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Characterization Of Ternary Semigroups By Quasi-Ideals

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***Abstract*— In this note we have presented some characterizations of ternary semigroups through quasi-ideals. We have presented some definitions and prepositions about quasi-ideals, minimal quasi-ideals and 0-minimal quasi-ideals in a ternary semigroup.**

Keywords—quasi-ideal; minimal quasi-ideal; 0-minimal quasi-ideal; ternary semigroup

I. INTRODUCTION

A ternary semigroup is an algebraic structure dates back to S. Banach [9] who showed by an example that a ternary semigroup does not necessarily reduce to an ordinary semigroup. J. Los [9]showed that any ternary semigroup may be embedded in an ordinary semigroup in such a way that the ternary operation in the ternary semigroup is a extension of the binary operation in the ordinary semigroup. D. H. Lehmer [8] has also considered ternary groups. F.M. Sioson [7] studied ideal theory in ternary semigroups and gave the definitions of ideals. He also introduced the notion of a regular ternary semigroup and quasi-ideal in a ternary semigroup.

II. PRELIMINARIES

DEFINITION 2.1. A ternary semigroup is a non-empty set S together with a ternary operation which has the property of association:

$$(abc)de = a(bcd)e = ab(cde)$$

for all a, b, c and d in S.

DEFINITION 2.2. A non-empty set T of a ternary semigroup S is called a subsemigroup of S iff $a \in T, b \in T$ and $c \in T$ imply $abc \in T$.

DEFINITION 2.3. An element e of a ternary semigroup S is called:

(i) left identity of S iff $eea = a$ for all a in S

(ii) right identity of S iff $aee = a$ for all a in S

(iii) lateral identity of S iff $eae = a$ for all a in S

(iv) two sided identity of S iff e is left and right identity of S

(v) identity of S iff e is left, right and lateral identity of S.

DEFINITION 2.4. An element z of ternary semigroup S is called a zero element of S iff $zab = zza = zaz = azb = abz = azz = z$ for all a, b in S.

Let S any ternary semigroup and 1 a fixed element in S. We extend the ternary operation of S to $S \cup 1$ defining $111 = 1$ and $11a = 1a1 = a11 = a$ for all a in S. In this way we have attached to S the identity element 1. Similarly we attach to S the zero element 0 defining $000 = 0ab = a0b = ab0 = 0$ for all a, b, c in S. So we have:

$$S^1 = \begin{cases} S & \text{if } S \text{ has an identity} \\ S \cup 1 & \text{otherwise} \end{cases} \qquad S^0 = \begin{cases} S & \text{if } S \text{ has a zero elemen} \\ S \cup 0 & \text{otherwise} \end{cases}$$

DEFINITION 2.5. An element e of a ternary semigroup S is called idempotent iff $eee = e$.

It is clear that a zero element and identity are idempotent elements. The converse is not true in general. The sets $\{0\}$, $\{1\}$ and $\{e\}$, where e is an idempotent element, are ternary subsemigroups of S.

DEFINITION 2.6. A non-empty subset A of a ternary semigroup S is called:

(i) left ideal of S iff $SSA \subseteq A$

(ii) right ideal of S iff $ASS \subseteq A$

(iii) lateral ideal of S iff $SAS \subseteq A$

(iv) two sided ideal of S iff A is a left and right ideal of S

(v) ideal of S iff A is a left, right and lateral ideal of S.

DEFINITION 2.7. An ideal I of a ternary semigroup S with a zero element 0 is called proper iff $I \neq \{0\}$ *dhe* $I \neq S$.

PREPOSITION 2.8. Let S be a ternary semigroup, L a left ideal of S, R a right ideal of S and M a lateral ideal of S. Then, LMR is a two sided ideal of S. Moreover $RML \subseteq R \cap M \cap L$.

