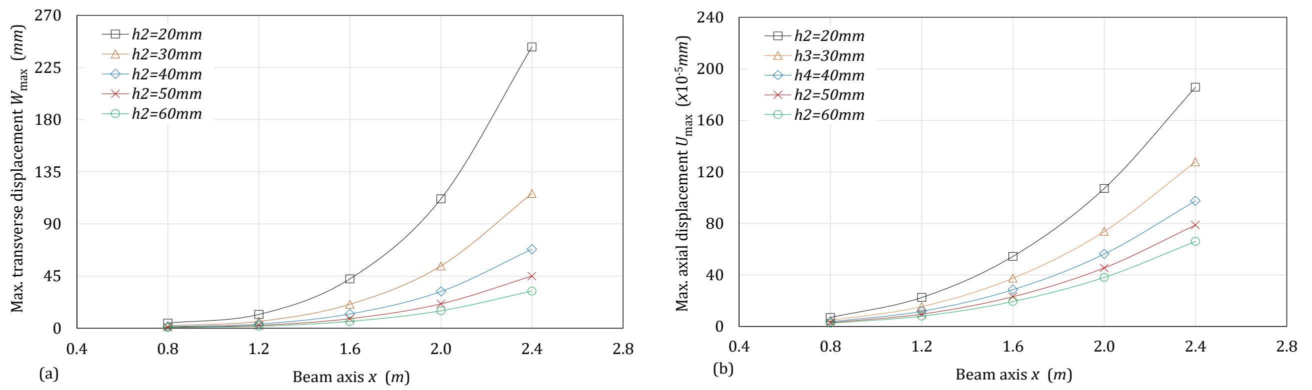

Fig. (11): *Steady state response of symmetric sandwich beam for different core thicknesses and sandwich lengths*

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