PROOF. Let $a \in L, b \in M, c \in R$ and $s_1, s_2 \in S$. Then, $s_1s_2(abc) = (s_1s_2a)bc \in LMR$ since $s_1s_2a \in L$ due to the fact that L is a left ideal of S. Thus LMR is a left ideal of S. We have also that $(abc)s_1s_2 = ab(cs_1s_2) \in LMR$ since $cs_1s_2 \in R$ due to the fact that R is a right ideal of S. Thus LMR is a right ideal of S. Now let we show that $RML \subseteq R \cap M \cap L$. Let $a \in R, b \in M$ and $c \in L$. Then, $abc \in R$ since R is a right ideal of S. We have also that $abc \in M$ since M is a lateral ideal of S. On the other side $abc \in L$ since L is a left ideal of S. This implies that $abc \in R \cap M \cap L$.

DEFINITION 2.9. A ternary semigroup S is called regular iff for all $a \in S$ exists $x, y \in S$ such that $a = axaya$.

III. QUASI-IDEALS IN TERNARY SEMIGROUPS

DEFINITION 3.1. A non-empty subset Q of a ternary semigroup S is called a quasi-ideal of S iff $QSS \cap SQS \cap SSQ \subseteq Q$ and $QSS \cap SSQSS \cap SSQ \subseteq Q$.

PREPOSITION 3.2. Every left ideal L of a ternary semigroup S is a quasi-ideal of S.

PROOF. Let $a \in LSS \cap SLS \cap SSL$. Then, it follows that $a \in SSL$. On the other hand $SSL \subseteq L$ since L is a left ideal of S. Hence $a \in L$, thus $LSS \cap SLS \cap SSL \subseteq L$. Then, it is clear that $LSS \cap SSLSS \cap SSL \subseteq L$.

In a similar way one can prove that:

PREPOSITION 3.3. Every right ideal R [lateral ideal M, ideal I] of a ternary semigroup S is a quasi-ideal of S.

PREPOSITION 3.4. Let S be a ternary semigroup with zero 0. Then, every quasi-ideal Q of S contains 0.

PROOF. For all $k \in Q$ we have $0 = 00k = 0k0 = k00 \in SSQ \cap SQS \cap QSS \subseteq Q$. Thus $0 \in Q$.

Let S is a ternary semigroup without a zero element 0. Then, a quasi-ideal Q is called **proper** iff $Q \neq S$.

PREPOSITION 3.5. Every quasi-ideal Q of a ternary semigroup S is a ternary subsemigroup of S.

PROOF. Since Q is a quasi-ideal of S we have $Q^3 \subseteq QSS \cap SQS \cap SSQ \subseteq Q$.

PREPOSITION 3.6. The intersection of a right ideal R, lateral ideal M and left ideal L of a ternary semigroup S is a quasi-ideal of S.

PROOF. By Preposition 2.8. $RML \subseteq R \cap M \cap L$. Whence, the intersection $R \cap M \cap L$ is not empty. Since $(R \cap M \cap L)SS \cap S(R \cap M \cap L)S \cap SS(R \cap M \cap L) \subseteq RSS \cap SMS \cap SSL \subseteq R \cap M \cap L$ and $(R \cap M \cap L)SS \cap SS(R \cap M \cap L)SS \cap SS(R \cap M \cap L) \subseteq RSS \cap SSMSS \cap SSL \subseteq R \cap M \cap L$ we have that $R \cap M \cap L$ is a quasi-ideal of S.

PREPOSITION 3.7. The intersection of a quasi-ideal Q and a ternary subsemigroup T of a ternary semigroup S is either empty or a quasi-ideal of T.

PROOF. If $T \cap Q$ is not empty, then $T \cap Q$ is a subset of T such that $(T \cap Q)TT \cap T(T \cap Q)T \cap TT(T \cap Q) \subseteq T^3 \subseteq T$, $(T \cap Q)TT \cap TT(T \cap Q)TT \cap TT(T \cap Q) \subseteq T^3 \subseteq T$, $(T \cap Q)TT \cap T(T \cap Q)T \cap TT(T \cap Q) \subseteq QTT \cap TQT \cap TTQ \subseteq Q$ and $(T \cap Q)TT \cap TT(T \cap Q)TT \cap TT(T \cap Q) \subseteq QTT \cap TTQTT \cap TTQ \subseteq Q$. These imply that $T \cap Q$ is a quasi-ideal of T.

PREPOSITION 3.8. The intersection of a quasi-ideal Q and a ternary subsemigroup B of a ternary semigroup S with zero 0, is a quasi-ideal of B.

The proof is similar to that of Preposition 3.8.

PREPOSITION 3.9. Let e be an idempotent element of a ternary semigroup S and R, M, L a right ideal, lateral ideal and left ideal of S, respectively. Then, Ree, eeL and $eeMee$ are quasi-ideals of S.

PROOF. It suffices to show that $Ree = R \cap (SeS \cup SSeSS) \cap See, eeL = eeS \cap (SeS \cup SSeSS) \cap L$ and $eeMee = eSS \cap M \cap SSe$. Let $x \in Ree$. Then, $x = aee, a \in R$. Thus, $x \in R$ and $x \in See$. Moreover, $x \in SeS$ and $x = aee = aeeee \in SSeSS$. It means that $x \in SeS \cup SSeSS$. Thus, $x \in R \cap (SeS \cup SSeSS) \cap See$ which imply that $Ree \subseteq R \cap (SeS \cup SSeSS) \cap See$. Now, let $x \in R \cap (SeS \cup SSeSS) \cap See$. It means that $x \in R \cap See$ and $x \in SeS \cup SSeSS$. Then, $x = aee, a \in S$. Hence, $x = aee = aeeee = xee \in Ree$. This imply $R \cap (SeS \cup SSeSS) \cap See \subseteq Ree$. Similarly, we show that $eeL = eeS \cap (SeS \cup SSeSS) \cap L$. Now, let we show that $eeMee = eSS \cap M \cap SSe$. Let $x \in eeMee$. Then, $x = eeaee, a \in M$. Whence, $x \in eSS \cap SSe$ and $x \in M$. It follows that $x \in eSS \cap M \cap SSe$. It implies that $eeMee \subseteq eSS \cap M \cap SSe$. Now, let $x \in eSS \cap M \cap SSe$. This imply $x \in eSS \cap SSe$ and $x \in M$. Thus, $x = eab$ and $x = cde$ with $a, b, c, d \in S$. Then, $eexee = eeeabee = eabee = cdeee = cde = x$ which imply that $x \in eeMee$. Hence, $eSS \cap M \cap SSe \subseteq eeMee$.

PREPOSITION 3.10. Every quasi-ideal Q of a ternary semigroup S is an intersection of a left ideal $Q \cup SSQ$, lateral ideal $Q \cup SQS \cup SSQSS$ and right ideal $Q \cup QSS$ of S.

PROOF. The inclusion $Q \subseteq (Q \cup SSQ) \cap (Q \cup SQS \cup SSQSS) \cap (Q \cup QSS)$ is clear.

Converseley, let a an element of the intersection $(Q \cup SSQ) \cap (Q \cup SQS \cup SSQSS) \cap (Q \cup QSS)$. Since Q is a quasi-ideal of S, the second case implies $a \in SSQ \cap (SQS \cup SSQSS) \cap QSS \subseteq Q$. Thus, $(Q \cup SSQ) \cap (Q \cup SQS \cup SSQSS) \cap (Q \cup QSS) \subseteq Q$.

PREPOSITION 3.11. A non-empty subset of a ternary semigroup S is a quasi-ideal of S iff and only iff it is an intersection of a left ideal, lateral ideal and right ideal of S.

PROOF. Let $L = \bigcup_{q \in Q}(q)_l$. For all $s_1, s_2 \in S$ and for all $a \in L$ we have $a \in (q)_l$ for any $q \in Q$. If $a = q$ then $s_1 s_2 a = s_1 s_2 q \in (q)_l \subseteq \bigcup_{q \in Q}(q)_l = L$. If $a \in SSq$ we have $a = s_3 s_4 q$ ku $s_3, s_4 \in S$. Hence, $s_1 s_2 a = s_1 s_2 (s_3 s_4 q) = (s_1 s_2 s_3) s_4 q \in SSq \subseteq (q)_l \subseteq \bigcup_{q \in Q}(q)_l = L$. Thus, L is a left ideal of S. Now, let $M = \bigcup_{q \in Q}(q)_t$. For all $s_1, s_2 \in S$ and for all $a \in M$ we have $a \in (q)_t$ for any $q \in Q$. If $a = q$ then $s_1 a s_2 = s_1 q s_2 \in SqS \subseteq (q)_t \subseteq \bigcup_{q \in Q}(q)_t = M$. If $a \in SqS$ then $a = s_3 q s_4$ with $s_3, s_4 \in S$. Then, we have $s_1 a s_2 = s_1 (s_3 q s_4) s_2 \in SSqSS \subseteq (q)_t \subseteq M$. If $a \in SSqSS$ then $a = s_3 s_4 q s_5 s_6$. Therefore, $s_1 a s_2 = s_1 (s_3 s_4 q s_5 s_6) s_2 = (s_1 s_3 s_4) q (s_5 s_6 s_2) \subseteq SqS \subseteq$

$(q)_t \subseteq M$. Thus, M is a lateral ideal of S. Let $R = \bigcup_{q\in Q}(q)_r$. For all $s_1, s_2 \in S$ and for all $a \in R$ we have $a \in (q)_r$ for any $q \in Q$. If $a = q$ then $as_1s_2 = qs_1s_2 \in qSS \subseteq (q)_r \subseteq R$. If $a \in qSS$ then $a = qs_3s_4$ with $s_3, s_4 \in S$. Whence, $as_1s_2 = (qs_3s_4)s_1s_2 = qs_3(s_4s_1s_2) \in qSS \subseteq (q)_r \subseteq R$. This mean that R is a right ideal of S. It is clear that $Q \subseteq L \cap M \cap R$. Let we show the next incusion. On the other hand, by set operations we have $L \cap M \cap L = (Q \cup QSS) \cap (Q \cup SQS \cup SSQSS) \cap (Q \cup SSQ) = Q \cup [QSS \cap (SQS \cup SSQSS) \cap SSQ] \subseteq Q$. Therefore, $Q = L \cap M \cap R$.

Conversely, suppose that $Q = L \cap M \cap R$ where L is a left ideal of S, M is a lateral ideal of S and R is a right ideal of S. Then, $SSQ \cap SQS \cap QSS = SS(L \cap M \cap R) \cap S(L \cap M \cap R)S \cap (L \cap M \cap R)SS \subseteq SSL \cap SMS \cap RSS \subseteq L \cap M \cap R = Q$ and $SSQ \cap SSQSS \cap QSS = SS(L \cap M \cap R) \cap SS(L \cap M \cap R)SS \cap (L \cap M \cap R)SS \subseteq SSL \cap SSMSS \cap RSS \subseteq L \cap M \cap R = Q$. Hence, $Q = L \cap M \cap R$ is a quasi-ideal of S.

PREPOSITION 3.12. If Q është is a proper quasi-ideal of a ternary semigroup S with zero 0 such that Q does not contain left [lateral, right] ideals of S, $SSQ \cap SQS \not\subset Q$, $SQS \cap QSS \not\subset Q$ and $SSQ \cap QSS \not\subset Q$, then SSQ [QSQ, QSS] is a proper left [lateral, right] ideal of S.

PROOF. Let we show that SSQ is a proper left ideal of S. Suppose that $SSQ = 0$. Then $SSQ = 0 \subseteq Q$ which is not possible because Q does not contain left ideals. If $SSQ = S$ then $SSQ \cap SQS \cap QSS = S \cap SQS \cap QSS = SQS \cap QSS \subseteq Q$ which is a contradiction. Now, let we show that SQS is a proper lateral ideal of S. If $SQS = 0$ then $SQS = 0 \subseteq Q$ which is a contradiction because Q does not contain lateral ideals of S. If $SQS = S$ then $SSQ \cap SQS \cap QSS = SSQ \cap S \cap QSS = SSQ \cap QSS \subseteq Q$ which is a contradiction Let we show that QSS is a proper right ideal. If $QSS = 0$ athen $QSS = 0 \subseteq Q$ which is not possible. If $QSS = S$ then $SSQ \cap SQS \cap QSS = SSQ \cap SQS \cap S = SSQ \cap SQS \subseteq Q$ which could not happen as it contradicts the given condition.

PREPOSITION 3.13. The intersection of any set of quasi-ideals of a ternary semigroup S is either empty or a quasi-ideal of S.

PROOF. Let $(Q_\lambda)_{\lambda\in\Lambda}$ a set of quasi-ideals of S. If $\bigcap_{\lambda\in\Lambda} Q_\lambda$ is not empty, then for every $Q_\mu, \mu \in \Lambda, D = SS(\bigcap_{\lambda\in\Lambda} Q_\lambda) \cap S(\bigcap_{\lambda\in\Lambda} Q_\lambda)S \cap (\bigcap_{\lambda\in\Lambda} Q_\lambda)SS \subseteq SSQ_\mu \cap SQ_\mu S \cap Q_\mu SS \subseteq Q_\mu$. Thus $D \subseteq \bigcap_{\lambda\in\Lambda} Q_\lambda$ which ends the proof.

PREPOSITION 3.14. The intersection of any set of quasi-ideals of a ternary semigroup S with zero 0 is a quasi-ideal of S.

PROOF. Since every quasi-ideal of S contains a zero element 0, the intersection of any set of quasi-ideals of S is not empty. The proof can be continued in a similar way to the proof of the Preposition 3.13.

Let X be a non-empty subset of a ternary semigroup S. **The quasi-ideal of S generated by X** is the intersection $(X)_q$ of all quasi-ideals of S containing X, which in fact it is a quasi-ideal of S. (Certainly, $(X)_q$ is contained in every quasi-ideal of S containing X). If the subset X consists in a single element x, then $(x)_q$ is called **the principal quasi-ideal of S generated by x.**

THEOREM 3.15. If X is any non-empty subset of a ternary semigroup S, then $(X \cup SSX) \cap (X \cup SXS \cup SSXSS) \cap (X \cup XSS)$ is a quasi-ideal $(X)_q$ of S generated by X.

PROOF. $D = (X \cup SSX) \cap (X \cup SXS \cup SSXSS) \cap (X \cup XSS)$ is a quasi-ideal of S containing X, therefore $(X)_q \subseteq D$. On the other hand, the quasi-ideal $Q = (X)_q$ of S has the form $(X)_q = Q = (Q \cup SSQ) \cap (Q \cup SQS \cup SSQSS) \cap (Q \cup QSS)$. This and the inclusion $X \subseteq Q$ imply $D = (X \cup SSX) \cap (X \cup SXS \cup SSXSS) \cap (X \cup XSS) \subseteq (Q \cup SSQ) \cap (Q \cup SQS \cup SSQSS) \cap (Q \cup QSS) = (X)_q$. Whence, we have $(X)_q = D$.

REMARK 3.16. 1. The principal quasi-ideal $(x)_q$ of a ternary semigroup S generated by the element x of S has the form $(x)_q = (x \cup SSx) \cap (x \cup SxS \cup SSxSS) \cap (x \cup xSS)$ $(*)$

2. Let X be a non-empty subset of a ternary semigroup S and x an element of S. Then, it easy to prove that $(X)_q = X \cup (SSX \cap (SXS \cup SSXSS) \cap XSS)$ and $(x)_q = x \cup (SSx \cap (SxS \cup SSxSS) \cap xSS)$. Since $(X)_l = X \cup SSX$, $(X)_m = X \cup SXS \cup SSXSS$ and $(X)_r = X \cup XSS$ are the left ideal, lateral ideal and the right ideal of S generated by X respectively, Preposition 3.11. implies $(X)_q = (X)_l \cap (X)_t \cap (X)_r$. Therefore. $(*)$ implies $(x)_q = (x)_l \cap (x)_t \cap (x)_r$ where $(x)_l = x \cup SSx$, $(x)_m = x \cup SxS \cup SSxSS$ and $(x)_r = x \cup xSS$ are the left principal ideal, lateral principal ideal and the right principal ideal of i S generated by x.

THEOREM 3.17. The family $(Q_i)_{i\in I}$ of all quasi-ideals of a ternary semigroup S is a complete lattice.

PROOF. It easy to show that the family $(Q_i)_{i\in I}$ of all quasi-ideals of a ternary semigroup S is partially ordered by set inclusion. Let we show that the infimum of $(Q_i)_{i\in I}$ is $\wedge_{i\in I} Q_i = \bigcap_{i\in I} Q_i$. It is clear that $\bigcap_{i\in I} Q_i \subseteq Q_i, \forall i \in I$. Now, let Q be a quasi − ideal of S such that $Q \subseteq Q_i, \forall i \in I$. It is evident that $Q \subseteq \bigcap_{i\in I} Q_i$. let we show that the supremum of $(Q_i)_{i\in I}$ is $\vee_{i\in I} Q_i = (\bigcup_{i\in I} Q_i)_r \cap (\bigcup_{i\in I} Q_i)_m \cap (\bigcup_{i\in I} Q_i)_l$. It is clear that $Q_i \subseteq \vee_{i\in I} Q_i, \forall i \in I$. Now, let Q be a quasi-ideal of S such that $Q_i \subseteq Q, \forall i \in I$. Then, $\vee_{i\in I} Q_i = (\bigcup_{i\in I} Q_i \cup (\bigcup_{i\in I} Q_i)SS) \cap (\bigcup_{i\in I} Q_i \cup S(\bigcup_{i\in I} Q_i)S \cup SS(\bigcup_{i\in I} Q_i)SS) \cap \left(\bigcup_{i\in I} Q_i \cup SS(\bigcup_{i\in I} Q_i)\right) = \bigcup_{i\in I} Q_i \cup \left((\bigcup_{i\in I} Q_i)SS \cap (S(\bigcup_{i\in I} Q_i)S \cup SS(\bigcup_{i\in I} Q_i)SS) \cap SS(\bigcup_{i\in I} Q_i)\right) \subseteq QSS \cap (SQS \cup SSQSS) \cap SSQ \subseteq Q$.

THEOREM 3.18. A subset Q of a regular ternary semigroup S is a quasi-ideal of S iff and only iff $\mathrm{QSQSQ} \cap \mathrm{QSSQSSQ} \subseteq \mathrm{Q}$.

PROOF. Let S be a regular ternary semigroup and Q a quasi-ideal of S. Then, $\mathrm{QSQSQ} \cap \mathrm{QSSQSSQ} \subseteq \mathrm{SSQ}$, $\mathrm{QSQSQ} \cap \mathrm{QSSQSSQ} \subseteq \mathrm{QSS}$ and $\mathrm{QSQSQ} \cap \mathrm{QSSQSSQ} \subseteq \mathrm{SQS} \cup \mathrm{SSQSS}$, therefore $\mathrm{QSQSQ} \cap \mathrm{QSSQSSQ} \subseteq \mathrm{SSQ} \cap (\mathrm{SQS} \cup \mathrm{SSQSS}) \cap \mathrm{QSS} \subseteq \mathrm{Q}$.

Conversely, let S be a regular ternary semigroup and Q a subset of S such that $\mathrm{QSQSQ} \cap \mathrm{QSSQSSQ} \subseteq \mathrm{Q}$.Then, $\mathrm{QSS} \cap (\mathrm{SQS} \cup \mathrm{SSQSS}) \cap \mathrm{SSQ} = \mathrm{QSS}(\mathrm{SQS} \cup \mathrm{SSQSS})\mathrm{SSQ} = (\mathrm{QSS})(\mathrm{SQS})(\mathrm{SSQ}) \cup (\mathrm{QSS})(\mathrm{SSQSS})(\mathrm{SSQ}) \subseteq \mathrm{QSQSQ} \cup \mathrm{QSSQSSQ} \subseteq \mathrm{Q}$.

THEOREM 3.19. If S is a regular ternary semigroup and $\mathrm{Q}_1, \mathrm{Q}_2, \mathrm{Q}_3$ are quasi-ideals of S, then $\mathrm{Q}_1\mathrm{Q}_2\mathrm{Q}_3$ is a quasi-ideal of S.

PROOF. $(\mathrm{Q}_1\mathrm{Q}_2\mathrm{Q}_3)\mathrm{S}(\mathrm{Q}_1\mathrm{Q}_2\mathrm{Q}_3)\mathrm{S}(\mathrm{Q}_1\mathrm{Q}_2\mathrm{Q}_3) \cup (\mathrm{Q}_1\mathrm{Q}_2\mathrm{Q}_3)\mathrm{SS}(\mathrm{Q}_1\mathrm{Q}_2\mathrm{Q}_3)\mathrm{SS}(\mathrm{Q}_1\mathrm{Q}_2\mathrm{Q}_3) = (\mathrm{Q}_1(\mathrm{Q}_2\mathrm{Q}_3\mathrm{S})\mathrm{Q}_1(\mathrm{Q}_2\mathrm{Q}_3\mathrm{S})\mathrm{Q}_1)\mathrm{Q}_2\mathrm{Q}_3 \cup (\mathrm{Q}_1(\mathrm{Q}_2\mathrm{Q}_3\mathrm{S})\mathrm{S}\mathrm{Q}_1(\mathrm{Q}_2\mathrm{Q}_3\mathrm{S})\mathrm{S}\mathrm{Q}_1)\mathrm{Q}_2\mathrm{Q}_3 \subseteq \mathrm{Q}_1\mathrm{Q}_2\mathrm{Q}_3$.

THEOREM 3.20. If for every quasi-ideal Q of a ternary semigroup S, $\mathrm{Q}^3 = \mathrm{Q}$, then S is a regular ternary semigroup.

PROOF. Let R a right ideal of S, M a lateral ideal of S, and L a left ideal of S. Since $\mathrm{R} \cap \mathrm{M} \cap \mathrm{L}$ is a quasi-ideal of S we have $\mathrm{R} \cap \mathrm{M} \cap \mathrm{L} = (\mathrm{R} \cap \mathrm{M} \cap \mathrm{L})^3 = (\mathrm{R} \cap \mathrm{M} \cap \mathrm{L})(\mathrm{R} \cap \mathrm{M} \cap \mathrm{L})(\mathrm{R} \cap \mathrm{M} \cap \mathrm{L}) \subseteq \mathrm{RML}$. On the other hand, $\mathrm{RML} \subseteq \mathrm{R} \cap \mathrm{M} \cap \mathrm{L}$. Thus, $\mathrm{RML} = \mathrm{R} \cap \mathrm{M} \cap \mathrm{L}$. This mean that S is regular.

IV. MINIMAL QUASI-IDEALS IN TERNARY SEMIGROUPS

DEFINITION 4.1 A quasi-ideal Q of a ternary semigroup S is called minimal iff Q does not contain proper quasi-ideals of S.

THEOREM 4.2. A quasi-ideal of a ternary semigroup S minimal iff and only iff it is an intersection of a minimal left ideal L , a minimal right ideal R, and a minimal lateral ideal M of S.

PROOF. Suppose that Q is a quasi-ideal of a ternary semigroup S and $Q = R \cap M \cap L$ where R, L and M are minimal right ideal, minimal left ideal and minimal lateral idealof a ternary semigroup S, respectively. Now, let we show that Q is minimal. If Q' is a quasi-ideal of a ternary semigroup S which is contained in Q,then $SSQ' \subseteq SSQ \subseteq SSL \subseteq L$, $Q'SS \subseteq QSS \subseteq RSS \subseteq R$ and $SQ'S \cup SSQ'SS \subseteq SQS \cup SSQSS \subseteq SMS \cup SSMSS \subseteq M$. Since SSQ', $Q'SS$ and $SQ'S \cup SSQ'SS$ are left ideal, right ideal and lateral ideal, respectively, then by minimality of L, R and M we have $SSQ' = L$, $Q'SS = R$ and $SQ'S \cup SSQ'SS = M$. Therefore, we find that $Q = R \cap M \cap L = SSQ' \cap (SQ'S \cup SSQ'SS) \cap Q'SS \subseteq Q'$.Then, we have $Q \subseteq Q'$ and $Q' \subseteq Q$, thus $Q = Q'$. Hence, Q is a minimal quasi-ideal of a ternary semigroup S.

Conversely, let $a \in Q$ where Q is a minimal quasi-ideal of a ternary semigroup S. We have $SSa \cap (SaS \cup SSaSS) \cap aSS$ is a quasi-ideal of S by Preposition 3.11. Then $SSa \cap (SaS \cup SSaSS) \cap aSS \subseteq SSQ \cap (SQS \cup SSQSS) \cap QSS \subseteq Q$. By the minimality of Q we find $SSa \cap (SS \cup SSaSS) \cap aSS = Q$.let we show that SSa is a minimal left ideal of S.If L is a left ideal of a ternary semigroup S which is contained in SSa, then $L \cap (SaS \cup SSaSS) \cap aSS \subseteq SSa \cap (SaS \cup SSaSS) \cap aSS = Q$, but $L \cap (SaS \cup SSaSS) \cap aSS$ is a quasi−ideal of a ternary semigroup S and by the minimality of Q we have $L \cap (SaS \cup SSaSS) \cap aSS = Q$, whence $Q \subseteq L$. Now, we have $SSa \subseteq SSQ \subseteq SSL \subseteq L$. Then, by inclusion $SSa \subseteq L$ and $L \subseteq SSa$ we have $SSa = L$. Thus, SSa is a minimal left ideal of a ternary semigroup S. Similarly, we can prove the minimality of the right ideal aSS and lateral ideal $SaS \cup SSaSS$.

PREPOSITION 4.3. Every minimal lateral ideal of a ternary semigroup S is a minimal ideal of S.

PROOF. Let M be a minimal lateral ideal of S. We have to show that M is a minimal ideal of$S$. Let $m \in M$. Then, $SmS \cup SSmSS$ is a lateral ideal of S and $SmS \cup SSmSS \subseteq SMS \cup SSMSS \subseteq M$. Since M is minimal we have $M = SmS \cup SSmSS$. Now, $MSS = (SmS \cup SSmSS)SS = (SmS)SS \cup (SSmSS)SS \subseteq SmS \cup SSmSS \subseteq M$ and $SSM = SS(SmS \cup SSmSS) = SS(SmS) \cup SS(SSmSS) \subseteq SmS \cup SSmSS \subseteq M$. This mean that M is a left and right ideal of S. Therefore, M is an ideal of S. Now, we have to show that M is a minimal ideal of S. Let M' be an ideal of S such that $M' \subseteq M$. Since M' is an ideal of S, then M' is a lateral ideal of S. Since M is a minimal lateral ideal of S we have $M' = M$. Thus, M is a minimal ideal of S.

COROLLARY 4.4. Every minimal quasi-ideal of a ternary semigroup S is contained in a minimal ideal of S.

PROOF. Let Q be a quasi-ideal of S. Then, $Q = R \cap M \cap L$ where R is a minimal right ideal, M is a minimal lateral ideal and L is a minimal left ideal of S. It is clear that $Q \subseteq M$. Then, it follows that M is a minimal ideal of S.

DEFINITION 4.5. A non-zero two sided [left, right, lateral, quasi] ideal I is called 0-minimal iff it does not contain any proper two sided [left, right, lateral, quasi] ideal in a ternary semigroup S^0.

DEFINITION 4.6. A quasi-ideal Q of a ternary semigroup S^0 is called rigorouzly 0-minimal iff $SSQSS$ is a 0-minimal ideal of S^0.

DEFINITION 4.7. A quasi-ideal Q of a ternary semigroup S^0 is called canonical iff $Q \neq 0$ and $Q = R \cap M \cap L$ where R, L and M is a right , left and lateral 0-minimal ideal, respectively.

THEOREM 4.8. If Q is a canonical quasi-ideal of a ternary semigroup S^0, then it is rigorouzly 0-minimal.

PROOF. Let Q a canonical quasi-ideal of a ternary semigroup S^0. Then, Q$\neq 0$ and $Q = R \cap M \cap L$ where R, L and M are right, left and lateral 0-minimal ideals of a ternary semigroup S^0. We have $SS(R \cap M \cap L)SS = (SSR \cap SSM \cap SSL)SS = (SSR)SS \cap S(SMS)S \cap (SSL)SS \subseteq RSS \cap SMS \cap LSS \subseteq R \cap M \cap L$. Since R, L and M are 0-minimal ideals, then $R \cap M \cap L$ is a 0-minimal quasi-ideal and by the minimality of $R \cap M \cap L$ we have $SS(R \cap M \cap L)SS = R \cap M \cap L$. This imply that $R \cap M \cap L$ is rigorouzly 0-minimal.

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